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On injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ with a three-element family $\mathscr{F}^3$ of inductive non-empty subsets of ω

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Let $\mathscr{F}$ be an ω-closed family of the subsets of the set of all non-negative integers ω and $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ be the bicyclic extension generated by $\mathscr{F}$, which are defined in [Visnyk Lviv Univ. Ser. Mech.-Mat. 2020, **90**, 5–19]. In [Visnyk Lviv Univ. Ser. Mech.-Mat. 2023, **95**, 28–45] we describe injective monoid endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ with the family $\mathscr{F}^3=\{[0),[1),[2)\}$ and show that for every injective monoid endomorphism ε of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ there exists a positive integer k such that $\varepsilon=\alpha_{[k]}$ and the mapping $\alpha_{[k]}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3}\to\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is defined by the formula

$$(i,j,[p))\alpha_{[k]}=\begin{cases}(ki,kj,[p)), & \text{if } p\in\{0,1\},\\ (k(i+1)-1,k(j+1)-1,[2)), & \text{if } p=2,\end{cases}$$

for all $i,j\in\omega$.

In this paper, we describe injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. In particular, we construct injective endomorphisms ϖ_3 and λ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ and describe the subsemigroup $\langle\lambda,\varpi_3\rangle$ of the semigroup $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, generated by ϖ_3 and λ. We show that for every injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ there exists an injective endomorphism $\iota\in\langle\lambda,\varpi_3\rangle$ such that $\varepsilon=\alpha_{[k]}\circ\iota$ for some positive integer k, where $\alpha_{[k]}$ is an injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Also, we prove that the submonoid of $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$, which is generated by the set $\{\alpha_{[k]}\colon k\in\mathbb{N}\}$ and by the endomorphism λ, is isomorphic to the semidirect product $(\mathbb{N},\cdot)\ltimes_{\mathfrak{h}}(\omega,+)$.

Key words and phrases: bicyclic monoid, inverse semigroup, bicyclic extension, endomorphism, semigroup of endomorphisms, inductive set, semidirect product.

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Introduction, motivation and main definitions

We shall follow the terminology of the monographs [1,2,16]. By ω we denote the set of all non-negative integers and by $\mathbb{N}$ the set of all positive integers. If $\mathfrak{f}\colon X\to Y$ is a map, then by $(x)\mathfrak{f}$ and $(A)\mathfrak{f}$ we denote the image of $x\in X$ and $A\subseteq X$ under $\mathfrak{f}$, respectively.

Let $\mathscr{P}(\omega)$ be the family of all subsets of ω. For any $F\in\mathscr{P}(\omega)$ and any integer n we put $n+F=\{n+k\colon k\in F\}$ if $F\neq\varnothing$ and $n+\varnothing=\varnothing$. A subfamily $\mathscr{F}\subseteq\mathscr{P}(\omega)$ is called *ω-closed* if $F_1\cap(-n+F_2)\in\mathscr{F}$ for all $n\in\omega$ and $F_1,F_2\in\mathscr{F}$. For any $a\in\omega$ we denote $[a)=\{x\in\omega\colon x\geqslant a\}$.

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A subset A of ω is said to be *inductive*, if $i \in A$ implies $i+1 \in A$. Obvious, that $\varnothing$ is an inductive subset of ω.

Remark 1 ([5]). (1) *By [4, Lemma 6], non-empty subset $F \subseteq \omega$ is inductive in ω if and only if $(-1+F) \cap F = F$.*

(2) *Since the set ω with the usual order is well-ordered, for any non-empty inductive subset F in ω there exists non-negative integer $n_F \in \omega$ such that $[n_F) = F$.*

(3) *Statement (2) implies that the intersection of an arbitrary finite family of non-empty inductive subsets in ω is a non-empty inductive subset of ω.*

For an arbitrary semigroup S any homomorphism $\alpha \colon S \to S$ is called an *endomorphism* of S. If the semigroup has the identity element 1_S, then the endomorphism α of S such that $(1_S)\alpha = 1_S$ is said to be a *monoid endomorphism* of S. A bijective endomorphism of S is called an *automorphism*.

A semigroup S is called *inverse* if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element x^{-1} is called the *inverse of* $x \in S$. If S is an inverse semigroup, then the function $\operatorname{inv} \colon S \to S$, which assigns to every element x of S its inverse element x^{-1}, is called the *inversion*.

If S is a semigroup, then we shall denote the subset of all idempotents in S by $E(S)$. If S is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ as a *band* (or the *band of* S). Then the semigroup operation on S determines the following partial order $\preccurlyeq$ on $E(S)$: $e \preccurlyeq f$ if and only if $ef = fe = e$. This order is called the *natural partial order* on $E(S)$.

If S is an inverse semigroup, then the semigroup operation on S determines the following partial order $\preccurlyeq$ on S: $s \preccurlyeq t$ if and only if there exists $e \in E(S)$ such that $s = te$. This order is called the *natural partial order* on S [20].

The *bicyclic monoid* $\mathscr{C}(p,q)$ is the semigroup with the identity 1 generated by two elements p and q subjected only to the condition $pq = 1$. The semigroup operation on $\mathscr{C}(p,q)$ is determined by

$$q^k p^l \cdot q^m p^n = q^{k+m-\min\{l,m\}} p^{l+n-\min\{l,m\}}.$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(p,q)$ is a group congruence [1].

On the set $\boldsymbol{B}_{\omega} = \omega \times \omega$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1) \cdot (i_2, j_2) = \begin{cases} (i_1 - j_1 + i_2, j_2), & \text{if } j_1 < i_2, \\ (i_1, j_2), & \text{if } j_1 = i_2, \\ (i_1, j_1 - i_2 + j_2), & \text{if } j_1 > i_2. \end{cases}$$

It is well known that the bicyclic monoid $\mathscr{C}(p,q)$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}$ by the mapping $\mathfrak{h} \colon \mathscr{C}(p,q) \to \boldsymbol{B}_{\omega}$, $q^k p^l \mapsto (k,l)$ (see [1, Section 1.12] or [18, Exercise IV.1.11(ii)]).

Next we shall describe the construction which is introduced in [4].

Let $\mathscr{F}$ be an ω-closed subfamily of $\mathscr{P}(\omega)$. On the set $\boldsymbol{B}_{\omega} \times \mathscr{F}$ we define the semigroup operation "$\cdot$" in the following way

$$(i_1, j_1, F_1) \cdot (i_2, j_2, F_2) = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + F_1) \cap F_2), & \text{if } j_1 < i_2, \\ (i_1, j_2, F_1 \cap F_2), & \text{if } j_1 = i_2, \\ (i_1, j_1 - i_2 + j_2, F_1 \cap (i_2 - j_1 + F_2)), & \text{if } j_1 > i_2. \end{cases}$$

In [4], it is proved that if the family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ is ω-closed, then $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$ is a semigroup. Moreover, if an ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ contains the empty set $\varnothing$, then the set $\boldsymbol{I} = \{(i, j, \varnothing) \colon i, j \in \omega\}$ is an ideal of the semigroup $(\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)$. For any ω-closed family $\mathscr{F} \subseteq \mathscr{P}(\omega)$ the following semigroup

$$\boldsymbol{B}_\omega^{\mathscr{F}} = \begin{cases} (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot)/\boldsymbol{I}, & \text{if } \varnothing \in \mathscr{F}, \\ (\boldsymbol{B}_\omega \times \mathscr{F}, \cdot), & \text{if } \varnothing \notin \mathscr{F} \end{cases}$$

is defined in [4]. Also, the structure of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is studied there.

Group congruences on the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ and its homomorphic retracts in the case when an ω-closed family $\mathscr{F}$ consists of inductive non-empty subsets of ω are studied in [5]. In [6], it is proven that an injective endomorphism ε of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is the indentity transformation if and only if ε has three distinct fixed points, which is equivalent to existence non-idempotent element $(i, j, [p)) \in \boldsymbol{B}_\omega^{\mathscr{F}}$ such that $(i, j, [p))\varepsilon = (i, j, [p))$.

In [3, 17], the algebraic structure of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ is established in the case when ω-closed family $\mathscr{F}$ consists of atomic subsets of ω. The structure of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}_n}$, for the family $\mathscr{F}_n$ which is generated by the initial interval $\{0, 1, \ldots, n\}$ of ω, is studied in [8]. The semigroup of endomorphisms of $\boldsymbol{B}_\omega^{\mathscr{F}_n}$ is described in [7, 19].

In [13], the semigroup $\mathbf{End}(\boldsymbol{B}_\omega)$ of all endomorphisms of the bicyclic semigroup $\boldsymbol{B}_\omega$ is described.

In the paper [9], injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}}$ with the two-elements family $\mathscr{F}$ of inductive non-empty subsets of ω are studies. In [10,12], the semigroup $\boldsymbol{End}^1(\boldsymbol{B}_\omega^{\mathscr{F}})$ of all monoid endomorphisms of the monoid $\boldsymbol{B}_\omega^{\mathscr{F}}$ is studied.

In [15], we study the semigroup $\overline{\boldsymbol{End}}(\boldsymbol{B}_\omega^{\mathcal{F}^2})$ of all endomorphisms of the bicyclic extension $\boldsymbol{B}_\omega^{\mathcal{F}^2}$ with the two-element family $\mathcal{F}^2$ of inductive non-empty subsets of ω. The submonoid $\langle \varpi \rangle^1$ of $\overline{\boldsymbol{End}}(\boldsymbol{B}_\omega^{\mathcal{F}^2})$ with the property that every element of the semigroup $\overline{\boldsymbol{End}}(\boldsymbol{B}_\omega^{\mathcal{F}^2})$ has the unique representation as the product of the monoid endomorphism of $\boldsymbol{B}_\omega^{\mathcal{F}^2}$ and the element of $\langle \varpi \rangle^1$ is constructed.

Later we assume that $\mathscr{F}^3$ is a family of inductive non-empty subsets of ω which consists of three sets. By [5, Proposition 1], for any ω-closed family $\mathscr{F}$ of inductive subsets in $\mathscr{P}(\omega)$ there exists an ω-closed family $\mathscr{F}^*$ of inductive subsets in $\mathscr{P}(\omega)$ such that $[0) \in \mathscr{F}^*$ and the semigroups $\boldsymbol{B}_\omega^{\mathscr{F}}$ and $\boldsymbol{B}_\omega^{\mathscr{F}^*}$ are isomorphic. Hence, without loss of generality, we may assume that the family $\mathscr{F}$ contains the set $[0)$, i.e. $\mathscr{F}^3 = \{[0), [1), [2)\}$.

In [14], we describe injective monoid endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$. Here we shown that for every injective monoid endomorphism ε of $\boldsymbol{B}_\omega^{\mathscr{F}^3}$ there exists a positive integer k such that $\varepsilon = \alpha_{[k]}$ and the mapping $\alpha_{[k]} \colon \boldsymbol{B}_\omega^{\mathscr{F}^3} \to \boldsymbol{B}_\omega^{\mathscr{F}^3}$ is defined by the formula

$$(i, j, [p))\alpha_{[k]} = \begin{cases} (ki, kj, [p)), & \text{if } p \in \{0, 1\}, \\ (k(i+1) - 1, k(j+1) - 1, [2)), & \text{if } p = 2, \end{cases}$$

for all $i, j \in \omega$ (see [14, Theorem 5]). Also we prove that the monoid $\boldsymbol{End}^1_{\mathsf{inj}}(\boldsymbol{B}_\omega^{\mathscr{F}^3})$ of all injective monoid endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$ is isomorphic to the multiplicative semigroup of positive integers.

In this paper, we describe injective endomorphisms of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$. In particular, we find endomorphisms ϖ_3 and λ of $\boldsymbol{B}_\omega^{\mathscr{F}^3}$ such that for every injective endomorphism ε of the semigroup $\boldsymbol{B}_\omega^{\mathscr{F}^3}$ there exists an injective endomorphism $\iota \in \langle \lambda, \varpi_3 \rangle$ such that $\varepsilon = \alpha_{[k]} \circ \iota$ for some positive integer k.

1 On some properties of injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}$

For any positive integer n we denote $\mathscr{F}^n = \{[0), \ldots [n-1)\}$.

Proposition 1. *For any positive integer n the following statements hold:*

(i) *the map $\lambda\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^n} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^n}$ defined by the formula*

$$(i,j,[p))\lambda = (i+1,j+1,[p)), \quad i,j\in\omega,\ p\in\{0,\ldots,n-1\},$$

is an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}$;

(ii) *the map $\varpi_n\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^n} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^n}$ defined by the formula*

$$(i,j,[p))\varpi_n = (i+p,j+p,[n-1-p)), \quad i,j\in\omega,\ p\in\{0,\ldots,n-1\},$$

is an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}$;

(iii) $\varpi_n^2 = \lambda^{n-1}$;

(iv) $\varpi_n\lambda = \lambda\varpi_n$.

Proof. The proofs of statements (i) and (ii) are same as [11, Proposition 6] and [11, Lemma 3], respectively.

(iii) For arbitrary $i,j\in\omega$ and $p\in\{0,\ldots,n-1\}$ we have

$$\begin{aligned}(i,j,[p))\varpi_n^2 &= (i+p,j+p,[n-1-p))\varpi_n\\ &= (i+p+n-1-p,j+p+n-1-p,[n-1-(n-1-p)))\\ &= (i+n-1,j+n-1,[p)),\end{aligned}$$

hence, $\varpi_n^2 = \lambda^{n-1}$.

(iv) For arbitrary $i,j\in\omega$ and $p\in\{0,\ldots,n-1\}$ we get

$$(i,j,[p))\varpi_n\lambda = (i+p,j+p,[n-1-p))\lambda = (i+p+1,j+p+1,[n-1-p))$$

and

$$(i,j,[p))\lambda\varpi_n = (i+1,j+1,[p))\varpi_n = (i+1+p,j+1+p,[n-1-p)),$$

which implies the equality $\varpi_n\lambda = \lambda\varpi_n$. $\square$

Recall that the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ is isomorphic to the bicyclic semigroup if and only if $\mathscr{F} = [p)$ for some $p\in\omega$.

For any non-negative integers s and t with $s < t \leqslant n$ we denote $\mathscr{F}^{[s,t]} = \{[s),\ldots,[t)\}$ and

$$\boldsymbol{B}_{\omega}^{\mathscr{F}^{[s,t]}} = \{(i,j,[p))\colon i,j\in\omega,\ p=s,\ldots,t\}.$$

Proposition 2. *Let s and t be non-negative integers such that $s < t \leqslant n$. Then the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^{[s,t]}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}$ is isomorphic to the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^m}$, $m\leqslant n$, if and only if $m = t-s+1$.*

Proof. Suppose that $m = t-s+1$. We define the map $\mathfrak{I}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^{[s,t]}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^m}$ by the formula

$$(i,j,[p+s))\mathfrak{I} = (i,j,[p)), \qquad i,j\in\omega,\ p\in\{0,\ldots,m-1\}.$$

For arbitrary $i_1, i_2, j_1, j_2 \in \omega$ and $p_1, p_2 \in \{0, \ldots, m-1\}$ we have

$$
\begin{aligned}
&((i_1, j_1, [p_1+s)) \cdot (i_2, j_2, [p_2+s)))\mathfrak{I} \\
&\qquad = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + [p_1+s)) \cap [p_2+s))\mathfrak{I}, & \text{if } j_1 < i_2, \\ (i_1, j_2, [p_1+s) \cap [p_2+s))\mathfrak{I}, & \text{if } j_1 = i_2, \\ (i_1, j_1 - i_2 + j_2, [p_1+s) \cap (i_2 - j_2 + [p_2+s)))\mathfrak{I}, & \text{if } j_1 > i_2 \end{cases} \\
&\qquad = \begin{cases} (i_1 - j_1 + i_2, j_2, (j_1 - i_2 + [p_1)) \cap [p_2))\mathfrak{I}, & \text{if } j_1 < i_2, \\ (i_1, j_2, [p_1) \cap [p_2))\mathfrak{I}, & \text{if } j_1 = i_2, \\ (i_1, j_1 - i_2 + j_2, [p_1) \cap (i_2 - j_2 + [p_2)))\mathfrak{I}, & \text{if } j_1 > i_2 \end{cases} \\
&\qquad = (i_1, j_1, [p_1)) \cdot (i_2, j_2, [p_2)) = (i_1, j_1, [p_1+s))\mathfrak{I} \cdot (i_2, j_2, [p_2+s))\mathfrak{I},
\end{aligned}
$$

because $[l+s) \cap [q+s) = ([l) \cap [q)) + s$ for all $l, q, s \in \omega$.

By [4, Theorem 2(iv)] the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^{[s,t]}}$ has $t-s+1$ $\mathscr{D}$-classes and $\boldsymbol{B}_{\omega}^{\mathscr{F}^m}$ has m $\mathscr{D}$-classes. If the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^{[s,t]}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^m}$ are isomorphic, then $\boldsymbol{B}_{\omega}^{\mathscr{F}^{[s,t]}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^m}$ have the same many $\mathscr{D}$-classes, and hence $m = t-s+1$. $\square$

For any positive integer m and $p \in \{0, 1, \ldots, n-1\}$ we define

$$
\boldsymbol{B}_{\omega}^{\mathscr{F}^n}[(m,m,[p))] = (m,m,[p)) \cdot \boldsymbol{B}_{\omega}^{\mathscr{F}^n} \cdot (m,m,[p)) = (m,m,[p)) \cdot \boldsymbol{B}_{\omega}^{\mathscr{F}^n} \cap \boldsymbol{B}_{\omega}^{\mathscr{F}^n} \cdot (m,m,[p)).
$$

Lemma 1. *Let n be an arbitrary positive integer. Then $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}[(m,m,[0))] = (\boldsymbol{B}_{\omega}^{\mathscr{F}^n})\lambda^m$ for any positive integer m.*

Proof. For an arbitrary positive integer m, any $i, j \in \omega$ and $p \in \{0, \ldots, n-1\}$ we have

$$
\begin{aligned}
&(m,m,[0)) \cdot (i,j,[p))) \cdot (m,m,[0)) \\
&\quad = \begin{cases} (i,j,(m-i+[0)) \cap [p)) \cdot (m,m,[0)), & \text{if } m < i, \\ (m,j,[0) \cap [p)) \cdot (m,m,[0)), & \text{if } m = i, \\ (m, m-i+j, [0) \cap (i-m+[p))) \cdot (m,m,[0)), & \text{if } m > i \end{cases} \\
&\quad = \begin{cases} (i,j,[p)) \cdot (m,m,[0)), & \text{if } m < i, \\ (m,j,[p)) \cdot (m,m,[0)), & \text{if } m = i, \\ (m,m-i+j,[0)) \cdot (m,m,[0)), & \text{if } m > i \text{ and } i-m+p \leqslant 0, \\ (m,m-i+j,(i-m+[p))) \cdot (m,m,[0)), & \text{if } m > i \text{ and } i-m+p > 0 \end{cases} \\
&\quad = \begin{cases} (i,j,[p)) \cdot (m,m,[0)), & \text{if } m \leqslant i, \\ (m,m-i+j,[0)) \cdot (m,m,[0)), & \text{if } m > i \text{ and } i-m+p \leqslant 0, \\ (m,m-i+j,(i-m+[p))) \cdot (m,m,[0)), & \text{if } m > i \text{ and } i-m+p > 0 \end{cases} \\
&\quad = \begin{cases} (i-j+m, m, (j-m+[p)) \cap [0)), & \text{if } m \leqslant i \text{ and } j < m, \\ (i,j,[p) \cap [0)), & \text{if } m \leqslant i \text{ and } j = m, \\ (i, j-m+m, [p) \cap (m-j+[0))), & \text{if } m \leqslant i \text{ and } j > m, \\ (m-m+i-j+m, m, (j-i+[0) \cap [0)), & \text{if } m > i, i+p \leqslant m \text{ and } m-i+j<m, \\ (m,m,[0) \cap [0)), & \text{if } m > i, i+p \leqslant m \text{ and } m-i+j=m, \\ (m, m-i+j, [0) \cap (i-j+[0))), & \text{if } m > i, i+p \leqslant m \text{ and } m-i+j>m, \\ (m-m+i-j+m, m, (j-m+[p)) \cap [0)), & \text{if } m > i, i+p > m \text{ and } m-i+j<m, \\ (m,m,(i-m+[p)) \cap [0)), & \text{if } m > i, i+p > m \text{ and } m-i+j=m, \\ (m, m-i+j, (i-m+[p)) \cap (i-j+[0))), & \text{if } m > i, i+p > m \text{ and } m-i+j>m \end{cases}
\end{aligned}
$$

$$= \begin{cases} (i-j+m, m, (j-m+[p]) \cap [0]), & \text{if } m \leqslant i \text{ and } j < m, \\ (i,j,[p]), & \text{if } m \leqslant i \text{ and } j = m, \\ (i,j,[p]), & \text{if } m \leqslant i \text{ and } j > m, \\ (m+i-j, m, [0]), & \text{if } m > i, i+p \leqslant m \text{ and } j < i, \\ (m,m,[0]), & \text{if } m > i, i+p \leqslant m \text{ and } j = i, \\ (m, m-i+j, [0]), & \text{if } m > i, i+p \leqslant m \text{ and } j > i, \\ (m+i-j, m, j-m+[p]), & \text{if } m > i, i+p > m \text{ and } j < i, \\ (m,m,i-m+[p]), & \text{if } m > i, i+p > m \text{ and } j = i, \\ (m, m-i+j, (i-m+[p])), & \text{if } m > i, i+p > m \text{ and } j > i, \end{cases}$$

and hence we get that $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}[(m,m,[0])] \subseteq (\boldsymbol{B}_{\omega}^{\mathscr{F}^n})\lambda^m$. Since for any $i,j \geqslant m$, $i,j \in \omega$ and $p \in \{0,\dots,n\}$ we have

$$(m,m,[0]) \cdot (i,j,[p])) \cdot (m,m,[0]) = (i,j,[p])) \cdot (m,m,[0]) = (i,j,[p])),$$

we conclude that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^n})\lambda^m \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^n}[(m,m,[0])]$. This implies the requested equality. $\square$

Proposition 1 (i) and Lemma 1 imply the following corollary.

Corollary 1. *Let n be an arbitrary positive integer. Then $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}[(m,m,[0])]$ are isomorphic semigroups for any positive integer m.*

We observe that the isomorphism of the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^n}[(m,m,[0])]$ is defined as the corestriction of the endomorphism $\lambda^m \colon \boldsymbol{B}_{\omega}^{\mathscr{F}^n} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^n}$ onto its image $(\boldsymbol{B}_{\omega}^{\mathscr{F}^n})\lambda^m$.

2 On endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$

In this section, for any $p \in \{0,1,2\}$ we denote $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_p} = \{(i,j,[p]) \colon i,j \in \omega\}$. Also we denote $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}} = \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0} \cup \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$, $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,2}} = \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0} \cup \boldsymbol{B}_{\omega}^{\mathscr{F}^3_2}$, and $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{1,2}} = \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1} \cup \boldsymbol{B}_{\omega}^{\mathscr{F}^3_2}$.

Lemma 2. *Let ε be an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Then the following statements hold:*

(i) $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \nsubseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}}$;

(ii) $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \nsubseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{1,2}}$;

(iii) $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \nsubseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,2}}$;

(iv) $(0,0,[0])\varepsilon \notin \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$.

Proof. (i) Suppose to the contrary that there exists an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}}$. If $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$, then by [4, Lemma 2] we have that $(0,0,[0])\varepsilon = (s,s,[0])$ for some $s \in \omega$. By Corollary 1 there exists an isomorphism $\mathfrak{I} \colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3}(s,s,[0]) \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Then the composition $\varepsilon \circ \mathfrak{I}$ is an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})(\varepsilon \circ \mathfrak{I}) \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}} \subset \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, which contradicts [14, Theorem 5]. Hence $(0,0,[0])\varepsilon \notin \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$.

If $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$, then by [4, Lemma 2] there exists $s\in\omega$ such that $(0,0,[0])\varepsilon=(s,s,[1])$. We consider the map $\varpi_2\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}} \subset \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ defined by the formula

$$(i,j,[p])\varpi_2=\begin{cases}(i,j,[1]), & \text{if } p=0,\\ (i+1,j+1,[0]), & \text{if } p=1.\end{cases}$$

By [15, Lemma 1], the map ϖ_2 is an injective endomorphism of the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, because by Corollary 1 the semigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^2}$ are isomorphic. Then the composition $\varepsilon\circ\varpi_2\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}} \subset \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$, which contradicts the above part of the proof of this statement. The obtained contradiction implies statement (i).

(ii) Suppose to the contrary that there exists an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{1,2}}$. By Corollary 1 there exists an isomorphism $\mathfrak{I}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{1,2}} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}}$. Then the composition $\varepsilon\circ\mathfrak{I}$ is an injective endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})(\varepsilon\circ\mathfrak{I}) \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{1,2}}$, which contradicts statement (i). The obtained contradiction implies statement (ii).

(iii) Suppose the contrary, i.e. there exists an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \subseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,2}}$. If $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$, then by [4, Lemma 2] we get that $(0,0,[0])\varepsilon=(s,s,[0])$ for some $s\in\omega$. Since $(0,0,[0])$ is the identity element of the monoid $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, $(0,0,[0])\varepsilon$ is the identity element of the subsemigroup $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. This implies that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon$ is a subsemigroup of the homomorphic image $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s$. By Corollary 1 there exists an isomorphism $\mathfrak{I}\colon (\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{1,2}})\lambda^s \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. This implies that the composition $\varepsilon\circ\mathfrak{I}$ is an injective monoid endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Then by [14, Theorem 5] there exists a positive integer k such that $\alpha_{[k]}=\varepsilon\circ\mathfrak{I}$. Then $\varepsilon=\alpha_{[k]}\circ\mathfrak{I}^{-1}$ and $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \nsubseteq \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,2}}$.

If $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_2}$, then by [4, Lemma 2] there exists $s\in\omega$ such that $(0,0,[0])\varepsilon=(s,s,[0])$. Then the composition $\varepsilon\circ\varpi_3$ is an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(0,0,[0])\varepsilon\circ\varpi_3 \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$. This contradicts the above part of proof of this statement. The obtained contradiction implies statement (iii).

(iv) Suppose to the contrary that there exists an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$. By statements (i) and (ii), neither the condition $(0,0,[1])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$ nor the condition $(0,0,[2])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$ holds.

If $(0,0,[1])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_2}$, then by statement (ii) we have that $(0,0,[2])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$. By [4, Lemma 2], we have that $(0,0,[0])\varepsilon=(s,s,[1])$ for some $s\in\omega$. Proposition 1 and Corollary 1 imply that the corestriction of the mapping $\lambda^s\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ onto the image $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s$ is an isomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ onto $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s$, and we denote this isomorphism by $\mathfrak{I}$. Then the composition $\varepsilon\circ\mathfrak{I}^{-1}$ is a monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Hence, without loss of generality, we may assume in what follows that $(0,0,[0])\varepsilon=(0,0,[1])$.

Since, by Corollary 1, the subsemigroups $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{0,1}}$ and $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_{1,2}}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ are isomorphic under the mapping $\mathfrak{I}\colon (i,j,[p]) \mapsto i,j,[p+1])$, $i,j\in\omega$, $p\in\{0,1\}$, it follows from [9, Theorem 1] that there exist a positive integer k and $p\in\{0,1\}$ such that

$$(i,j,[0])\varepsilon=(ki,kj,[0]), \qquad (i,j,[1])\varepsilon=(p+ki,p+kj,[1]), \quad \text{for all } i,j\in\omega.$$

By [4, Lemma 5], we have

$$(0,0,[2]) \preccurlyeq (0,0,[1]) \preccurlyeq (0,0,[0]) \qquad \text{and} \qquad (2,2,[0]) \preccurlyeq (1,1,[1]) \preccurlyeq (0,0,[2]).$$

By [16, Proposition 1.4.21 (6)], we get

$$(0,0,[2])\varepsilon \preccurlyeq (0,0,[1])\varepsilon = (p,p,[2]) \preccurlyeq (0,0,[0])\varepsilon = (0,0,[1])$$

and

$$(2,2,[0])\varepsilon = (2k,2k,[1]) \preccurlyeq (1,1,[1])\varepsilon(k+p,k+p,[2]) \preccurlyeq (0,0,[2])\varepsilon.$$

Since $(0,0,[2])\varepsilon \preccurlyeq (0,0,[1])$, it follows from [4, Lemma 2] that there exists a positive integer x such that $(0,0,[2])\varepsilon = (x,x,[0])$. By the equality $(1,1,[0]) \cdot (0,0,[2]) = (1,1,[1])$, we have

$$(1,1,[0])\varepsilon \cdot (0,0,[2])\varepsilon = (1,1,[1])\varepsilon.$$

Hence, we get the equality

$$(k,k,[1]) \cdot (x,x,[0]) = (k+p,k+p,[2]),$$

which contradicts the semigroup operation on $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. The obtained contradiction implies that the conditions $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$, $(0,0,[1])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_2}$, and $(0,0,[2])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$ do not hold.

Suppose that there exists an injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ such that $(0,0,[0])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$, $(0,0,[1])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$, and $(0,0,[2])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_2}$. By Proposition 1 (ii), ϖ_3 is an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ and hence the composition $\varepsilon \circ \varpi_3$ is an injective endomorphism of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, as well. Then we get $(0,0,[0])(\varepsilon \circ \varpi_3) \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$, $(0,0,[1])(\varepsilon \circ \varpi_3) \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_2}$, and $(0,0,[2])(\varepsilon \circ \varpi_3) \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_0}$, which contradicts the above part of the proof.

Therefore we obtain that $(0,0,[0])\varepsilon \notin \boldsymbol{B}_{\omega}^{\mathscr{F}^3_1}$. $\square$

If S is a semigroup and $a,b \in S$, then by $\langle a,b \rangle$ we denote the subsemigroup of S which is generated by elements a and b. Also, by λ and ϖ_3 we denote the injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, defined in Proposition 1 for $n = 3$.

Theorem 1. *For every injective endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ there exists an injective endomorphism $\iota \in \langle \lambda, \varpi_3 \rangle$ such that $\varepsilon = \alpha_{[k]} \circ \iota$ for some positive integer k.*

Proof. Lemma 2 implies that for every endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ only one of the following conditions holds:

(1) $(0,0,[p])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_p}$ for all $p \in \{0,1,2\}$;

(2) $(0,0,[p])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_{2-p}}$ for all $p \in \{0,1,2\}$.

Suppose condition (1) holds. By [4, Proposition 3], for any $p \in \{0,1,2\}$ the subsemigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3_p}$ of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ is isomorphic to the bicyclic semigroup. Hence, by [5, Proposition 4], we have $(i,j,[p])\varepsilon \in \boldsymbol{B}_{\omega}^{\mathscr{F}^3_p}$ for all $i,j \in \omega$ and $p \in \{0,1,2\}$. Since $(0,0,[0])$ is an idempotent of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, it follows from [4, Lemma 2] that there exists $s \in \omega$ such that $(0,0,[0])\varepsilon = (s,s,[0])$. This implies that $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\varepsilon \subseteq (\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s$. By Proposition 1 and Corollary 1 the corestriction $\lambda^s\restriction_{(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s} \colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3} \to (\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s$ of the endomorphism λ^s onto $(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s$ is an isomorphism, and hence we get that the composition $\varepsilon\circ(\lambda^s\restriction_{(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s})^{-1}$ is an injective monoid endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. This implies that there exists a positive integer k such that $\varepsilon\circ(\lambda^s\restriction_{(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s})^{-1}=\alpha_{[k]}$. The injectivity of the map λ^s implies $\varepsilon=\alpha_{[k]}\circ\lambda^s\restriction_{(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})\lambda^s}=\alpha_{[k]}\circ\lambda^s$.

Suppose condition (2) holds. Then $\varepsilon\circ\varpi_3$ is an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, which satisfies condition (1). By the previous part of the proof, there exists a positive integer k such that $\varepsilon\circ\varpi_3=\alpha_{[k]}\circ\lambda^s$. We remark that the above equality holds if and only if $\varepsilon\circ\varpi_3\circ\varpi_3=\alpha_{[k]}\circ\lambda^s\circ\varpi_3$, because ε, ϖ_3, $\alpha_{[k]}$, and λ^s are injective maps. By Proposition 1 (iii), we get

$$\varepsilon\circ\lambda^{k-1}=\varepsilon\circ\varpi_3\circ\varpi_3=\alpha_{[k]}\circ\lambda^s\circ\varpi_3. \tag{1}$$

Suppose $(0,0,[0])\varepsilon=(x,x,[2])$ for some $x\in\omega$. Then we have

$$(0,0,[0])(\varepsilon\circ\lambda^{k-1})=(x,x,[2])\lambda^{k-1}=(x+k-1,x+k-1,[2])$$

and

$$(0,0,[0])(\alpha_{[k]}\circ\lambda^s\circ\varpi_3)=(0,0,[0])(\lambda^s\circ\varpi_3)=(s,s,[0])\varpi_3=(s,s,[2]).$$

This implies that $k-1\leqslant s$. Then equality (1), Proposition 1, and the injectivity of above maps imply that

$$\varepsilon=\alpha_{[k]}\circ\lambda^s\circ\varpi_3\circ(\lambda^{k-1})^{-1}=\alpha_{[k]}\circ\varpi_3\circ\lambda^s\circ(\lambda^{k-1})^{-1}=\alpha_{[k]}\circ\varpi_3\circ\lambda^{s-k+1}=\alpha_{[k]}\circ\lambda^{s-k+1}\circ\varpi_3.$$

This completes the proof of the theorem. ☐

Lemma 3. *Let X be a non-empty set and $\mathfrak{a}$ and $\mathfrak{b}$ be transformations of the set X. Then for an arbitrary injective transformation $\mathfrak{h}$ of X the equality $\mathfrak{a}\circ\mathfrak{h}=\mathfrak{b}\circ\mathfrak{h}$ implies that $\mathfrak{a}=\mathfrak{b}$.*

Proof. Suppose to the contrary that there exist transformation $\mathfrak{a}$ and $\mathfrak{b}$ of the set X such that $\mathfrak{a}\circ\mathfrak{h}=\mathfrak{b}\circ\mathfrak{h}$ and $\mathfrak{a}\neq\mathfrak{b}$. Then there exists $x\in X$ such that $(x)\mathfrak{a}\neq(x)\mathfrak{b}$. Since $\mathfrak{h}$ is an injective transformation of X, we get that $(x)(\mathfrak{a}\circ\mathfrak{h})\neq(x)(\mathfrak{b}\circ\mathfrak{h})$, which contradicts the assumption. The obtained contradiction implies the statement of the lemma. ☐

We denote by $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ the semigroup of all injective endomorphisms of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$.

Proposition 3. *Let k be an arbitrary positive integer. Then in $\boldsymbol{End}_{inj}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ the following equalities hold:*

(i) $\lambda\circ\alpha_{[k]}=\alpha_{[k]}\circ\lambda^k$;

(ii) $\varpi_3\circ\alpha_{[k]}\circ\varpi_3=\alpha_{[k]}\circ\lambda^{k+1}$;

(iii) $\varpi_3\circ\alpha_{[k]}=\alpha_{[k]}\circ\varpi_3\circ\lambda^{k-1}$.

Proof. (i) For any $i,j\in\omega$ and $p\in\{0,1,2\}$ we have

$$\begin{aligned}
(i,j,[p])(\lambda\circ\alpha_{[k]}) &= (i+1,j+1,[p])\alpha_{[k]}\\
&= \begin{cases}(k(i+1),k(j+1),[p]), & \text{if } p\in\{0,1\},\\ (k(i+2)-1,k(j+2)-1,[2]), & \text{if } p=2\end{cases}\\
&= \begin{cases}(ki+k,kj+k,[p]), & \text{if } p\in\{0,1\},\\ (k(i+1)-1+k,k(j+1)-1+k,[2]), & \text{if } p=2\end{cases}
\end{aligned}$$

and

$$(i,j,[p])(\alpha_{[k]}\circ\lambda^k)=\begin{cases}(ki,kj,[p])\lambda^k, & \text{if } p\in\{0,1\},\\ (k(i+1)-1,k(j+1)-1,[2])\lambda^k, & \text{if } p=2\end{cases}$$
$$=\begin{cases}(ki+k,kj+k,[p]), & \text{if } p\in\{0,1\},\\ (k(i+1)-1+k,k(j+1)-1+k,[2]), & \text{if } p=2.\end{cases}$$

This implies that the equality $\lambda\circ\alpha_{[k]}=\alpha_{[k]}\circ\lambda^k$ holds in $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$.

(ii) For any $i,j\in\omega$ and $p\in\{0,1,2\}$ we have

$$(i,j,[p])(\varpi_3\circ\alpha_{[k]}\circ\varpi_3)=(i+p,j+p,[2-p])(\alpha_{[k]}\circ\varpi_3)$$
$$=\begin{cases}(i,j,[2])(\alpha_{[k]}\circ\varpi_3), & \text{if } p=0,\\ (i+1,j+1,[1])(\alpha_{[k]}\circ\varpi_3), & \text{if } p=1,\\ (i+2,j+2,[0])(\alpha_{[k]}\circ\varpi_3), & \text{if } p=2\end{cases}$$
$$=\begin{cases}(k(i+1)-1,k(j+1)-1,[2])\varpi_3, & \text{if } p=0,\\ (k(i+1),k(j+1),[1])\varpi_3, & \text{if } p=1,\\ (k(i+2),k(j+2),[0])\varpi_3, & \text{if } p=2\end{cases}$$
$$=\begin{cases}(k(i+1)-1+2,k(j+1)-1+2,[2-2]), & \text{if } p=0,\\ (k(i+1)+1,k(j+1)+1,[2-1]), & \text{if } p=1,\\ (k(i+2),k(j+2),[2-0]), & \text{if } p=2\end{cases}$$
$$=\begin{cases}(ki+k+1,kj+k+1,[0]), & \text{if } p=0,\\ (ki+k+1,kj+k+1,[1]), & \text{if } p=1,\\ (k(i+1)+k,k(j+1)+k,[2]), & \text{if } p=2\end{cases}$$

and

$$(i,j,[p])(\alpha_{[k]}\circ\lambda^{k+1})=\begin{cases}(ki,kj,[p])\lambda^{k+1}, & \text{if } p\in\{0,1\},\\ (k(i+1)-1,k(j+1)-1,[2])\lambda^{k+1}, & \text{if } p=2\end{cases}$$
$$=\begin{cases}(ki+k+1,kj+k+1,[p]), & \text{if } p\in\{0,1\},\\ (k(i+1)-1+k+1,k(j+1)-1+k+1,[2]), & \text{if } p=2\end{cases}$$
$$=\begin{cases}(ki+k+1,kj+k+1,[p]), & \text{if } p\in\{0,1\},\\ (k(i+1)+k,k(j+1)+k,[2]), & \text{if } p=2,\end{cases}$$

and hence $\varpi_3\circ\alpha_{[k]}\circ\varpi_3=\alpha_{[k]}\circ\lambda^{k+1}$ in $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$.

(iii) The equality $\varpi_3\circ\alpha_{[k]}\circ\varpi_3=\alpha_{[k]}\circ\lambda^{k+1}$ implies that $\varpi_3\circ\alpha_{[k]}\circ\varpi_3\circ\varpi_3=\alpha_{[k]}\circ\lambda^{k+1}\circ\varpi_3$. By Proposition 1 (iii), we get that $\varpi_3^2=\lambda^2$, and hence $\varpi_3\circ\alpha_{[k]}\circ\lambda^2=\alpha_{[k]}\circ\lambda^{k+1}\circ\varpi_3$. Since k is a positive integer and λ^2 is an injective endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, Lemma 3, Proposition 1 (iv), and the equality $\varpi_3\circ\alpha_{[k]}\circ\lambda^2=\alpha_{[k]}\circ\lambda^{k+1}\circ\varpi_3$ imply that $\varpi_3\circ\alpha_{[k]}=\alpha_{[k]}\circ\varpi_3\circ\lambda^{k-1}$. □

Remark 2. *By [14, Theorem 5], for any injective monoid endomorphism ε of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ there exists a positive integer k such that $\varepsilon=\alpha_{[k]}$. Also (see [14]), we have that $\alpha_{[k_1]}\circ\alpha_{[k_2]}=\alpha_{[k_1\cdot k_2]}$ in the monoid $\boldsymbol{End}^1_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ of all injective monoid endomorphisms of $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ for all positive integers k_1 and k_2.*

By $\langle\alpha_{[\bullet]}\rangle$ we denote the subsemigroup of $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$, which is generated by endomorphisms $\alpha_{[k]}$, $k\in\mathbb{N}$, of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. According to [14, Theorem 6], the semigroup $\langle\alpha_{[\bullet]}\rangle$ is isomorphic to the multiplicative monoid of positive integers $(\mathbb{N},\cdot)$.

Later, without loss of generality, we may assume that λ^0 is an identity endomorphism of the semigroup $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, and it is clear to see that $\lambda^0 = \alpha_{[1]}$. It is obvious that the semigroup $\lambda^+ = \{\lambda^m \colon m \in \mathbb{N}\}$ is isomorphic to the additive semigroup of positive integers $(\mathbb{N}, +)$ and the monoid $\lambda^* = \{\lambda^m \colon m \in \omega\}$ is isomorphic to the additive monoid of non-negative integers $(\omega, +)$.

Let S and T be semigroups and $\mathfrak{h} \colon S \to \boldsymbol{End}(T)$ be a homomorphism from S into the semigroup $\boldsymbol{End}(T)$ of all endomorphisms of T. By $\mathfrak{h}_{s_2}$ we denote the image of s_2 under homomorphism $\mathfrak{h}$. The Cartesian product $S \times T$ with the following semigroup operation

$$(s_1, t_1) \cdot (s_2, t_2) = (s_1 s_2, (t_1)\mathfrak{h}_{s_2} t_2)$$

is called the *semidirect product* of S by T, and it is denoted by $S \ltimes_{\mathfrak{h}} T$.

Theorem 2. *In the monoid $\boldsymbol{End}_{inj}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ the following statements hold:*

(i) $(\alpha_{[k_1]} \circ \lambda^{m_1}) \circ (\alpha_{[k_2]} \circ \lambda^{m_2}) = \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + m_2}$ *for any $k_1, k_2 \in \mathbb{N}$ and $m_1, m_2 \in \omega$;*

(ii) *the submonoid $\langle \langle \alpha_{[\bullet]} \rangle, \lambda^* \rangle$ of $\boldsymbol{End}_{inj}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$, which is generated by the sets $\langle \alpha_{[\bullet]} \rangle$ and λ^*, is isomorphic to the semidirect product $(\mathbb{N}, \cdot) \ltimes_{\mathfrak{h}} (\omega, +)$, where the homomorphism $\mathfrak{h} \colon (\mathbb{N}, \cdot) \to \boldsymbol{End}(\omega, +)$ is defined by the formula $(n)\mathfrak{h}_k = kn$;*

(iii) $(\alpha_{[k_1]} \circ \lambda^{m_1} \circ \varpi_3) \circ (\alpha_{[k_2]} \circ \lambda^{m_2}) = \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + k_2 + m_2} \circ \varpi_3$ *for any $k_1, k_2 \in \mathbb{N}$ and $m_1, m_2 \in \omega$;*

(iv) $(\alpha_{[k_1]} \circ \lambda^{m_1} \circ \varpi_3) \circ (\alpha_{[k_2]} \circ \lambda^{m_2} \circ \varpi_3) = \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + k_2 + m_2 + 2}$ *for any $k_1, k_2 \in \mathbb{N}$ and $m_1, m_2 \in \omega$.*

Proof. (i) By Proposition 3 (i), for any $k_1, k_2 \in \mathbb{N}$ and $m_1, m_2 \in \omega$ we have

$$\alpha_{[k_1]} \circ \lambda^{m_1} \circ \alpha_{[k_2]} \circ \lambda^{m_2} = \alpha_{[k_1]} \circ \alpha_{[k_2]} \circ \lambda^{k_2 \cdot m_1} \circ \lambda^{m_2} = \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + m_2}.$$

(ii) By Proposition 3 (i), every element of the monoid $\langle \langle \alpha_{[\bullet]} \rangle, \lambda^* \rangle$ is presented as a composition $\alpha_{[k]} \circ \lambda^m$ for some $k \in \mathbb{N}$ and $m \in \omega$. It is obvious that such a presentation is unique. We define an isomorphism $\mathfrak{I} \colon \langle \langle \alpha_{[\bullet]} \rangle, \lambda^* \rangle \to (\mathbb{N}, \cdot) \ltimes_{\mathfrak{h}} (\omega, +)$ by the formula $(\alpha_{[k]} \circ \lambda^m)\mathfrak{I} = (k, m)$ for all $k \in \mathbb{N}$ and $m \in \omega$. Statement (i) implies that so defined map $\mathfrak{I}$ is an isomorphisms of the semigroups $\langle \langle \alpha_{[\bullet]} \rangle, \lambda^* \rangle$ and $(\mathbb{N}, \cdot) \ltimes_{\mathfrak{h}} (\omega, +)$.

(iii) By Proposition 3 and statement (i), for any $k_1, k_2 \in \mathbb{N}$ and $m_1, m_2 \in \omega$ we have

$$\begin{aligned}(\alpha_{[k_1]} \circ \lambda^{m_1} \circ \varpi_3) \circ (\alpha_{[k_2]} \circ \lambda^{m_2}) &= \alpha_{[k_1]} \circ \lambda^{m_1} \circ \alpha_{[k_2]} \circ \varpi_3 \circ \lambda^{k_2} \circ \lambda^{m_2} \\ &= \alpha_{[k_1]} \circ \alpha_{[k_2]} \circ \lambda^{k_2 \cdot m_1} \circ \varpi_3 \circ \lambda^{k_2 + m_2} \\ &= \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + k_2 + m_2} \circ \varpi_3.\end{aligned}$$

(iv) By statement (ii) and Proposition 1 (iii), for any $k_1, k_2 \in \mathbb{N}$ and $m_1, m_2 \in \omega$ we get

$$\begin{aligned}(\alpha_{[k_1]} \circ \lambda^{m_1} \circ \varpi_3) \circ (\alpha_{[k_2]} \circ \lambda^{m_2} \circ \varpi_3) &= \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + k_2 + m_2} \circ \varpi_3^2 \\ &= \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + k_2 + m_2} \circ \lambda^2 \\ &= \alpha_{[k_1 \cdot k_2]} \circ \lambda^{k_2 \cdot m_1 + k_2 + m_2 + 2}.\end{aligned}$$

$\square$

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Гутік О.В., Серівка М.В. *Про ін'єктивні ендоморфізми напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ з триелементною сім'єю $\mathscr{F}^3$ індуктивних непорожніх підмножин у ω* // Карпатські матем. публ. — 2026. — Т.18, №2. — С. 434–446.

Нехай $\mathscr{F}$ — ω-замкнена сім'я підмножин множини усіх невід'ємних цілих чисел ω і $\boldsymbol{B}_{\omega}^{\mathscr{F}}$ — біциклічне розширення, породжене сім'єю $\mathscr{F}$, які означені в [Visnyk Lviv Univ. Ser. Mech.-Mat. 2020, **90**, 5–19]. У [Visnyk Lviv Univ. Ser. Mech.-Mat. 2023, **95**, 28–45] ми описуємо ін'єктивні моноїдальні ендоморфізми моноїда $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ з $\mathscr{F}^3 = \{[0),[1),[2)\}$ і доводимо, що для кожного ін'єктивного моноїдального ендоморфізму ε напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ існує таке натуральне число k, що $\varepsilon = \alpha_{[k]}$ і відображення $\alpha_{[k]}\colon \boldsymbol{B}_{\omega}^{\mathscr{F}^3} \to \boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ визначається за формулою

$$(i,j,[p))\alpha_{[k]} = \begin{cases} (ki,kj,[p)), & \text{якщо } p \in \{0,1\}, \\ (k(i+1)-1, k(j+1)-1, [2)), & \text{якщо } p = 2, \end{cases}$$

для всіх $i,j \in \omega$.

У цій праці ми описуємо ін'єктивні ендоморфізми напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$. Зокрема, побудовано ін'єктивні ендоморфізми ϖ_3 і λ напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ і описана піднапівгрупа $\langle \lambda, \varpi_3 \rangle$ напівгрупи $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$ усіх ін'єктивних ендоморфізмів напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$, яка породжена ϖ_3 і λ. Доведено, що для кожного ін'єктивного ендоморфізму ε напівгрупи $\boldsymbol{B}_{\omega}^{\mathscr{F}^3}$ існує такий ін'єктивний ендоморфізм $\iota \in \langle \lambda, \varpi_3 \rangle$, що $\varepsilon = \alpha_{[k]} \circ \iota$ для деякого натурального числа k. Також ми доводимо, що підмоноїд моноїда $\boldsymbol{End}_{\mathsf{inj}}(\boldsymbol{B}_{\omega}^{\mathscr{F}^3})$, породжений множиною $\{\alpha_{[k]}\colon k \in \mathbb{N}\}$ і ендоморфізмом λ, ізоморфний напівпрямому добутку $(\mathbb{N},\cdot) \ltimes_{\mathfrak{h}} (\omega,+)$.

Ключові слова і фрази: біциклічний моноїд, інверсна напівгрупа, біциклічне розширення, ендоморфізм, напівгрупа ендоморфізмів, індуктивна множина, напівпрямий добуток.