HOMOMORPHISMS OF THE ALGEBRA OF SYMMETRIC ANALYTIC FUNCTIONS ON $\ell_1$

The algebra $H_{bs}(\ell_1)$ of symmetric analytic functions of bounded type is investigated. In particular, we study continuity of some homomorphisms of the algebra of symmetric polynomials on $\ell_p$ and composition operators of the algebra of symmetric analytic functions. The paper contains several open questions.

Key words and phrases: polynomials and analytic functions on Banach spaces, symmetric polynomials, spectra of algebras.

INTRODUCTION

Let $X$ be a complex Banach space. By a symmetric function on $X$ we mean a function which is invariant with respect to a semigroup of isometric operators on $X$. In the case $X = \ell_p$ by a symmetric function on $\ell_p$ we mean a function which is invariant under any reordering of a sequence in $\ell_p$.

Let us denote by $P(\ell_p)$ the algebra of all polynomials on $\ell_p$, $1 \leq p < \infty$, and by $P_s(\ell_p)$ the algebra of all symmetric polynomials on $\ell_p$. The completion of $P(\ell_p)$ in the metric of uniform convergence on bounded sets coincides with the algebra of entire analytic functions of bounded type $H_b(\ell_p)$ on $\ell_p$. We use the notations $H_{bs}(\ell_p)$ for the subalgebra of all symmetric analytic functions in $H_b(\ell_p)$. Also we use the notation $M_{bs}(\ell_p)$ for the spectrum (the set of all non-null continuous complex-valued homomorphisms) of the algebra $H_{bs}(\ell_p)$.

Symmetric polynomials on rearrangement-invariant function spaces were studied in [7, 8]. Spectra of algebras of analytic functions were studied in [2, 3, 9, 10]. The spectrum of the algebra $H_{bs}(\ell_p)$ was investigated in [4–6].

Recall that for any $\varphi, \theta \in M_{bs}(\ell_p)$ and $f \in H_{bs}(\ell_p)$, the symmetric convolution $\varphi \ast \theta$ was defined in [4] as follows

$$(\varphi \ast \theta)(f) = \varphi(\theta[T^*_y(f)]),$$

form an algebraic basis in the algebra of all symmetric polynomials on $\ell_p$, where $\lfloor p \rfloor$ is the smallest integer that is greater than or equal to $p$.

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where $T^y_1(f)(x) = f(x \bullet y) := (x_1, y_1, x_2, y_2, \ldots), x, y \in \ell_p, x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)$.

Let $x, y \in \ell_p, x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots)$. In [6] the multiplicative intertwining of $x$ and $y$, $x \odot y$, was defined as the resulting sequence of ordering the set $\{x_i, y_j : i, j \in \mathbb{N}\}$ with one single index in some fixed order. It enabled us to define the multiplicative convolution operator as a mapping $f \mapsto M_y(f)$, where $M_y(f)(x) = f(x \odot y)$. And for arbitrary $\varphi, \theta \in M_{bs}(\ell_p)$ in [6] it was defined their multiplicative convolution $\varphi \odot \theta$ according to

$$(\varphi \odot \theta)(f) = \varphi(\theta[M_x(f)]) \text{ for every } f \in H_{bs}(\ell_p).$$

Using the symmetric convolution operation and the multiplicative convolution operator in the spectrum of the algebra $H_{bs}(\ell_1)$, a representation of $M_{bs}(\ell_1)$ in terms of entire functions of exponential type was obtained.

In this paper we continue to investigate the algebra $H_{bs}(\ell_1)$ of all symmetric analytic functions on $\ell_1$ that are bounded on bounded sets. In particular, we study continuity of some homomorphisms (linear multiplicative operators) of the algebra of symmetric polynomials on $\ell_p$ and composition operators of the algebra of symmetric analytic functions.

1 CONTINUOUS AND DISCONTINUOUS HOMOMORPHISMS

Let us recall that in [5] it was constructed a family $\{\psi_\lambda : \lambda \in \mathbb{C}\}$ of elements of the set $M_{bs}(\ell_p)$ such that $\psi_\lambda(F_p) = \lambda$ and $\psi_\lambda(F_k) = 0$ for $k > p$.

**Proposition 1.** The homomorphism $\Gamma : P_s(\ell_1) \to P_s(\ell_1)$, such that $\Gamma : F_n \mapsto F_{n-1}, (\text{in particular}, \Gamma : F_1 \mapsto 0)$, is discontinuous.

**Proof.** Since $\psi_\lambda \circ F_1 = \lambda$ and $\psi_\lambda \circ F_k = 0$ when $k \neq 1$, we have that $\psi_\lambda \circ \Gamma(F_2) = \lambda$ and $\psi_\lambda \circ \Gamma(F_k) = 0, k \neq 2$. It follows that $\psi_\lambda \circ \Gamma$ is discontinuous and we obtain that $\Gamma$ is discontinuous too.

Note that $\Gamma$ acts in the natural way from $P_s(\ell_2)$ into $P_s(\ell_1)$.

**Question 1.** Does the homomorphism $\Gamma : P_s(\ell_2) \to P_s(\ell_1)$ is discontinuous?

**Proposition 2.** The homomorphism $\Delta : P_s(\ell_1) \to P_s(\ell_1), \Delta : F_{n-1} \mapsto F_n$, is discontinuous.

**Proof.** Let us define

$$m(P(x)) := P(-x) = (-1)^{\deg P} P(x),$$

where $P$ is a homogeneous polynomial. It is easy to see that $m$ is continuous and $m(F_k) = (-1)^k F_k$.

We have $m \circ \Delta \circ m \circ \Delta(F_n) = -F_{n+2}$. Let $x \in \ell_1, x \neq 0$. Let us define

$$\Theta_x := \delta_x \circ m \circ \Delta \circ m \circ \Delta.$$

Then $\Theta_x(F_n) = -F_{n+2}(x)$.

Let $x_0 = (-1, 0, 0, \ldots)$. It is easy to see that $\delta_{x_0}(F_n) = \left\{\begin{array}{ll} -1, & \text{if } n = 2k - 1, \\ 1, & \text{if } n = 2k. \end{array}\right.$

We have $\Theta_{x_0}(F_n) : (F_1, F_2, \ldots) \mapsto (0, 0, 1, -1, 1, -1, \ldots)$. According to [5, Theorem 1.6] we have that

$$(\delta_{x_0} \circ \Theta_{x_0})(F_1) = \delta_{x_0}(F_1) + \Theta_{x_0}(F_1) = -1 + 0 = -1.$$
Hence we obtain that \( \Delta \) is discontinuous.

**Remark 1.** Propositions 1 and 2 are also true for homomorphisms \( \Gamma : \mathcal{P}_s(\ell_p) \to \mathcal{P}_s(\ell_p) \) and \( \Delta : \mathcal{P}_s(\ell_p) \to \mathcal{P}_s(\ell_p) \).

## 2 Composition Operators

In this section we consider some homomorphisms which are composition operators, and study their continuity.

1. Let \( R : \mathbb{C}^m \to \mathbb{C}^m \) be an analytic mapping, \( R = (R_1, \ldots, R_m) \). Let us define \( T_R : (F_1, \ldots, F_m) \mapsto (R_1(F_1, \ldots, F_m), \ldots, R_m(F_1, \ldots, F_m)) \), that is

\[
T_R(F_k) = R_k(F_1, \ldots, F_m).
\]

Let \( P \) be a symmetric polynomial of degree \( m \) on \( \ell_1 \). Then, as it was mentioned above, there exists a polynomial \( q \) on \( \mathbb{C}^m \) such that \( P(x) = q(F_1(x), \ldots, F_m(x)) \). Applying \( T_R \) we obtain that

\[
T_R(P) = q(R_1(F_1, \ldots, F_m), \ldots, R_m(F_1, \ldots, F_m)).
\]

**Proposition 3.** If \( R : t_n \mapsto a_n t_n + c_n \), where \( a_n = \varphi(F_n) \) for some \( \varphi \in \mathcal{M}_{bs} \) and \( c_n = \psi(F_n) \) for some \( \psi \in \mathcal{M}_{bs} \), then \( T_R \) is continuous.

In this case \( T_R(f) = (\delta_x \circ \varphi) \ast \psi(f) \) for every \( f \in \mathcal{H}_{bs}(\ell_1) \).

**Question 2.** For which more \( R \) the mapping \( T_R \) is continuous?

2. Let us consider now an analytic function of one variable \( h(t) \) and define

\[
T_h(F_k(x)) := \sum_{n=1}^{\infty} (h(x_n))^k.
\]

**Proposition 4.** The operator \( T_h \) is continuous.

**Proof.** The continuity of \( T_h \) can be proved directly.

3. Let \( \{P_n\}_{n=1}^{\infty} \) be a sequence of symmetric polynomials such that for every \( x \in \ell_1 \) the sequence \( (P_1(x), \ldots, P_n(x), \ldots) \in \ell_1 \).

Let us denote by \( P \) a mapping \( x \mapsto (P_1(x), \ldots, P_n(x), \ldots) \). Also for every \( f \in \mathcal{H}_{bs}(\ell_1) \) we define

\[
C_P(f)(x) := f \circ P(x).
\]

**Proposition 5.** The composition operator \( C_P(f) \) is continuous.

**Theorem 1.** Let \( G : \ell_1 \to \ell_1 \) be an analytic operator of bounded type. \( G \) commutes with permutation operators (in the sense that \( G(\sigma_1 x) = \sigma_2 G(x) \), where \( \sigma_1, \sigma_2 \) are permutations on the set of positive integers) if and only if the operator \( C_G(f)(x) := f \circ G(x) \), where \( x \in \ell_1, f \in \mathcal{H}_{bs}(\ell_1) \), is homomorphism.
Proposition 6. \(U\) and let \(H\) homomorphism on \(G\). Then \(f\) is the Gelfand transform of \(F\) such that 

\[
\delta_x \circ F(f) = F(f)(x) = \Lambda(\delta_x)(f) = \hat{f}(\Lambda(\delta_x)).
\]

It is easy to see that not every mapping \(\Lambda : \mathcal{M}_{bs}(\ell_1) \to \mathcal{M}_{bs}(\ell_1)\) generates a continuous homomorphism on \(\mathcal{H}_{bs}(\ell_1)\) by the formula (2). We denote by \(\mathcal{M}(\ell_1)\) the class of all mappings which generate continuous homomorphisms.

Question 3. How can we describe the class \(\mathcal{M}(\ell_1)\)?

From the properties of the operations \(\star\) and \(\diamond\) immediately follows the next theorem.

Theorem 3. Let \(\varphi \in \mathcal{M}_{bs}(\ell_1)\) and mappings \(\Lambda_1, \Lambda_2 : \mathcal{M}_{bs}(\ell_1) \to \mathcal{M}_{bs}(\ell_1)\) belong to \(\mathcal{M}(\ell_1)\). Define 

\[
\Lambda_\star(\varphi) := \Lambda_1(\varphi) \star \Lambda_2(\varphi),
\]

\[
\Lambda_\diamond(\varphi) := \Lambda_1(\varphi) \diamond \Lambda_2(\varphi).
\]

Then \(\Lambda_\star\) and \(\Lambda_\diamond\) belong to \(\mathcal{M}(\ell_1)\) as well. In other words, the class \(\mathcal{M}(\ell_1)\) is closed with respect to symmetric operations \(\star\) and \(\diamond\).
REFERENCES


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Досліджується алгебра \( \mathcal{H}_{bs}(\ell_1) \) цілих симметричних аналітичних функцій з \( \ell_1 \) в \( C \), що є обмеженими на обмежених множинах. Зокрема, вивчається неперервність деяких гомоморфізмів алгебри симметричних поліномів на просторі \( \ell_p \) та операторів композиції на алгебрі симметричних аналітичних функцій. В статті поставлено декілька відкритих питань.

Ключові слова і фрази: поліноми та аналітичні функції на банахових просторах, симетричні поліноми, спекти алгебр.

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В работе исследуется алгебра \( \mathcal{H}_{bs}(\ell_1) \) целых симметрических аналитических функций ограниченного типа с \( \ell_1 \) в \( C \). В частности, изучается непрерывность некоторых гомоморфизмов алгебры симметрических полиномов на пространстве \( \ell_p \) и операторов композиции на алгебре симметрических аналитических функций. В статье сформулировано несколько открытых вопросов.

Ключевые слова и фразы: полиномы и аналитические функции на банаховых пространствах, симметрические полиномы, спектры алгебр.