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Namioka property of generalized ordered spaces

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A topological space X is called *Namioka*, if for every compact space K and every separately continuous function $f : X \times K \to \mathbb{R}$ there exists a dense G_δ-set $A \subseteq X$ such that f is jointly continuous at every point of $A \times K$.

We introduce a notion of a topological space with *countable resolvable set condition* (i.e. in every separable subspace of such space each well-ordered strictly increasing (or decreasing) family of resolvable sets is at most countable) and prove that every hereditarily Baire perfect generalized ordered space with countable resolvable set condition is Namioka.

Key words and phrases: generalized ordered space, Namioka property, countable resolvable set condition, Baire-one function.

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1 Introduction

A linearly ordered topological space (LOTS) is a triple $(X, \tau, \leq)$, where $(X, \leq)$ is a linearly ordered set and τ is the usual open-interval topology of the order $\leq$. Note that a subspace of LOTS is not necessarily LOTS as the example of Sogrenfrey line shows: it is a subspace of the double arrow space which is compact and linearly ordered, but the Sogrenfrey line is not LOTS. A subspace of a linearly ordered topological space is called *a generalized ordered space* or *GO-space*.

A topological space X is called *Namioka*, if for every compact space K and every separately continuous function $f : X \times K \to \mathbb{R}$ there exists a dense G_δ-set $A \subseteq X$ such that f is jointly continuous at every point of $A \times K$. The classical result of I. Namioka [9, Theorem 1.2] states that every Čech-complete space is Namioka. J. Saint-Raymond [10, Theorem 3] proved that every completely regular Namioka space is Baire. Moreover, he showed that in the class of all metrizable spaces, Namioka and Baire spaces coincide and that every separable Baire space is Namioka.

In [2], it was proved that every quarter-stratifiable Baire space is Namioka and the Sorgenfrey line is a hereditarily Namioka space. Namioka property of GO-spaces in terms of topological games was considered by V. Mykhaylyuk [8]. He introduced a modification of the classical weak Choquet σ-game (see [8, Section 2]) and showed that this new game coincides with the σ-game in the class of GO-spaces.

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Theorem 1 ([8, Theorem 5.3]). *Let X be a GO-space which can be covered by a sequence $(A_\xi)_{0\le\xi<\omega_1}$ of nowhere dense subsets $A_\xi \subseteq X$. Then X is Namioka if and only if the player β has no winning strategy in o-game.*

The next open question was formulated in [8, Question 6.3] and became the starting point for the investigations of current manuscript.

Question 1. *Let X be a Namioka GO-space. Is it true that the player β has no winning strategy in o-game?*

In the given note we show that every hereditarily Baire perfect GO-space X with countable resolvable set condition (see Definition 1) is Namioka. In view of Theorem 1 one can reformulate Question 1 in the following form.

Question 2. *Let X be a hereditarily Baire perfect GO-space with countable resolvable set condition. Can X be covered by a sequence $(A_\xi)_{0\le\xi<\omega_1}$ of nowhere dense subsets $A_\xi \subseteq X$?*

The last question is still open and we will investigate it in our future research.

2 Auxiliary facts

Recall that a topological space X without isolated points is called *crowded*.

Lemma 1. *Let $\mathscr{A}$ be a partition of a first countable Hausdorff space X by nowhere dense sets. Then there exists a countable crowded subspace $Q \subseteq X$ such that $|A \cap Q| \le 1$ for all $A \in \mathscr{A}$.*

Proof. Let $\mathscr{A} = \{A_s : s \in S\}$ be a partition of X by nowhere dense sets, $A_s \ne A_t$ if $s \ne t$. For every $x \in X$ we consider a countable base $\{U_m^x : m \in \mathbb{N}\}$ of a point x such that $U_{m+1}^x \subseteq U_m^x$ for each m.

We take an arbitrary point $x_0 \in X$ and let $x_0 \in A_{s_0}$ for a unique $s_0 \in S$. The set $B_1 = X \setminus A_{s_0}$ is dense in X, hence $U_1^{x_0} \cap B_1 \ne \varnothing$. Let $x_1 \in U_1^{x_0} \cap B_1$. There exists $s_1 \ne s_0$ such that $x_1 \in A_{s_1}$. We put $m_1 = 1$. Since X is Hausdorff, there is $m_2 > m_1$ such that $U_{m_2}^{x_0} \cap U_{m_2}^{x_1} = \varnothing$. Moreover, there exist $x_{10} \in U_{m_2}^{x_0} \cap B_2$ and $x_{11} \in U_{m_2}^{x_1} \cap B_2$, where the set $B_2 = X \setminus (A_{s_0} \cup A_{s_1})$ is everywhere dense.

Assume we have chosen points $x_{\mathbf{i}_1}, \dots, x_{\mathbf{i}_n}$ and an increasing sequence $1 = m_1 < \dots < m_n$ of integers for some $n \ge 1$ such that

(i) $\mathbf{i}_k \in \{0,1\}^k$, $1 \le k \le n$;

(ii) $x_{\mathbf{i}_{k+1}} = x_{\mathbf{i}_k}$, if $\mathbf{i}_{k+1} = (0, \mathbf{i}_k)$, $1 \le k < n$;

(iii) $x_{(1,\mathbf{i}_k)} \in U_{m_{k+1}}^{x_{\mathbf{i}_k}}$, $1 \le k < n$;

(iv) $|A \cap Q_n| \le 1$ for all $A \in \mathscr{A}$, where $Q_n = \bigcup\limits_{k=1}^{n} \{x_{\mathbf{i}_k} : \mathbf{i}_k \in \{0,1\}^k\}$.

Now we choose 2^{n+1} points $x_{\mathbf{i}_{n+1}}$, where $\mathbf{i}_{n+1} \in \{0,1\}^{n+1}$. We put $x_{\mathbf{i}_{n+1}} = x_{\mathbf{i}_n}$, if $\mathbf{i}_{n+1} = (0, \mathbf{i}_n)$. Since X is Hausdorff, there exist $m_{n+1} > m_n$ such that $U_{m_{n+1}}^{x_{\mathbf{i}_n}} \cap U_{m_{n+1}}^{x_{\mathbf{j}_n}} = \varnothing$ for all distinct $\mathbf{i}_n, \mathbf{j}_n \in \{0,1\}^n$. Let $x_{\mathbf{i}_n} \in A_{s_{\mathbf{i}_n}}$ for some $s_{\mathbf{i}_n} \in S$ and notice that $A_{s_{\mathbf{i}_n}} \ne A_{s_{\mathbf{j}_n}}$ for all distinct

$\mathbf{i}_n, \mathbf{j}_n \in \{0,1\}^n$. Since the set $B_{n+1} = X \setminus \bigcup_{\mathbf{i}_n \in \{0,1\}^n} A_{s_{\mathbf{i}_n}}$ is everywhere dense in X, there exists $x_{(1,\mathbf{i}_n)} \in U^{x_{\mathbf{i}_n}}_{m_{n+1}} \cap B_{n+1}$.

It is easy to see that we get the sequence $x_{\mathbf{i}_1}, \ldots, x_{\mathbf{i}_n}, \ldots$ satisfying (i)–(iv) for every $n \geq 1$. We put $Q = \bigcup_{n=1}^{\infty} \{x_{\mathbf{i}_n} : \mathbf{i}_n \in \{0,1\}^n\}$. Note that property (iii) implies that Q is crowded. □

A subset A of a topological space X is an *H-set* or *resolvable in the sense of Hausdorff*, if there exists a decreasing sequence $(F_\xi)_{\xi \in [0,\alpha)}$ of closed subsets of X such that

$$A = \bigcup_{\xi < \alpha,\ \xi \text{ is odd}} (F_\xi \setminus F_{\xi+1}).$$

Notice that each open or closed set in a topological space is resolvable and the class of all H-sets in a hereditarily Baire perfect paracompact space coincides with the class of all sets, which are F_σ and G_δ simultaneously (see [5, Theorem 1, Proposition 3.1]). More relations between resolvable and Borel sets in topological spaces can be found in [11].

Definition 1. *We say that a topological space X satisfies Countable Resolvable Set Condition or is CRSC, if in every separable subspace of X each well-ordered strictly increasing (or decreasing) family of H-sets is at most countable.*

By [7, §24] every metric separable space is CRSC. Notice that if X is CRSC, then every its separable subspace should satisfy the Countable Chain Condition. A common example of a separable non-metrizable topological space that fails the Countable Chain Condition is the Niemytzki plane. Another classic example often used in set theory is a Suslin Line, which is a totally ordered space with the lexicographical order that is separable but fails Countable Chain Condition because it contains an uncountable antichain of points, each with disjoint neighborhoods.

Lemma 2. *Let X be a hereditarily Baire perfect GO-space with Countable Resolvable Set Condition, $\mathscr{A}$ be a σ-disjoint family of sets with empty interior such that $\cup\mathscr{A}'$ is F_σ in X for any $\mathscr{A}' \subseteq \mathscr{A}$. Then the union $\cup\mathscr{A}$ has empty interior.*

Proof. Assume to the contrary that $G = \mathrm{int}(\cup\mathscr{A}) \neq \varnothing$. Then we can also assume that $\mathscr{A}$ covers X, because otherwise we can replace X by a hereditarily Baire perfect GO-space G and $\mathscr{A}$ by the family $(A \cap G)_{A \in \mathscr{A}}$.

Let $\mathscr{A} = \bigcup_n \mathscr{A}_n$ and every family $\mathscr{A}_n$ consists of mutually disjoint sets. Notice that every $A_n = \bigcup \mathscr{A}_n$ is an F_σ-set in X and $(A_n)_{n \in \mathbb{N}}$ is a covering of a Baire space X. Then there is $N \in \mathbb{N}$ such that $X_0 = \mathrm{int}\, A_N \neq \varnothing$. Let us observe that X_0 is a hereditarily Baire perfect GO-space. We put

$$\mathscr{A}_0 = (A \cap X_0)_{A \in \mathscr{A}_N}.$$

Let $(A_\xi)_{\xi < \alpha}$ be any well-ordering of $\mathscr{A}_0$. Since $\bigcup \mathscr{A}'$ is F_σ in X_0 for any subfamily $\mathscr{A}' \subseteq \mathscr{A}_0$ and $\mathscr{A}_0$ is a partition of X_0, every set A_ξ is F_σ and G_δ simultaneously. Moreover, $\mathrm{int}_{X_0} A_\xi = \varnothing$, because X_0 is an open subset of X and $\mathrm{int}_X A_\xi = \varnothing$.

Fix $\xi < \alpha$. Then there exist sequences (U_n) and (F_n) of subsets in X_0 such that

$$A_\xi = \bigcup_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} F_n,$$

each U_n is open and each F_n is closed and nowhere dense in X_0. Assume that A_ξ is somewhere dense and find an open set $V \subseteq X_0$ such that $A_\xi \cap V$ is dense in V. Then for every n the set

$$B_n = (X_0 \setminus \overline{V}) \cup (U_n \setminus F_n)$$

is open dense subset of X_0. Since X_0 is Baire, $\bigcap\limits_{n=1}^{\infty} B_n$ is dense in X_0 (and in V). From the other hand, we have

$$\left(\bigcap_{n=1}^{\infty} B_n\right) \cap V \subseteq \left(\bigcap_{n=1}^{\infty} U_n \setminus F_n\right) \cap V = \varnothing.$$

Hence, our assumption is not valid, which follows that A_ξ is nowhere dense in X_0.

Therefore, $\mathscr{A}_0$ is a partition of X_0 by nowhere dense sets. Note that every perfect GO-space is first countable [1, Theorem 3.1]. By Lemma 1 there exists a countable crowded set $Q \subseteq X$ such that

$$\text{every } A_\xi \text{ contains at most one point from } Q. \qquad (*)$$

We put

$$Y = \overline{Q} \quad \text{and} \quad B_\xi = A_\xi \cap Y$$

for all $\xi < \alpha$. Suppose that there is $\xi < \alpha$ such that $W = \mathrm{int}_Y B_\xi \neq \varnothing$. Then $W \subseteq A_\xi$ and it contains infinitely many points from Q, which contradicts to $(*)$. Hence, every A_ξ has empty interior in Y. Therefore, $\mathscr{B} = (B_\xi)_{\xi<\alpha}$ is a partition of Y by sets with empty interior such that every union of members of $\mathscr{B}$ is F_σ in Y. We put

$$C_\xi = \bigcup_{\eta \le \xi} B_\eta$$

for all $\xi < \alpha$ and obtain strictly increasing sequence of F_σ- and G_δ-sets C_ξ such that the family $(C_\xi)_{\xi<\alpha}$ covers a hereditarily Baire perfect *GO*-space Y.

Since every C_ξ is an H-set [5, Proposition 3.1] and Y has CRSC, there exists $\beta < \omega_1$ such that $C_\xi = C_{\xi+1} = \dots$ for all $\xi \ge \beta$. Hence, $(C_\xi)_{\xi \ge \beta}$ is at most countable covering of Y by nowhere dense sets. This is a contradiction, since Y is Baire. ☐

3 Classification of maps on GO-spaces

The definition of Namioka space is equivalent to the following: X is Namioka if and only if for every compact space K any continuous function $f : X \to C_p(K)$ is continuous at every point of a dense G_δ-subset of X as a function with values in $C_u(K)$, where $C_p(K)$ and $C_u(K)$ are spaces of all continuous functions between K and $\mathbb{R}$ equipped with the topologies of pointwise and uniform convergences, respectively.

Recall that a function $f : X \to Y$ between topological spaces X and Y has the *point of continuity property*, if for every nonempty closed set $F \subseteq X$ the restriction $f|_F$ has a point of continuity.

Theorem 2. *Let X be a hereditarily Baire perfect GO-space with Countable Resolvable Set Condition, (Y, ϱ) be a metric space. Then every Borel 1 function $f : X \to Y$ has point of continuity property.*

Proof. Fix any F_σ-measurable map $f : X \to Y$, $\varepsilon > 0$ and a closed non-empty set $F \subseteq X$. Since X is hereditarily Baire, it is sufficient to assume that $F = X$ and to prove that there exists an open set G in X such that $\mathrm{diam} f(G) < \varepsilon$.

Since Y is paracompact, we choose a σ-disjoint covering $\mathscr{V}$ of Y by open sets of diameters less than ε. Then the family $\mathscr{A} = \{f^{-1}(V) : V \in \mathscr{V}\}$ is a σ-disjoint covering of X. Moreover, $\bigcup \mathscr{A}'$ is an F_σ-set in X for all $\mathscr{A}' \subseteq \mathscr{A}$, since f is Borel-one map. Lemma 2 implies that there exists $V \in \mathscr{V}$ such that $G = \mathrm{int} f^{-1}(V) \neq \varnothing$. Then $\mathrm{diam}\, f(G) \leq \mathrm{diam}\, V < \varepsilon$. □

Lemma 3. *Let X be a topological space and $(Y, |\cdot - \cdot|)$ be a metric space. Let $C_p = C_p(X,Y)$ and $C_u = C_u(X,Y)$ be spaces of all continuous maps between X and Y equipped with the topologies of pointwise and uniform convergence, respectively. Then the identity map $\mathrm{id} : C_p \to C_u$ is Borel-one map.*

Proof. Consider the metric ϱ on C_u, $\varrho(f,g) = \sup_{x\in X} |f(x) - g(x)|$, which generates the topology on uniform convergence. Fix any open set $G \subseteq C_u$. Then

$$G = \bigcup_{n\in\mathbb{N}} \{x \in C_p : \varrho(x, C_p \setminus G) \leq \tfrac{1}{n}\}.$$

It is easy to see that the function $d : C_p \to [0, +\infty)$, defined by $d(x) = \varrho(x, C_p \setminus G)$ for all $x \in C_p$, is lower semi-continuous on C_p. This implies that the set $\{x \in C_p : \varrho(x, C_p \setminus G) \leq \frac{1}{n}\}$ is closed in C_p. Hence, G is an F_σ-set in C_p. □

Theorem 3. *Let X be a hereditarily Baire perfect GO-space with Countable Resolvable Set Condition, Y be a topological space, Z be a metric space and $f : X \to C_p(Y,Z)$ be a continuous map. If Z is equiconnected or* $\mathrm{dim} X = 0$, *then f belongs to the first Baire class as a map with values in $C_u(Y,Z)$.*

Proof. Lemma 3 implies that $f : X \to C_u(Y,Z)$ is F_σ-measurable. By Theorem 2 we get that $f : X \to C_u(Y,Z)$ is barely continuous. It remains to apply [2, Theorem 4.3] in case when $\mathrm{dim} X = 0$, and [2, Proposition 2.2] as well as [3, Theorem 10] in case when Z is equiconnected. □

Remark. *In general, the identity* $\mathrm{id} : C_p(X) \to C_u(X)$ *is not of the first Baire class. Let $X = 2^\omega$ be the compact space of all $\{0,1\}$-valued sequences. Assume that* $\mathrm{id} : C_p(2^\omega) \to C_u(2^\omega)$ *is a Baire-one map. Then it should be strongly functionally σ-discrete [4, Theorem 2.5], which implies that there exist a sequence $(\mathscr{B}_n)_{n\in\omega}$ of families of sets in $C_p(2^\omega)$ and a sequence $(\mathscr{U}_n)_{n\in\omega}$ of discrete families $\mathscr{U}_n = (U_B)_{B\in\mathscr{B}_n}$ of open sets such that $B \subseteq U_B$ for all $B \in \mathscr{B}_n$, $n \in \omega$, and the family $\bigcup_{n\in\omega} \mathscr{B}_n$ is a base for f (this means that the preimage $f^{-1}(V)$ of any open set $V \subseteq C_u(2^\omega)$ is a union of some members of families $\mathscr{B}_n$).*

Notice that $C_p(2^\omega)$ has Countable Chain Condition. Then every family $\mathscr{U}_n$ (and every family $\mathscr{B}_n$) is at most countable. Since $C_u(2^\omega)$ is not separable, there exists a discrete set $D \subseteq C_u(2^\omega)$ of cardinality $\aleph_1$. Consider the restriction $\mathrm{id}|_D$. *Then the base $\mathscr{B}$ for* $\mathrm{id}|_D$ *contains all singletons $\{y\}$ for $y \in D$, which implies that $|D| < \aleph_1$, a contradiction.*

Theorem 4. *Every hereditarily Baire perfect GO-space X with Countable Resolvable Set Condition is Namioka.*

Proof. It is sufficient to prove that for a compact Hausdorff space K any continuous map $f : X \to C_p(K)$ is continuous at every point of a dense G_δ-subset of X as a function with values in $C_u(K)$. So, let K be a compact Hausdorff space and $f : X \to C_p(K)$ be a continuous map. Lemma 3 and Theorem 2 imply that $f : X \to C_u(K)$ is barely continuous. Then the discontinuity point set of f is an F_σ-set of the first category in X by [6, Theorem 2.3]. Since X is Baire, the set of all continuity points of f is dense and G_δ in X. □

The following problem arises naturally.

Question 3. *Does every hereditarily Baire perfect GO-space satisfy the Countable Resolvable Set Condition?*

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Топологічний простір X називають *наміоковим*, якщо для кожного компактного простору K і кожної нарізно неперервної функції $f : X \times K \to \mathbb{R}$ існує всюди щільна G_δ-множина $A \subseteq X$ така, що f є сукупно неперервною в кожній точці множини $A \times K$.

Ми вводимо поняття топологічного простору з умовою зліченності розкладних множин (тобто, в кожному його сепарабельному підпросторі довільна цілком впорядкована строго зростаюча (або спадна) сім'я розкладних множин є не більше, ніж зліченною) і доводимо, що кожний досконалий спадково берівський узагальнений впорядкований простір з умовою зліченності розкладних множини є наміоковим.

Ключові слова і фрази: узагальнений впорядкований простір, властивість Наміоки, умова зліченності розкладних множин, функція першого класу Бера.