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Subsymmetric functions on Banach spaces with subsymmetric bases

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Properties of subsymmetric polynomials, analytic functions and some their generalizations on Banach spaces with subsymmetric bases are considered. We prove that if a polynomial on a complex infinite-dimensional Banach space X has a subsymmetric set of zeros, then it is subsymmetric. From here we deduce that the algebra $\mathcal{P}_{\mathfrak{S}}(X)$ of subsymmetric polynomials on X is factorial. We consider conditions when a subsymmetric function on a Banach space can be approximated by subsymmetric analytic functions or polynomials. In addition we construct some weighted backward shift-like mappings on the metric space of point evaluation functionals on $\mathcal{P}_{\mathfrak{S}}(X)$ and prove their topological transitivity.

Key words and phrases: polynomial on infinite-dimensional spaces, set of polynomials zeros, symmetric polynomial, topologically transitive operator.

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Introduction

Symmetric (invariant) functions with respect to a group or semigroup of operators on a linear space are important in the invariant theory [34, 48], function theory [4], nonlinear functional analysis [23], topological algebras [5, 14, 15], quantum physics [11, 16, 30, 41, 42], neural nets [51], approximation theory [29], and other branches of mathematics and applications.

Let X be a real or complex Banach space and S be a semigroup of continuous operators. A function f on X is said to be S-*invariant* (or S-*symmetric*) if $f(\sigma(x)) = f(x)$ for every $\sigma \in S$ and $x \in X$. The classic invariant theory investigates S-invariant polynomials on a finite-dimensional complex space $X = \mathbb{C}^n$ for a group S of linear operators on $\mathbb{C}^n$. The fundamental problems of invariant theory are the following (see, e.g., [47]):

– *find a set of generators of the algebra of S-invariant polynomials;*
– *describe the relations (the syzygies) among the generators;*
– *write an arbitrary S-invariant polynomial as a polynomial in the generators.*

If X is an infinite-dimensional Banach space, then it make sense to consider semigroups of operators on X that are not groups. Also, it is interesting to consider algebras of S-invariant analytic functions on X. In particular, the algebra $H_{bS}(X)$ of S-invariant entire analytic functions of bounded type (bounded on bounded subsets of X) is important in the theory of topological algebras. It leads to the following additional problems.

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- *Does every S-invariant analytic functions on X belong to $H_{bS}(X)$?*
- *What is the spectrum (the set of maximal ideals) of $H_{bS}(X)$ if X is a complex Banach space?*
- *Is the algebra of S-invariant polynomials on X factorial?*
- *Let f be an S-invariant function on X. Can we approximate f by S-invariant analytic functions if X is a real Banach space?*

S-invariant polynomials with respect to various groups and semigroups of operators on Banach spaces ℓ_p and L_p were firstly considered in [21,26–28,40]. In these papers, some generators and linear bases in corresponding algebras of S-symmetric polynomials were constructed. These investigations were continued for different groups and semigroups by many authors (see [10,12,20,35,44,45] and references therein). Algebras $H_{bS}(X)$ of S-invariant entire analytic functions of bounded type and their spectra $M_{bS}(X)$ (the sets of maximal ideals) were investigated in [5,12,14,15,35,46] and other papers.

For every $x \in X$ the functional $\delta_x \colon x \mapsto f(x)$ belongs to the spectrum $M_{bS}(X)$, that is, δ_x is a linear, multiplicative, and continuous functional on $H_{bS}(X)$. We have the following relation of equivalence on X, generated by $M_{bS}(X)$: vectors x and y are equivalent if $\delta_x = \delta_y$. Let $\mathcal{M}^S_X$ be the set of equivalence classes. The case when S is the group of permutations of basis vectors, $\mathcal{M}^S_X$ admits some complete metrizable topologies (see [22,33]). In this work, we consider a metric space $\mathfrak{M}_X = \mathcal{M}^S_X$, where $S = \mathfrak{S}$ is a semigroup of subsymmetric translations (the definition in the main body of the paper).

Analytic functions of unbounded type were considered in [1,2,50]. The zero-sets of polynomials on Banach spaces were investigated by many authors (see, e.g., [6,9,36] and references therein). Approximation of continuous functions by analytic ones were studied in [7,8,13,37].

In this paper, we investigate properties of subsymmetric polynomials, analytic functions and some their generalizations on Banach spaces with subsymmetric bases. In Section 1, we provide some basic definitions an preliminary results. In Section 2, we prove that if a polynomial on a complex infinite-dimensional Banach space X has a subsymmetric set of zeros, then it is subsymmetric. From here we deduce that the algebra $\mathcal{P}_{\mathfrak{S}}(X)$ of subsymmetric polynomials on X is factorial. These results are extended for polynomials of some more general form. In Section 3, we consider conditions when a subsymmetric function on a Banach space can be approximated by subsymmetric analytic functions or polynomials. In Section 4, we construct some weighted backward shift-like mappings on the metric space of point evaluation functionals on $\mathcal{P}_{\mathfrak{S}}(X)$ and prove their topological transitivity.

1 Definitions and preliminaries

Let X and Y be Banach spaces. A mapping $P_n \colon X \to Y$ is an *n-homogeneous polynomial* if there is an n-linear map $\overline{P}_n$ from the Cartesian product $X^n = X \times \cdots \times X$ to Y such that $P(x) = \overline{P}_n(x, \ldots, x)$. A finite sum $P(x) = P_0(x) + P_1(x) + \cdots + P_m(x)$ of homogeneous polynomials is a *polynomial* of degree m, providing $P_m \neq 0$. Here P_0 is a constant in Y. It is well known that a polynomial is continuous if and only if it is bounded on any bounded subset of X. The norm of continuous polynomials from X to Y is defined by

$$\|P\| = \sup_{\|x\| \le 1} \|P(x)\|.$$

Let $\mathcal{O}$ be an open set of X. A mapping $f\colon X \to Y$ is *analytic* (or *holomorphic*) if for every $x \in \mathcal{O}$ there is a sequence of n-homogeneous continuous polynomials g_n such that

$$g(y) := f(x+y) = \sum_{n=0}^{\infty} g_n(y) \tag{1}$$

for every y, $\|y\| < r_x$, where $r_x > 0$ is the *radius of uniform convergence* of the Taylor series expansion (1), namely

$$r_x = \Big(\limsup_{n\to\infty} \|f_n\|^{1/n}\Big)^{-1}.$$

If X and Y are complex Banach spaces, then g is bounded on the ball $r\mathcal{B} = \{y\colon \|y\| < r\}$ for every $r < r_x$ and unbounded (or undefined) on $r\mathcal{B}$ if $r > r_x$.

A function on a complex Banach space X is *entire* if it is analytic on the whole space X. An entire function is of *bounded type* if it is bounded on bounded subsets of X. Equivalently, f is entire if the radius of the uniform convergence of f at zero is equal to infinity. If f is not of bounded type, then we say that f is of *unbounded type*. It is well-known [1, 2] that every infinite-dimensional Banach space admits a function of unbounded type. The space $H_b(X)$ of functions of bounded type is a Fréchet algebra endowed with the topology of uniformly convergence on bounded subsets (see [3]). For a given semigroup S of continuous operators on X we denote by $H_{bS}(X)$ the subalgebra of $H_b(X)$ of S-invariant functions. It is easy to check that $H_{bS}(X)$ is a closed (Fréchet) subalgebra of $H_b(X)$.

Any real Banach space X can be isometrically embedded into the complexification $X^{\mathbb{C}}$ of X consisting of vectors $\{(x,y) = x + iy\colon x,\, y \in X\}$ and endowed with the norm

$$\|(x,y)\| = \|x+iy\| = \sup_{\substack{\phi\in X^*,\\ \|\phi\|=1}} \sqrt{(\phi(x))^2 + (\phi(y))^2}$$

(see [13, 39]). Moreover, if h is a real function defined on an open set of the real space X that is analytic at some point x with radius of convergence $r_x > 0$ at this point, then there exists a complex-valued function $h^{\mathbb{C}}$ defined on an open set of the complexification $X^{\mathbb{C}}$ that is holomorphic at the point $(x,0) \in X^{\mathbb{C}}$ with the radius of convergence at least $r_x/2e$ and such that the restriction of $h^{\mathbb{C}}$ to X equals h (see [13]).

If f is analytic at some neighbourhood of zero $r\mathcal{B}$, then it has the following Taylor series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n(x)$$

and n-homogeneous polynomials f_n can be expressed as

$$f_n(x) = \frac{n!}{2\pi}\int_{-\pi}^{\pi} e^{in\theta} f\big(e^{i\theta}x\big)\,d\theta. \tag{2}$$

If f is S-invariant with respect to a semigroup S of linear operators on X, then from (2) it follows that each n-homogeneous polynomial f_n is S-invariant as well.

A sequence of vectors (e_n), $n \in \mathbb{N}$, is a *Schauder basis* of X if it is linearly independent and every $x \in X$ can be uniquely represented as

$$x = \sum_{n=1}^{\infty} x_n e_n, \quad x_n \in \mathbb{C}, \tag{3}$$

and series (3) converges in X. Clearly that if X has a Schauder basis, then it is separable. However, for nonseparable spaces can be used the concept of an uncountable Schauder basis (see, e.g., [25]). A Schauder basis (e_n) is *unconditional* if (3) converges unconditionally for $x \in X$. A Schauder basis (e_n) is *symmetric* if for every bijective map (permutation) $\sigma\colon \mathbb{N} \to \mathbb{N}$ the basis $(e_{\sigma(n)})$ is equivalent, that is, $\sum_{n=1}^{\infty} x_n e_n$ converges if and only if $\sum_{n=1}^{\infty} x_n e_{\sigma(n)}$ converges. A Schauder basis (e_n) is called *spreading invariant* if for every increasing mapping $\sigma\colon \mathbb{N} \to \mathbb{N}$ the basis $(e_{\sigma(n)})$ is equivalent. Finally, a Schauder basis (e_n) is *subsymmetric* if it is spreading invariant and unconditional (see [20,38] for details).

Let us denote by $\mathfrak{S}$ the semigroup generated by the following sequence of linear operators

$$\mathcal{C}_j\colon (x_1,\ldots,x_n,\ldots) \mapsto (x_1,\ldots,x_{j-1},0,x_j,x_{j+1}\ldots), \quad j \in \mathbb{N},$$

on a Banach space X with a subsymmetric basis.

For a given subsequence $\mathbf{n} = (n_1,\ldots,n_k,\ldots) \subset \mathbb{N}$, we denote

$$\mathcal{C}_{\mathbf{n}}\colon x = \sum_{n=1}^{\infty} x_n e_n \mapsto \sum_{n=1}^{\infty} x_n e_{n_k}.$$

Let $\overline{\mathfrak{S}}$ be the semigroup generated by all operators $\mathcal{C}_{\mathbf{n}}$. A function $f\colon X \to \mathbb{C}$ is said to be *subsymmetric* or $\mathfrak{S}$-*symmetric* if $f \circ A(x) = f(A(x)) = f(x)$ for every $A \in \mathfrak{S}$, and f is said to be $\overline{\mathfrak{S}}$-*symmetric* if $f \circ A(x) = f(A(x)) = f(x)$ for every $A \in \overline{\mathfrak{S}}$. Clearly, $\mathfrak{S} \subset \overline{\mathfrak{S}}$ and so each $\overline{\mathfrak{S}}$-symmetric function is subsymmetric.

Proposition 1. *Let f be a continuous subsymmetric function on X. Then f is $\overline{\mathfrak{S}}$-symmetric.*

Proof. Let $\mathbf{n} = (n_1,\ldots,n_k,\ldots)$, $x \in X$ and $z = \mathcal{C}_{\mathbf{n}}(x)$. For every $N \in \mathbb{N}$, we denote

$$x^N = \sum_{k=1}^{N} x_k e_k \quad \text{and} \quad z^N = \sum_{k=1}^{N} x_k e_{n_k}.$$

It is easy to see that for each N there is $A_N \in \mathfrak{S}$ such that $z^N = A_N(x^N)$. Clearly, $x^N \to x$, and $z^N \to z$ as $N \to \infty$. Since f is continuous, $f(x^N) \to f(x)$ and $f(z^N) \to f(z) = f\big(\mathcal{C}_{\mathbf{n}}(x)\big)$ as $N \to \infty$. On the other hand, since f is subsymmetric, we have

$$f(x^N) = f(A_N(x^N)) = f(z^N)$$

and so $f(x^N) \to f\big(\mathcal{C}_{\mathbf{n}}(x)\big)$ as $N \to \infty$. Thus, $f(x) = f\big(\mathcal{C}_{\mathbf{n}}(x)\big)$ for every subsequence $\mathbf{n}$. ☐

It is known [27] that the following (so-called *standard*) subsymmetric polynomials

$$P_{\alpha_1,\ldots,\alpha_n}(x) = \sum_{i_1<\cdots<i_n} x_{i_1}^{\alpha_1} \cdots x_{i_n}^{\alpha_n}, \quad \alpha_j \geq \lceil p \rceil$$

form a linear basis in the linear space of subsymmetric polynomials on ℓ_p, $1 \leq p < \infty$, where $\lceil p \rceil$ is the ceiling of p. Also, c_0 does not admit any nonconstant subsymmetric polynomial [27]. On the other hand, there are subsymmetric continuous functions on c_0. For example, $f(x) = \|x\|$ is symmetric (and so subsymmetric as well) on c_0. The following function

$$g(x) = \max_{i\in\mathbb{N}} |x_i| \|(x_{i+1},x_{i+2},\ldots)\|^2 = \max_{i<j} |x_i||x_j|^2$$

is subsymmetric but not symmetric on c_0.

For a given $x \in X$ we denote by $\operatorname{supp}(x)$ the *support* of x defined as

$$\operatorname{supp}(x) = \{k \in \mathbb{N} \colon x_k \neq 0\}.$$

The space

$$c_{00} = \{(x_1, \ldots, x_m, 0, \ldots) \colon m \in \mathbb{N}\} = \{x \in X \colon \operatorname{supp}(x) < \infty\}$$

is a dense subspace of any Banach space X with an unconditional basis. In [12], it was considered the following scalar-valued homomorphisms on $\mathcal{P}_{\mathfrak{S}}(X)$

$$\delta_{x \triangleleft y}(P) = P(x \triangleleft y) := \lim_{n \to \infty} P(x_1, \ldots, x_n, y_1, y_2, \ldots),$$

associated with some $x, y \in X$. If $x \in c_{00}$, that is, $x = (x_1, \ldots, x_m, 0, \ldots)$ for some m, then there is $z = (x_1, \ldots, x_m, y_1, y_2, \ldots)$ in X such that $f(z) = f(x \triangleleft y)$ for every subsymmetric function f.

In the general case, $\delta_{x \triangleleft y} \colon P \mapsto P(x \triangleleft y)$ is a linear multiplicative functional on $\mathcal{P}_{\mathfrak{S}}(X)$. Moreover, if P is n-homogeneous, then

$$|P(x \triangleleft y)| \leq \|x + y\|^n.$$

From here it follows (see [12]) that $\delta_{x \triangleleft y}$ can be continuously extended to the algebra $H_{b\mathfrak{S}}(X)$ of all subsymmetric functions of bounded type for the case of complex Banach space X. Also, as it is observed in [12], if $X = \ell_p$, $1 \leq p < \infty$, then

$$P_{\alpha_1, \ldots, \alpha_m}(x \triangleleft y) = \sum_{j=0}^{m+1} P_{\alpha_1, \ldots, \alpha_j}(x) P_{\alpha_{j+1}, \ldots, \alpha_m}(y), \quad x, y \in X. \tag{4}$$

From (4) it follows that $f((x \triangleleft y) \triangleleft z) = f(x \triangleleft (y \triangleleft z))$ for all $x, y, z \in \ell_p$ and every $f \in H_{b\mathfrak{S}}(\ell_p)$.

Let M be a metric space. A continuous mapping $T \colon M \to M$ is called *topologically transitive* if for any nonempty open subsets U and V in M there exists a number n such that

$$T^n(U) \bigcap V \neq \varnothing.$$

2 Algebras of subsymmetric polynomials on Banach spaces

2.1 Zeros of subsymmetric polynomials and factoriality

Lemma 1. *Let $Q_1, \ldots, Q_m$ be a finite sequence of mutually different polynomials on X such that*

$$\{Q_1 \circ A, \ldots, Q_m \circ A\} = \{Q_1, \ldots, Q_m\}$$

for every $A \in \mathfrak{S}$. Then all polynomials $Q_1, \ldots, Q_m$ are subsymmetric.

Proof. We suppose that Q_i is not subsymmetric for some $1 \leq i \leq m$. If j and k are natural numbers such that $Q_i \circ \mathcal{C}_k = Q_j$, then

$$Q_i(x) = Q_j(x) \quad \text{for every } x \text{ of the form } x = (x_1, \ldots, x_{k-1}, 0, 0, \ldots).$$

Indeed, $\mathcal{C}_k((x_1, \ldots, x_{k-1}, 0, 0, \ldots)) = (x_1, \ldots, x_{k-1}, 0, 0, \ldots)$ by the definition of $\mathcal{C}_k$. Let r be a maximal natural number such that $Q_i(x) = Q_s(x)$ for some $s \neq i$ and every x of the form $x = (x_1, \ldots, x_{r-1}, 0, 0, \ldots)$. Since all polynomials $Q_1, \ldots, Q_m$ mutually different, we get $r < \infty$. Then $Q_i \circ \mathcal{C}_l \notin \{Q_1, \ldots, Q_m\} \setminus \{Q_i\}$ for every $l > r$. Indeed, if $Q_i \circ \mathcal{C}_l = Q_s$ for some $s \neq i$, then $Q_i(x) = Q_s(x)$ for every x of the form $x = (x_1, \ldots, x_{l-1}, 0, 0, \ldots)$, which contradicts the maximality of r. Thus, $Q_i \circ \mathcal{C}_l = Q_i$ for every $l > r$. Let

$$A_{r-1} = \underbrace{\mathcal{C}_1 \circ \cdots \circ \mathcal{C}_1}_{r-1}.$$

Then $Q_i \circ A_{r-1}$ is subsymmetric. Indeed,

$$(Q_i \circ A_{r-1}) \circ \mathcal{C}_j(x) = Q_i(\underbrace{0,\ldots,0}_{r-1}, x_1, x_2, \ldots, x_{j-1}, 0, x_j, \ldots) = Q_i(\underbrace{0,\ldots,0}_{r-1}, x_1, x_2, \ldots)$$

for every $j \in \mathbb{N}$. Since $Q_i \circ A_{r-1} \in \{Q_1, \ldots, Q_m\}$, there is a number $1 \le s \le m$ such that $Q_i \circ A_{r-1} = Q_s$, Q_s is subsymmetric, and $i \neq s$ (because Q_i is not subsymmetric).

Let $1 \le h \le m$ be such that $Q_h \circ A_{r-1} = Q_i$. Then $h \neq s$, $h \neq i$, and $(Q_h \circ A_{r-1}) \circ A_{r-1} = Q_s$ is a subsymmetric polynomial in $\{Q_1, \ldots, Q_m\}$. Since this set of polynomials is finite, we will find a map $A \in \mathfrak{S}$ and $1 \le t \le m$ such that $Q_t \circ A$ is subsymmetric and Q_t is not in the set $\{Q_1 \circ A, \ldots, Q_m \circ A\}$. A contradiction with the condition of the lemma. ☐

A subset $V \subset X$ is called *subsymmetric* if for very $x \in V$ and $A \in \mathfrak{S}$ we have $A(x) \in V$.

Theorem 1. *Let P be a nonzero polynomial on X such that* $\ker P$ *is a subsymmetric subset of X. Then P is subsymmetric. Moreover, if $P = Q_1^{k_1} \cdots Q_m^{k_m}$ for some mutually different irreducible polynomials $Q_1, \ldots, Q_m$, then all polynomials $Q_1, \ldots, Q_m$ are subsymmetric.*

Proof. Let us suppose first that P is irreducible. Since $\ker P$ is a subsymmetric set, we get $\ker P = \ker P \circ \mathcal{C}_j$ and, according to the Hilbert Nullstellensatz for infinite-dimensional spaces (see, e.g., [49]), we obtaine

$$P \circ \mathcal{C}_j = a_j P$$

for some constant $a_j \neq 0$. We claim that $a_j = a_n$ for every $n \in \mathbb{N}$. Indeed, denoting $a = a_1$, we have

$$P \circ \mathcal{C}_1^j = P \circ (\underbrace{\mathcal{C}_1, \ldots, \mathcal{C}_1}_{j}) = a^j P.$$

On the other hand,

$$P \circ \mathcal{C}_1^j = (P \circ \mathcal{C}_j) \circ \mathcal{C}_1^{j-1} = a_j P \circ \mathcal{C}_1^{j-1} = a_j a^{j-1}.$$

Thus $a_j = a$ for every $j \in \mathbb{N}$.

Clearly that for every finite vector $x = (x_1, \ldots, x_n, 0, \ldots)$, $x = \mathcal{C}_{n+1}(x)$ and so

$$P(x) = P \circ \mathcal{C}_{n+1}(x) = aP(x).$$

Since the set of finite vectors is dense in X and P is nonzero, $a = 1$. Therefore, P is subsymmetric.

Let P be a radical polynomial. That is, $P = Q_1 \cdots Q_m$ for some mutually different irreducible polynomials $Q_1, \ldots, Q_m$. Thus $\ker P$ is the union of the algebraic sets $\ker Q_1, \ldots, \ker Q_m$. The condition that $\ker P$ is subsymmetric and the Hilbert Nullstellensatz imply that for every $A \in \mathfrak{S}$ there is a constant $a \neq 0$ such that $P = aP \circ A$. Reasoning as above, we can see that $a = 1$. Hence,

$$\{Q_1 \circ A, \ldots, Q_m \circ A\} = \{Q_1, \ldots, Q_m\}$$

for every $A \in \mathfrak{S}$. By Lemma 1, all polynomials $Q_1, \ldots, Q_m$ are subsymmetric and so P is subsymmetric.

In the general case, let $P = Q_1^{k_1} \cdots Q_m^{k_m}$ for some mutually different irreducible polynomials $Q_1, \ldots, Q_m$. Then, as we proved above, $\mathrm{Rad} P = Q_1 \cdots Q_m$ is subsymmetric and all polynomials $Q_1, \ldots, Q_m$ are subsymmetric. Thus, P must be subsymmetric. ☐

Corollary 1. *The algebra of subsymmetric polynomials on X is factorial.*

Proof. Let P is a subsymmetric polynomial on X and $P = Q_1^{k_1} \cdots Q_m^{k_m}$ for some mutually different irreducible polynomials $Q_1, \ldots, Q_m$. Then $\ker P$ is a subsymmetric subset in X and by Theorem 1, $Q_1, \ldots, Q_m$ are subsymmetric. Since the ring of polynomials on X has a unique representation as a product of irreducible polynomials, for every representation $P = P_1 P_2$ we have that polynomials P_1 and P_2 are irreducible. ☐

2.2 Proto-subsymmetric functions

We say that a functions f defined on an S-symmetric subset V is *proto-S-invariant* if there is an operator $A \in S$ such that $f \circ A$ is S-invariant on V. Also, f is *L-proto-S-invariant* on V if there is a sequence A_j of operators in S such that the limit

$$h(x) = \lim_{j\to\infty} (f \circ A_j)(x)$$

exists for every $x \in V$ and the h is S-invariant on V. It is easy to see that if S is a group, then each proto-S-invariant function is S-invariant. As we mentioned in Proposition 1, a continuous function is subsymmetric (that is, $\mathfrak{S}$-symmetric) if and only if it is $\overline{\mathfrak{S}}$-symmetric. The situation is different for proto-S-invariant functions.

Example 1. *Let us consider the following functions:*

$$W_k(x) = \sum_{i=2}^{\infty} x_i^k;$$

$$Q_2(x) = \sum_{i<j+1} x_i x_j^2 = x_1 \sum_{j=2}^{\infty} x_i x_j^2 + x_2 \sum_{j=3}^{\infty} x_i x_j^2 + \cdots + x_n \sum_{j=n+2}^{\infty} x_i x_j^2 + \cdots;$$

$$u(x) = \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \sum_{i=n+1}^{\infty} x_i^n.$$

Then the polynomial W_k is not subsymmetric but proto-$\mathfrak{S}$-invariant on ℓ_p, $k \geq \lceil p \rceil$, because

$$W_k(\mathcal{C}_1(x)) = \sum_{i=1}^{\infty} x_i^k$$

is subsymmetric on ℓ_p. It is easy to check that the function Q_2 is not proto-$\mathfrak{S}$-invariant, but it is proto-$\overline{\mathfrak{S}}$-invariant. Indeed, let $\mathbf{n} = (1, 3, \ldots, 2n-1, \ldots)$, then

$$\mathcal{C}_{\mathbf{n}}(x) = (x_1, 0, x_2, 0, x_3, \ldots,)$$

and

$$Q_2(\mathcal{C}_{\mathbf{n}}(x)) = \sum_{i<j} x_i x_j^2$$

is a subsymmetric polynomial and so $\overline{\mathfrak{S}}$-invariant. Note that Q_2 is L-proto-$\mathfrak{S}$-invariant as well. Indeed, for $A_n = \mathcal{C}_{2n} \circ \mathcal{C}_{2(n-1)} \circ \cdots \circ \mathcal{C}_2$ we have

$$\lim_{n\to\infty} (Q_2 \circ A_n)(x) = \sum_{i<j} x_i x_j^2.$$

The function u is an analytic function of unbounded type on ℓ_1 which is bounded on the closed unit ball of ℓ_1 and unbounded on any ball centered at zero of radius $r > 1$. Each polynomial u_n is proto-$\mathfrak{S}$-invariant, but u is not proto-$\overline{\mathfrak{S}}$-invariant. The restriction of u to the closed unit ball is L-proto-$\mathfrak{S}$-invariant. Indeed, setting $A_j = \mathcal{C}_1^j$, we have that

$$\lim_{j\to\infty} (u \circ A_j)(x) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} x_i^n$$

is well-defined and subsymmetric for $\|x\| \leq 1$.

It is easy to check that the set of proto-S-invariant analytic functions of bounded type form an algebra which, however, is not complete in the general case.

Example 2. *Let* $f \in H_{\mathfrak{S}b}(X)$. *We set*

$$s_m(x) = \sum_{n=1}^{m} \frac{f(x_{n+1}, x_{n+2}, \ldots)}{2^n}.$$

For every m *the function* s_m *is proto-*$\mathfrak{S}$*-invariant because*

$$(s_m \circ \mathcal{C}_1^m)(x) = f(x) \sum_{n=1}^{m} \frac{1}{2^n} = \frac{2^m - 1}{2^m} f(x)$$

is subsymmetric. But the limit function

$$s(x) = \lim_{n \to \infty} s_n(x) = \sum_{n=1}^{\infty} \frac{f(x_{n+1}, x_{n+2}, \ldots)}{2^n}$$

is not proto-$\mathfrak{S}$*-invariant. Note that* s *is* L*-proto-*$\mathfrak{S}$*-invariant since*

$$\lim_{m \to \infty} (s \circ \mathcal{C}_1^m)(x) = f(x).$$

The following theorem is a generalization of Corollary 1.

Theorem 2. *Let a semigroup* S *of operators on* X *be such that the algebra* $\mathcal{P}_S(X)$ *is factorial. Then the algebra of proto-*S*-invariant polynomials is factorial as well. In particular, the algebra of proto-subsymmetric polynomial is factorial.*

Proof. Suppose that $P = Q_1 \cdots Q_m$ is a proto-S-invariant polynomial on X. Then we get $P \circ A \in \mathcal{P}_S(X)$ for some $A \in S$. On the other hand, we have $P \circ A = (Q_1 \circ A) \cdots (Q_m \circ A)$. Thus, each multiplier $Q_i \circ A$ is S-invariant, that is, each Q_i is proto-S-invariant. ☐

2.3 Super-subsymmetric polynomials

A polynomial P of two vector variables $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ is said to be *supersymmetric* if it is symmetric with respect to basis vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$ (separately) and if

$$P\big((a, x_2, \ldots, x_n), (a, y_2, \ldots, y_m)\big)$$

does not depend on $a \in \mathbb{C}$. Supersymmetric polynomials on $\mathbb{C}^n \times \mathbb{C}^m$ were studied in [32, 43]. Generalizations of the concept of supersymmetric polynomials for infinite-dimensional Banach spaces were considered in [17–19, 24, 31, 35]. Next we propose a general definition of supersymmetric polynomials associated with a given semigroup S of operators.

Definition 1. *Let* X *and* Y *be real or complex Banach spaces with unconditional bases and* S_1 *(resp.* S_2*) be a semigroup of continuous operators on* X *(resp. on* Y*). We say that a function* F *on* $X \times Y$ *is left super-*(S_1, S_2)*-symmetric if*

(i) $F(\cdot, y)$ *is* S_1*-symmetric on* X *for every* $y \in Y$*;*

(ii) $F(x, \cdot)$ *is* S_2*-symmetric on* Y *for every* $x \in X$*;*

(iii) $F\big((a, x_1, x_2, \ldots), (a, y_1, y_2, \ldots)\big) = F\big((x_1, x_2, \ldots), (y_1, y_2, \ldots)\big)$ *for every* $x = (x_1, x_2, \ldots)$ *in* X, $y = (y_1, y_2, \ldots)$ *in* Y*, and* $a \in \mathbb{C}$*.*

A function F on $X \times Y$ is right super-(S_1, S_2)-symmetric if instead of (iii) we have

(iii') $\lim\limits_{n\to\infty} F\big((x_1,\ldots,x_n,a,0,\ldots),(y_1,\ldots,y_n,a,0,\ldots)\big) = F\big((x_1,x_2,\ldots),(y_1,y_2,\ldots)\big)$ for every $x = (x_1,x_2,\ldots) \in X$, $y = (y_1,y_2,\ldots) \in Y$, and $a \in \mathbb{C}$ and the limit exists and finite.

If a function is both left and right super-(S_1, S_2)-symmetric, we will call it super-(S_1, S_2)-symmetric. If $X = Y$ is a Banach space with a symmetric basis and $\mathcal{S} = S_1 = S_2$ is the group of all permutations of basis vectors, then F is left super-$\mathcal{S}$-symmetric if and only if it is right super-$\mathcal{S}$-symmetric. In this case we say that F is supersymmetric. If $X = Y$ is a Banach space with a subsymmetric basis and $\mathfrak{S} = S_1 = S_2$ is the semigroup of subsymmetric translations, then F is a left (or right) super-subsymmetric function.

From the definition and properties of subsymmetric polynomials we have the following proposition.

Proposition 2. *A polynomial P is left super-subsymmetric on $X \times X$ if and only if*

$$P(z \lhd x, z \lhd y) = P(x,y),$$

and P is right super-subsymmetric on $X \times X$ if and only if

$$P(x \lhd z, y \lhd z) = P(x,y)$$

for all x, y, and $z \in X$.

Example 3. *The following polynomial*

$$Q_{k_1,k_2}(x,y) = P_{k_1,k_2}(x) + P_{k_1}(x)P_{k_2}(y) + P_{k_2,k_1}(y) - P_{k_1,k_2}(y) - P_{k_1}(y)P_{k_2}(x) - P_{k_2,k_1}(x)$$

is left super-subsymmetric on ℓ_1. Indeed

$$\begin{aligned} Q_{k_1,k_2}(z \lhd x, z \lhd y) &= P_{k_1,k_2}(z \lhd x) + P_{k_1}(z \lhd x)P_{k_2}(z \lhd y) + P_{k_2,k_1}(z \lhd y) \\ &\quad - P_{k_1,k_2}(z \lhd y) - P_{k_1}(z \lhd y)P_{k_2}(z \lhd x) - P_{k_2,k_1}(z \lhd x) \\ &= P_{k_1,k_2}(z) + P_{k_1}(z)P_{k_2}(x) + P_{k_1,k_2}(x) + (P_{k_1}(z) + P_{k_1}(x))(P_{k_2}(z) + P_{k_2}(y)) \\ &\quad + P_{k_2,k_1}(z) + P_{k_2}(z)P_{k_1}(y) + P_{k_2,k_1}(y) \\ &\quad - P_{k_1,k_2}(z) - P_{k_1}(z)P_{k_2}(y) - P_{k_1,k_2}(y) - (P_{k_1}(z) + P_{k_1}(y))(P_{k_2}(z) + P_{k_2}(x)) \\ &\quad - P_{k_2,k_1}(z) - P_{k_2}(z)P_{k_1}(x) - P_{k_2,k_1}(x) = Q_{k_1,k_2}(x,y). \end{aligned}$$

It is easy to check that $Q_{k_1,k_2}(x,y)$ is not right super-subsymmetric. By the similar calculations, it is possible to check that the following polynomial

$$R_{k_1,k_2}(x,y) = P_{k_2,k_1}(x) - P_{k_1,k_2}(x) + P_{k_1,k_2}(y) - P_{k_2,k_1}(y) + P_{k_1}(x)P_{k_2}(y) - P_{k_2}(x)P_{k_1}(y)$$

is right super-subsymmetric but not left super-subsymmetric.

Clearly, the sets of left (right) super-subsymmetric polynomials form algebras. The properties of these algebras, their spectra and generators, are subjects for further investigation.

3 Approximation of subsymmetric functions

3.1 The complex case

If f is an analytic function on a domain $\mathcal{O}$ of a complex Banach space X, then, by the definition, for every $x \in \mathcal{O}$ there is a neighbourhood U of x, $U \subset \mathcal{O}$ and homogeneous continuous polynomials g_n such that

$$f(x+y) = \sum_{n=0}^{\infty} g_n(y) \tag{5}$$

for every $y \in U$ and the series converges absolutely and uniformly on U. But, if f is S-invariant, then for some fixed $x \in X$ the function $y \mapsto f(x+y)$ is not need to be S-invariant, and polynomials g_n are not need to be S-invariant.

Example 4. *Let* $X = \ell_1$, $f(x) = P_2(x) = \sum_{i=1}^{\infty} x_i^2$. *Then* f *is subsymmetric (even symmetric) while*

$$f(x+y) = P_2(x+y) = \sum_{i=1}^{\infty} x_i^2 + \sum_{i=1}^{\infty} y_i^2 + 2\sum_{i=1}^{\infty} x_i y_i$$

is a subsymmetric of y *only if* $x = 0$.

It is easy to see that if f is a subsymmetric analytic function in a domain $\mathcal{O}$ containing zero, then polynomials g_n of the Taylor series expansion (5) are subsymmetric if $x = 0$ and not necessary subsymmetric otherwise.

From the definition of subsymmetric functions we have the following proposition.

Proposition 3. *If* f *is a subsymmetric function on a subsymmetric subset* $\Omega \subset X$, *then for every* x, y *and* z *in* X *such that* $y \lhd x \lhd z \in \Omega$ *the function* $x \mapsto f(y \lhd x \lhd z)$ *is subsymmetric.*

Taking into account the continuity of standard polynomials and equation (4) we have the following corollary.

Corollary 2. *If* $P \in \mathcal{P}_{\mathfrak{S}}(X)$, *then for every* y *and* z *in* X *the function* $x \mapsto P(y \lhd x \lhd z)$ *is in* $\mathcal{P}_{\mathfrak{S}}(X)$.

Theorem 3. *Let* f *be an analytic function on a subsymmetric open subset* $\mathcal{O}$ *of* X. *For every* $x \in \mathcal{O} \cap c_{00}$ *there exists* $\varepsilon > 0$ *such that the function* $g(y) = f(x \lhd y)$ *is analytic and subsymmetric in the open ball* $\varepsilon\mathcal{B}_X = \{y \in X \colon \|y\| < \varepsilon\}$.

Proof. Let $x = (x_1, \ldots, x_m, 0 \ldots)$ for some $m \in \mathbb{N}$. Then the function $z \mapsto f(x+z)$ is analytic in $\varepsilon\mathcal{B}_X$ for some $\varepsilon > 0$. Let $f(x+z) = \sum_{n=0}^{\infty} h_n(z)$ be the Taylor series expansion of $f(x+\cdot)$ in $\varepsilon\mathcal{B}_X$. If $y \in \varepsilon\mathcal{B}_X$, then

$$z = \Big(\underbrace{0, \ldots, 0}_{m}, y_1, y_2, \ldots\Big) \in \varepsilon\mathcal{B}_X.$$

On the other hand, $x + z = x \lhd y$, and so

$$f(x \lhd y) = \sum_{n=0}^{\infty} h_n(z) = \sum_{n=0}^{\infty} h_n\Big(\underbrace{0, \ldots, 0}_{m}, y_1, y_2, \ldots\Big) = \sum_{n=0}^{\infty} g_n(y),$$

where $g_n(y) = h_n\Big(\underbrace{0, \ldots, 0}_{m}, y_1, y_2, \ldots\Big)$. Since the function $y \mapsto f(x \lhd y)$ is subsymmetric, each n-homogeneous polynomial g_n is subsymmetric. ☐

Let $\mathfrak{A}$ be an infinite set of indexes. We denote by $\ell_p(\mathfrak{A})$, $1 \le p < \infty$, the normed space of functions $z\colon \mathfrak{A} \to \mathbb{C}$ such that

$$\|z\| := \Big(\sum_{\gamma\in\mathfrak{A}} |z(\gamma)|^p\Big)^{1/p} < \infty.$$

Clearly that the finiteness of this norm implies that $z(\gamma) \ne 0$ only for a countable amount of $\gamma \in \mathfrak{A}$. It is well-known that $\ell_p(\mathfrak{A})$ is a Banach space with respect to this norm and functions e_β, $\beta \in \mathfrak{A}$, defined by

$$\mathfrak{e}_\beta(\gamma) = \begin{cases} 1, & \text{if } \beta = \gamma, \\ 0, & \text{otherwise} \end{cases}$$

form a Schauder basis (not necessary countable) in $\ell_p(\mathfrak{A})$. That is, every $z \in \ell_p(\mathfrak{A})$ can be uniquely represented by

$$z = \sum_{\gamma\in\mathfrak{A}} z_\gamma \mathfrak{e}_\gamma,$$

where $z_\gamma = z(\gamma)$ are coordinates of z with respect to the basis $(\mathfrak{e}_\gamma)_{\gamma\in\mathfrak{A}}$. A set $\mathfrak{A}$ is *well-ordered* if it is linearly ordered and every nonempty subset of $\mathfrak{A}$ has a minimal element. In [12], it is proved that if $(\mathfrak{A}, \prec)$ is a well-ordered infinite set, then there exists an isomorphic embedding

$$J_{\mathfrak{A}}\colon \ell_p \hookrightarrow \ell_p(\mathfrak{A})$$

such that

$$J_{\mathfrak{A}}\colon e_k \mapsto \mathfrak{e}_{\gamma_k}$$

and $k < m$ implies $\gamma_k \prec \gamma_m$. Moreover, each standard polynomial $P_{\alpha_1,\ldots,\alpha_n}$ on ℓ_p can be extended to a polynomial $P^{\mathfrak{A}}_{\alpha_1,\ldots,\alpha_n}$ on $\ell_p(\mathfrak{A})$ by

$$P^{\mathfrak{A}}_{\alpha_1,\ldots,\alpha_n}(z) = \sum_{\gamma_1\prec\cdots\prec\gamma_n} z^{\alpha_1}_{\gamma_1}\cdots z^{\alpha_n}_{\gamma_n},$$

and the operator of extension is linear and multiplicative. From the fact that every polynomial in $\mathcal{P}_{\mathfrak{S}}(\ell_p)$ is a finite linear combination of standard polynomials it follows that each subsymmetric polynomial P on ℓ_p can be extended to a polynomial $P^{\mathfrak{A}}$ on $\ell_p(\mathfrak{A})$ and, it is easy to check that $\|P\| = \|P^{\mathfrak{A}}\|$. Also, in [12] it was observed that $\mathfrak{A}$ and $J_{\mathfrak{A}}$ can be chosen so that for given x and y in ℓ_p there are $\widetilde{x}$ and $\widetilde{y}$ in $\ell_p(\mathfrak{A})$ such that

$$P(x \lhd y) = P^{\mathfrak{A}}(\widetilde{x} + \widetilde{y}) \tag{6}$$

for every $P \in \mathcal{P}_{\mathfrak{S}}(\ell_p)$. For example, let $\mathfrak{A} = \mathbb{N}_1 \cup \mathbb{N}_2$, where $\mathbb{N}_1$ and $\mathbb{N}_2$ are copies on natural numbers with the usual order, and if $i \in \mathbb{N}_1$ and $j \in \mathbb{N}_2$, then $i \prec j$. If x and y are in ℓ_p, then

$$\widetilde{x} + \widetilde{y} = (x_1, \ldots, x_i, \ldots) + (y_1, \ldots, y_j, \ldots) \in \ell_p(\mathbb{N}_1 \cup \mathbb{N}_2), \quad i \in \mathbb{N}_1,\ j \in \mathbb{N}_2.$$

Therefore, $P(x \lhd y) = P^{\mathfrak{A}}(\widetilde{x} + \widetilde{y})$.

Taking into account the representation (6) and proceeding as in the proof of Theorem 3, we obtain the following corollary.

Corollary 3. *Let f be an analytic function on a subsymmetric open subset $\mathcal{O}$ of ℓ_p with $1 \le p < \infty$. For every $x \in \mathcal{O}$ there exists $\varepsilon > 0$ such that the function $g(y) = f(x \lhd y)$ is analytic and subsymmetric in the open ball $\varepsilon\mathcal{B}_{\ell_p} = \{y \in X\colon \|y\| < \varepsilon\}$.*

3.2 The real case

M.C. Boiso and P. Hájek in [13] extended Kurzweil's result in [37] to a *real* separable Banach space X admitting a uniformly analytic and separating function. According to [13], a real analytic function g on X is *uniformly analytic*, if the radius of uniform convergence of g at any point $x \in X$ is greater than or equal to R_g, for some $R_d > 0$. A real analytic function g on X is *separating* if the set $\{x \in X\colon d(x) < \alpha\}$ is a nonempty subset of the unit ball $\mathcal{B}_X$ of X.

It is easy to see that if X admits a separating polynomial, then it admits a uniformly analytic and separating function (see [13, p. 93]). On the other hand, any closed subspace of c_0 admits a uniformly analytic and separating function (see [13]) while c_0 does not admit a separating polynomial. In particular,

$$g(x) = \sum_{n=1}^{\infty} x_n^{2n}$$

is a uniformly analytic and separating function on c_0.

Theorem 4 ([13])**.** *Let X be a real separable Banach space admitting a uniformly analytic and separating function, O be an open set of X, and f be a uniformly continuous mapping defined on O and with values in a closed convex set C of an arbitrary Banach space Y. Then, for every $\varepsilon > 0$, there exists an analytic mapping h defined on O and having its values in C such that*

$$\|f(x) - h(x)\| < \varepsilon \quad \textit{for any} \quad x \in O.$$

In [27, Theorem 2.3], it is proved that for a given real Banach space X with a subsymmetric basis (e_n) and an N-homogeneous polynomial on X, there is an N-homogeneous subsymmetric polynomial P^* on X such that for each $\varepsilon > 0$ there exists an infinite set H of integers such that $\|P - P^*\|_{X_H} \le \varepsilon$, where X_H is the closed subspace generated by $\{e_n\colon n \in H\}$.

Moreover, for every $k \in \mathbb{N}$, we have

$$P^*\Big(\sum_{i=1}^{k} x_i e_i\Big) = \lim_{\substack{n_1 < \cdots < n_k, \\ n_j \in H}} P\Big(\sum_{i=1}^{k} x_i e_{n_i}\Big)$$

(see [27]) and so $\|P^*\| = \|P\|$, and the mapping $P \mapsto P^*$ is an algebra homomorphism [14].

Theorem 5. *Let X be a real Banach space with a subsymmetric basis and $f\colon X \to \mathbb{R}$ a uniformly continuous subsymmetric function. If there are numbers $r > 0$, $\varepsilon > 0$ and an analytic uniformly continuous function $h\colon r\mathcal{B}_X \to \mathbb{R}$ such that*

$$\sup_{\|x\| \le r} |f(x) - h(x)| < \varepsilon,$$

then there is a subsymmetric analytic function $u\colon R\mathcal{B}_X \to \mathbb{R}$ such that

$$\sup_{x \in R\mathcal{B}_X} |f(x) - u(x)| < \varepsilon,$$

where $R = \frac{r}{2e}$.

Proof. Let $f^{\mathbb{C}}$ and $h^{\mathbb{C}}$ be the complexifications of f and h, respectively. Then they are analytic in the $\frac{r}{2e}$-neighbourhood of $(x, 0) \in X^{\mathbb{C}}$. Let H be a subset in $\mathbb{N}$ such that

$$\big|(h^{\mathbb{C}})^*(x) - h^{\mathbb{C}}(x)\big| \le \frac{\varepsilon}{2}, \quad x \in R\mathcal{B}_{X^{\mathbb{C}}} \bigcap X_H^{\mathbb{C}}.$$

Then

$$|h^*(x) - h(x)| \le \frac{\varepsilon}{2}, \quad x \in R\mathcal{B}_X \bigcap X_H$$

as well. Since f is subsymmetric and continuous, it is $\mathfrak{S}$-symmetric (by Proposition 1) and so

$$f(x) = f(\mathcal{C}_H(x)) = f\Big(\sum_{k=1}^{\infty} x_i e_{i_k}\Big), \quad (i_1, \ldots, i_k, \ldots) = H.$$

Note that $x \in R\mathcal{B}_X$ implies $\mathcal{C}_H(x) \in R\mathcal{B}_X$. Thus,

$$\begin{aligned} \sup_{x \in R\mathcal{B}_X} |f(x) - h^*(x)| &= \sup_{x \in R\mathcal{B}_X \cap X_H} |f(x) - h^*(x)| \\ &\le \sup_{x \in R\mathcal{B}_X \cap X_H} |f(x) - h(x)| + \sup_{x \in R\mathcal{B}_X \cap X_H} |h^*(x) - h(x)| < \varepsilon. \end{aligned}$$

Therefore, we have the required inequality for $u = h^*$. ☐

4 Metric space of ordered multisets

4.1 A metric on the space of ordered multisets

Let us define the following relation of equivalence on X: we say that $x \simeq y$ if there is a bijection $\sigma\colon \operatorname{supp}(x) \to \operatorname{supp}(y)$ such that $\sigma(i) < \sigma(j)$ whenever $i < j$, and $x_i = y_{\sigma(i)}$ for every $i \in \operatorname{supp}(x)$. Clearly, $x \simeq y$ if and only if there are τ_1 and τ_2 in $\overline{\mathfrak{S}}$ such that $\tau_1(x) = \tau_2(y)$. Let $[x]$ be the equivalence class containing x. We define the *canonical* representative of $[x]$ as

$$\widehat{x} = (x_{i_1}, x_{i_2}, \ldots, x_{i_m}, 0, 0, \ldots) \in X,$$

where $i_1, i_2, \ldots, i_m$ are in $\operatorname{supp}(x)$ and $1 \le m \le \infty$. In other words either all coordinates $\widehat{x}_i$ of $\widehat{x}$ are nonzero or if $\widehat{x}_i = 0$, then $\widehat{x}_j = 0$ for every $j > i$. It is easy to check that the canonical representative always exists and unique. For a given $x \in X$, $\widehat{x}$ can be obtained removing all zero coordinates x_i for $i < M$, where M is the maximal number such that $x_M \ne 0$ or $M = \infty$ if x has infinite many nonzero coordinates.

If is easy to see that $f(x) = f(\widehat{x})$ for every subsymmetric function f and $x \in X$. Conversely, if f_0 is a function on the quotient set $X/\simeq$, then it can be lifted to a subsymmetric function f on X by $f(x) = f_0(\widehat{x})$, $x \in X$.

The following propositions shows that the mapping $x \mapsto \widehat{x}$ is discontinuous in X.

Proposition 4. *The mapping $w\colon x \mapsto \widehat{x}$ acting from X to X is discontinuous at any point x of the form $x = (0, x_2, \ldots, x_m, 0, \ldots)$, where $x_i \ne 0$, $2 \le i \le m \le \infty$.*

Proof. Let $x^{(n)} = (\varepsilon_n, x_2, \ldots, x_m, 0, \ldots)$ be a sequence of vectors in X, such that $\varepsilon_n \ne 0$, and $\varepsilon_n \to 0$ as $n \to \infty$. Since $w(x^{(n)}) = (\varepsilon_n, x_2, \ldots, x_m, 0, \ldots)$ and $w(x) = (x_2, x_3, \ldots, x_m, 0, \ldots)$, we obtain $\|x^{(n)} - x\| = |\varepsilon_n| \to 0$, that is, $x^{(n)} \to x$. On the other hand, we have $w(x^{(n)}) - w(x) = (\varepsilon_n - x_2, x_2 - x_3, \ldots, x_m, 0, \ldots)$. Thus, we get $\|w(x^{(n)}) - w(x)\| \ge |x_2| > 0$ and so $w(x^{(n)}) \not\to w(x)$. Hence, the map w is discontinuous at x. ☐

Let us denote by $\mathfrak{M}_X$ the quotient set $X/\simeq$. The set $\mathfrak{M}_X$ can be thought of as a set of ordered multisets of nonzero numbers (nonzero coordinates of vectors $x \in X$). We introduce the following metric d on $\mathfrak{M}_X$ by

$$d([x], [y]) = \|\widehat{x} - \widehat{y}\|, \quad x, y \in X.$$

Lemma 2. *The function d is a metric on $\mathfrak{M}_X$.*

Proof. Let $x \simeq x'$ and $y \simeq y'$. Since each equivalence class $[x]$ has a unique representative $\widehat{x}$, we get $\widehat{x} = \widehat{x'}$, and $\widehat{y} = \widehat{y'}$. Hence, $\|\widehat{x} - \widehat{y}\| = \|\widehat{x'} - \widehat{y'}\|$ and so the distance d is well-defied on $\mathfrak{M}_X$.

Since $\|\cdot\|$ is a norm on X, we have $d([x],[y]) = \|\widehat{x} - \widehat{y}\| \geq 0$ for all $[x],[y] \in \mathfrak{M}_X$. Also,

$$d([x],[y]) = 0 \iff \|\widehat{x} - \widehat{y}\| = 0 \iff \widehat{x} = \widehat{y} \iff x \simeq y \iff [x] = [y].$$

For any $[x],[y] \in \mathfrak{M}_X$ we have $d([x],[y]) = \|\widehat{x} - \widehat{y}\| = \|\widehat{y} - \widehat{x}\| = d([y],[x])$.

Let us check the triangle inequality. If $[x],[y],[z] \in \mathfrak{M}_X$, then $\widehat{x} - \widehat{z} = (\widehat{x} - \widehat{y}) + (\widehat{y} - \widehat{z})$, and from the triangle inequality for the norm $\|\cdot\|$ we obtain

$$\|\widehat{x} - \widehat{z}\| = \|\widehat{x} - \widehat{y} + \widehat{y} - \widehat{z}\| \leq \|\widehat{x} - \widehat{y}\| + \|\widehat{y} - \widehat{z}\|.$$

Thus,

$$d([x],[z]) \leq d([x],[y]) + d([y],[z]).$$

☐

It is well-known that every metric space M is completely Hausdorff, that is, for every u and v in M such that $u \neq v$ there is a continuous function f on M such that $f(u) \neq f(v)$. Let $C(M)$ be the algebra of all continuous functions on M endowed with the topology of pointwise convergence. In other words, the topology of $C(M)$ is the weakest topology such that all functionals $\delta_u\colon u \mapsto u(x)$ are continuous. Thus, if $M = \mathfrak{M}_X$, then $x \simeq y$ if and only if $\delta_x = \delta_y$. In [12], it was observed that for $X = \ell_p$, $x \simeq y$ if and only if $f(x) = f(y)$ for every $f \in H_{b\mathfrak{S}}(\ell_p)$.

Proposition 5. *Every continuous function f on $(\mathfrak{M}_X, d)$ can be lifted to a continuous subsymmetric function $\widetilde{f}$ on X by $\widetilde{f}(x) = f([x])$. If f is bounded on bounded subsets of $\mathfrak{M}_X$, then $\widetilde{f}$ is bounded on bounded subsets of X.*

Proof. Clearly, $\widetilde{f}$ is subsymmetric. If f is continuous, then $[x_n] \to [x_0]$ implies $f([x_n]) \to f([x_0])$ as $n \to \infty$. But $\widetilde{f}(x_n) = f([x_n])$ and $\widetilde{f}(x_0) = f([x_0])$. Thus, $\widetilde{f}$ is continuous.

If $U \subset X$ is a bounded subset and $[U] = \{[x]\colon x \in U\}$, then $[U]$ is bounded. Thus, if $\widetilde{f}$ is unbounded on U, then f must be unbounded on $[U]$. Hence, $\widetilde{f}$ is bounded on any bounded subset of X if f is bounded on bounded subsets of $\mathfrak{M}_X$. ☐

4.2 The completeness of $(\mathfrak{M}_X, d)$

Theorem 6. *The metric space $(\mathfrak{M}_X, d)$ is not complete for any infinite-dimensional Banach space X with a subsymmetric basis. The completion of $(\mathfrak{M}_X, d)$ is isometric to X endowed with the metric generated by the norm of X.*

Proof. Let $\widehat{I}\colon [x] \mapsto \widehat{x} \in X$ be the natural embedding of $\mathfrak{M}_X$ to X, and $\widehat{X} = \widehat{I}(\mathfrak{M}_X)$. Then $\widehat{I}$ is isometric and $\widehat{X}$ is dense in X. Indeed, for a given vector $x = (x_1, x_2, \ldots) \neq 0$ in X we consider the following sequence of vectors $x^{(n)} = \big(x_1^{(n)}, x_2^{(n)}, \ldots\big)$ such that

$$x_j^{(n)} = \begin{cases} \frac{1}{2^{n+j}}, & \text{if } j \notin \operatorname{supp}(x), \\ x_j, & \text{if } j \in \operatorname{supp}(x). \end{cases}$$

It is easy to see that $x^{(n)} \to x$ in X as $n \to \infty$, and $x^{(n)} = \widehat{x^{(n)}}$ for every n.

Since X is complete, it is a completion of $\widehat{X}$. Thus X is isometric to the completion of $(\mathfrak{M}_X, d)$, and sice $\widehat{X} \neq X$, the metric space $(\mathfrak{M}_X, d)$ is not complete. ☐

4.3 Topological transitivity of the backward shift on $(\mathfrak{M}_X, d)$

Let $\mathfrak{M}_{00}$ denotes the subset of $\mathfrak{M}_X$ consisting of elements $[x]$ such that $\operatorname{supp}(x)$ is a finite set for every $x \in [x]$. Clearly that $|\operatorname{supp}(x)| = |\operatorname{supp}(y)|$ for every $y \in [x]$. Thus, if $x \in \mathfrak{M}_{00}$, then we can write $\widehat{x} = (x_1, \dots, x_m, 0, \dots)$ for some $m < \infty$. In this case, we will write $\widehat{x} = (x_1, \dots, x_m)$ if $x_i \neq 0$, $i = 1, \dots, m$, and $\widehat{x} = \mathbf{0}$ if $x_i = 0$ for all $i \in \mathbb{N}$.

Let $[x]$, $[y] \in \mathfrak{M}_{00}$ be such that $\widehat{x} = (x_1, \dots, x_m)$ and $\widehat{y} = (y_1, \dots, y_j)$. We introduce the following algebraic operation on $\mathfrak{M}_{00}$ by

$$[x] \lhd [y] = [(x_1, \dots, x_m, y_1, \dots, y_j)].$$

Proposition 6. *Let $[x] \in \mathfrak{M}_{00}$ and $[y^{(n)}]$ be a sequence in $\mathfrak{M}_{00}$ convergent to $\mathbf{0}$ in $(\mathfrak{M}_{00}, d)$ as $n \to \infty$. Then $[x] \lhd [y^{(n)}] \to x$ as $n \to \infty$, while $[y^{(n)}] \lhd [x] \nrightarrow x$ as $n \to \infty$ in the general case.*

Proof. Let $\widehat{x} = (x_1, \dots, x_m)$ and $\widehat{y^{(n)}} = (y_1^{(n)}, \dots, y_{j(n)}^{(n)})$. Then

$$\|\widehat{x \lhd y^{(n)}} - \widehat{x}\| = \|(x_1, \dots, x_m, y_1^{(n)}, \dots, y_{j(n)}^{(n)}) - (x_1, \dots, x_m)\| = \|(y_1^{(n)}, \dots, y_{j(n)}^{(n)})\| \to 0$$

as $n \to \infty$. On the other hand, it is not difficult to find $[x]$ and $[y^{(n)}]$ in $\mathfrak{M}_{00}$ such that

$$\|\widehat{y^{(n)} \lhd x} - \widehat{x}\| = \|(y_1^{(n)}, \dots, y_{j(n)}^{(n)}, x_1, \dots, x_m) - (x_1, \dots, x_m)\| \nrightarrow 0$$

as $n \to \infty$. ☐

Let us denote by B_λ the weighted backward shift operator on X, defined by

$$B_\lambda \colon (x_1, \dots, x_n, \dots) \mapsto \lambda(x_2, \dots, x_n, \dots),$$

where $\lambda \in \mathbb{C}$. For a given $\lambda \in \mathbb{C}$ we define $\widetilde{B}_\lambda \colon \mathfrak{M}_X \to \mathfrak{M}_X$ by $\widetilde{B}_\lambda([x]) = B_\lambda(\widehat{x})$. Note that $d(\widetilde{B}_\lambda([x]), \widetilde{B}_\lambda([y])) \leq \lambda d([x], [y])$ and so $\widetilde{B}_\lambda$ is continuous on $\mathfrak{M}_X$.

Theorem 7. *If $|\lambda| > 1$, then $\widetilde{B}_\lambda$ is topologically transitive on $\mathfrak{M}_X$.*

Proof. For given open subsets U and V in $(\mathfrak{M}_X d)$ we choose $[x] \in U \cap \mathfrak{M}_{00}$, and $[y] \in \mathfrak{M}_{00}$. It is possible because $\mathfrak{M}_{00}$ is dense in $\mathfrak{M}_X$. We assume that $\widehat{x} = (x_1, \dots, x_m)$, that is, $|\operatorname{supp}(x)| = m$. Clearly, $m = m(x)$ is a function of x. For every $k \in \mathbb{N}$ we define the map $S_{\lambda,m,k} \colon \mathfrak{M}_X \to \mathfrak{M}_X$ by

$$S_{\lambda,m,k}([y]) = \frac{1}{\lambda^{k+m}}[(\underbrace{1, 1 \dots, 1}_{m}, y_1, y_2, \dots)].$$

Note that

$$\|S_{\lambda,m,k}([y])\| \leq \frac{k\|y\|}{|\lambda^{k+m}|} \to 0 \quad \text{as} \quad k \to \infty$$

for every $[y] \in \mathfrak{M}_X$. Let $k \in \mathbb{N}$ be such that $[x] \lhd S_{\lambda,m,k}([y]) \in U$. Such a number k can be chosen since, by Proposition 6, we have $[x] \lhd S_{\lambda,m,k}([y]) \to [x]$ as $n \to \infty$.

Let $[z] = [x] \lhd S_{\lambda,m,k}([y])$. Since $[x] \lhd S_{\lambda,m,k}([y]) \in U$, $[z] \in U$. Let us compute $\widetilde{B}_\lambda^{k+m}([z])$:

$$\widetilde{B}_\lambda^{k+m}([z]) = \widetilde{B}_\lambda^{k+m}\Bigg[\Bigg(x_1, \dots, x_m, \underbrace{\frac{1}{\lambda^{m+k}}, \dots, \frac{1}{\lambda^{m+k}}}_{k}, \frac{y_1}{\lambda^{m+k}}, \dots, \frac{y_m}{\lambda^{m+k}}, 0, \dots\Bigg)\Bigg]$$

$$= \widetilde{B}_\lambda^{k}\Bigg[\Bigg(\underbrace{\frac{1}{\lambda^{m+k}}, \dots, \frac{1}{\lambda^{m+k}}}_{k}, \frac{y_1}{\lambda^{k}}, \dots, \frac{y_m}{\lambda^{k}}, 0, \dots\Bigg)\Bigg] = [y].$$

Therefore, we have that for every pairs of open sets U and V there exist $[z] \in U$ and $n = k + m \in \mathbb{N}$ such that $\widetilde{B}_\lambda^{k+m}([z]) = [y] \in V$. Hence, $\widetilde{B}_\lambda$ topologically transitive. ☐

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Розглянуто властивості субсиметричних поліномів, аналітичних функцій та деяких їхніх узагальнень на банахових просторах із субсиметричними базисами. Ми доводимо, що якщо поліном на комплексному нескінченновимірному банаховому просторі X має субсиметричну множину нулів, то він є субсиметричним. Звідси ми робимо висновок, що алгебра $\mathcal{P}_{\mathfrak{S}}(X)$ субсиметричних поліномів на X є факторіальною. Ми розглядаємо умови, коли субсиметричну функцію на банаховому просторі можна апроксимувати субсиметричними аналітичними функціями або поліномами. Крім того, ми будуємо деякі зважені відображення типу "зсув назад" на метричному просторі функціоналів значень в точках на просторі $\mathcal{P}_{\mathfrak{S}}(X)$ та доводимо їх топологічну транзитивність.

Ключові слова і фрази: поліном на нескінченновимірних просторах, множина нулів поліномів, симетричний поліном, топологічно транзитивний оператор.