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Conjugate harmonic Poisson integral for a half-plane and some of its approximation properties

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At present, the approximation properties of harmonic Poisson integrals for the upper half-plane, which satisfy the Laplace equation in Cartesian coordinates, have not been sufficiently studied. This paper studies the approximation properties of conjugate harmonic Poisson integrals for the upper half-plane on classes of Hölder functions. An exact equality is found for the upper bound of the deviation of the conjugate harmonic Poisson integral from its boundary value in a uniform metric.

Key words and phrases: Laplace equation, conjugate harmonic Poisson integral for the upper half-plane, class of Hölder functions, analytic function.

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Introduction

Approximation theory methods play an increasingly important role in modern applied mathematics [2–5, 17, 18, 29]. Boundary value problems for partial differential equations [22–24], as well as the properties of their solutions, are also quite interesting from the viewpoint of applied research. Among the most studied are solutions to Laplace-type equations, which have been investigated, for example, in [9, 20, 26]. It is known that any mathematical object of this type can be a real or imaginary part of a complex variable function. However, the emergence and development of approximation theory have demonstrated that solutions to Laplace equations of a certain form can serve as approximating aggregates [7, 19, 28].

For example, one of such aggregates is the harmonic Poisson integral

$$\mathcal{P}_\rho\left(f;x\right) = \frac{1-\rho^2}{2\pi}\int_{-\pi}^{\pi}\frac{f\left(x+t\right)}{1-2\rho\cos t+\rho^2}\,dt, \qquad 0\le\rho<1, \tag{1}$$

that satisfies the Laplace equation in a unit disk.

At the same time, the harmonic Poisson integral (1) can also be considered as a linear method for summing Fourier series given by a set of functions of a real argument (see, e.g., [27]). In function theory, the study of the approximation properties of various linear methods for summing Fourier series has been and remains an important area of research (see, e.g., [1, 10, 25, 29]).

Returning to the harmonic Poisson integral (1), we note that the study of its approximation properties was initiated in the papers [21, 30]. Later, this research was extended in the papers [7, 13, 16, 19, 28], etc.

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However, the interpretation of integral (1) in the context of analytic function theory is no less interesting. Namely, the representation of the real part of a complex variable function in the form of a harmonic Poisson integral (1) indicates that the imaginary part is already related to the conjugate harmonic Poisson integral

$$\overline{\mathcal{P}}_{\rho}\left(f;x\right)=-\frac{\rho}{\pi}\int_{-\pi}^{\pi}\frac{f\left(x+t\right)\sin\left(t\right)}{1-2\rho\cos t+\rho^{2}}\,dt$$

for a unit disk, whose approximation properties were studied in [6,12].

At the same time, the approximation properties of conjugate harmonic Poisson integrals

$$\overline{\mathcal{P}}\left(f;x,y\right)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{tf\left(x+t\right)}{t^{2}+y^{2}}\,dt \tag{2}$$

for the upper half-plane, i.e. for $y>0$, remain completely unexplored.

As in [15], by H^1 we denote the class of Hölder functions defined on the entire real axis, consisting of all continuous functions satisfying the condition

$$\left|f\left(x_{1}\right)-f\left(x_{2}\right)\right|\leq\left|x_{1}-x_{2}\right| \tag{3}$$

for all $x_1, x_2 \in \mathbb{R}$.

The aim of our paper is to study the approximation properties of conjugate harmonic Poisson integral (2) in the upper half-plane for functions of Hölder classes H^1 in the metric of the space C, where $\|f\|_C = \sup\limits_{x\in\mathbb{R}} |f\left(x\right)|$.

1 Estimate of the conjugate harmonic Poisson integral deviation from its boundary values

Theorem 1. *For all functions of the Hölder class H^1 and conjugate harmonic Poisson integral* (2)*, the following exact equality*

$$\mathcal{E}\left(H^{1};\overline{\mathcal{P}}(y)\right)_{C}=\sup_{f\in H^{1}}\left\|\,\overline{f}\left(x\right)-\overline{\mathcal{P}}\left(f;x,y\right)\right\|_{C}=y \tag{4}$$

holds for all $y>0$.

Proof. If the harmonic Poisson integral

$$\mathcal{P}\left(f;x,y\right)=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f\left(x+t\right)}{t^{2}+y^{2}}\,dt \tag{5}$$

for the upper half-plane satisfies the Laplace equation, then it can be considered as a function $U\left(x,y\right)$ in the formula $W(\zeta) = U(x,y) + iV(x,y)$ for a function $W(\zeta)$ of the complex variable $\zeta = x + iy$.

There is a relationship between the two-dimensional functions $U(x,y)$ and $V(x,y)$ in the form of the Cauchy-Riemann conditions

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y}\qquad\text{and}\qquad\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}. \tag{6}$$

We can rewrite the differential equations (6) by replacing the real part $U(x,y)$ with the harmonic Poisson integral $\mathcal{P}(f;x,y)$. As for the imaginary part $V(x,y)$, it must be replaced by the conjugate harmonic Poisson integral $\overline{\mathcal{P}}(f;x,y)$. Then we get

$$\frac{\partial}{\partial x}\mathcal{P}(f;x,y)=\frac{\partial}{\partial y}\overline{\mathcal{P}}(f;x,y),\qquad \frac{\partial}{\partial y}\mathcal{P}(f;x,y)=-\frac{\partial}{\partial x}\overline{\mathcal{P}}(f;x,y). \tag{7}$$

The usage of the differential equations (7) for constructing the conjugate harmonic Poisson integral proves to be more expedient after introducing a new integration variable $z=x+t$ in the harmonic Poisson integral (5). Therefore, it acquires the following form

$$\mathcal{P}(f;x,y)=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(z)}{(z-x)^2+y^2}\,dz. \tag{8}$$

The result (8) indicates that the dependence of the harmonic Poisson integral on the variable x is determined only by its delta-like integral kernel [8,11,14]. Such a circumstance allows us to differentiate the integral (8) with respect to the variable x without imposing differentiability conditions on the function $f(x)$. Then it turns out that

$$\frac{\partial}{\partial x}\mathcal{P}(f;x,y)=\frac{2y}{\pi}\int_{-\infty}^{+\infty}\frac{(z-x)f(z)}{\left((z-x)^2+y^2\right)^2}\,dz. \tag{9}$$

By returning to the substitution $z=x+t$, we transform the integral (9) into the following one

$$\frac{\partial}{\partial x}\mathcal{P}(f;x,y)=\frac{2y}{\pi}\int_{-\infty}^{+\infty}\frac{tf(x+t)}{\left(t^2+y^2\right)^2}\,dt. \tag{10}$$

Finding the antiderivative for the right-hand side of (10) with respect to the variable y, gives us a relationship

$$\frac{\partial}{\partial x}\mathcal{P}(f;x,y)=-\frac{1}{\pi}\frac{\partial}{\partial y}\left(\int_{-\infty}^{+\infty}\frac{tf(x+t)}{t^2+y^2}\,dt\right).$$

By comparing the obtained result with the first condition in (7), we obtain the conjugate harmonic Poisson integral (2) for the upper half-plane, whose approximation properties for the Hölder function class H^1 will be studied in the current research.

However, the approximation properties of the integral (2) are more conveniently studied by replacing integration over the entire real axis with integration over the half-axis. To do this, we will split integral (2) into two integrals by applying the identity

$$f(x+t)=\frac{1}{2}\big\{f(x+t)+f(x-t)\big\}-\frac{1}{2}\big\{f(x-t)-f(x+t)\big\}.$$

Using the properties of integrals of even and odd functions over symmetric intervals, we obtain the following integral representation

$$\overline{\mathcal{P}}(f;x,y)=\frac{1}{\pi}\int_0^{+\infty}\frac{t\left\{f(x-t)-f(x+t)\right\}}{t^2+y^2}\,dt. \tag{11}$$

The boundary value of the conjugate harmonic Poisson integral (11) at the upper half-plane boundary is given by the definition $\overline{f}(x)=\overline{\mathcal{P}}(f;x,y)\Big|_{y=0}$ and has the following form

$$\overline{f}(x)=\frac{1}{\pi}\int_0^{+\infty}\frac{f(x-t)-f(x+t)}{t}\,dt. \tag{12}$$

Similarly to [31], we can consider the difference of the integrals (11) and (12) in order to establish an integral representation

$$\overline{f}(x) - \overline{\mathcal{P}}(f;x,y) = \frac{y^2}{\pi} \int_0^{+\infty} \frac{f(x-t) - f(x+t)}{t(t^2+y^2)}\, dt \tag{13}$$

for the quantity $\overline{f}(x) - \overline{\mathcal{P}}(f;x,y)$.

The inequality (3) and the integral representation (13) enable us to write down an estimate

$$\mathcal{E}\big(H^1;\overline{\mathcal{P}}(y)\big)_C = \sup_{f\in H^1} \big\|\overline{f}(x) - \overline{\mathcal{P}}(f;x,y)\big\|_C \le \frac{2y^2}{\pi} \int_0^{+\infty} \frac{dt}{t^2+y^2}\,.$$

After calculating the above improper integral, we obtain the following result

$$\mathcal{E}\big(H^1;\overline{\mathcal{P}}(y)\big)_C = \sup_{f\in H^1} \big\|\overline{f}(x) - \overline{\mathcal{P}}(f;x,y)\big\|_C \le y. \tag{14}$$

Of particular interest is the question of the existence of the so-called extremal function in the Hölder class H^1, which transforms the inequality (14) into an equality. It should be noted that the equality

$$\mathcal{E}\big(H^1;\overline{\mathcal{P}}(y)\big)_C = \sup_{f\in H^1} \big\|\,\overline{f}(0) - \overline{\mathcal{P}}(f;0,y)\big\|_C$$

is also valid. Its right-hand side coincides with the right-hand side of the inequality (14) for the case when

$$\overline{\mathcal{P}}\,(f;0,y) - \overline{f}\,(0) = y. \tag{15}$$

The application of the integral representation (13) for the quantity $\overline{f}\,(x) - \overline{\mathcal{P}}\,(f;x,y)$ indicates that the equation (15) is the following integral equation

$$\frac{y}{\pi} \int_0^{+\infty} \frac{f(t) - f(-t)}{t\,(t^2+y^2)}\, dt = 1 \tag{16}$$

for the extremal function $f(t)$.

The process of finding the solution to the integral equation (16) becomes more convenient after applying the Fourier integral

$$f(t) = \int_{-\infty}^{+\infty} \mathcal{C}(q) e^{iqt}\, dq.$$

Using this definition, we write down the difference

$$f(t) - f(-t) = 2i \int_{-\infty}^{+\infty} \mathcal{C}(q) \sin(qt)\, dq. \tag{17}$$

The use of the identity

$$\mathcal{C}(q) = \frac{1}{2}\{\mathcal{C}(q) + \mathcal{C}(-q)\} + \frac{1}{2}\{\mathcal{C}(q) - \mathcal{C}(-q)\}$$

in the Fourier integral (17) necessitates the use of the properties of integrals of even and odd functions over symmetric intervals.

Thus, we obtain the Fourier sine transform

$$f(t) - f(-t) = \int_0^{+\infty} \mathcal{B}(q) \sin(qt)\, dq, \tag{18}$$

where $\mathcal{B}(q) = 2i\,\{\mathcal{C}(q) - \mathcal{C}(-q)\}$.

Next, we need to substitute the integral representation (18) into the integral equation (16) and change the order of integration in the resulting double integral. Then we get the integral equation

$$\frac{y}{\pi} \int_0^{+\infty} \mathcal{B}\,(q)\, \mathcal{I}\,(q, y)\, dq = 1 \tag{19}$$

with the integral kernel

$$\mathcal{I}(q, y) = \int_0^{+\infty} \frac{\sin(qt)}{t\,(t^2 + y^2)}\, dt. \tag{20}$$

The integral in (20) can be conveniently calculated by parameterization method. If we rewrite the right-hand side of the identity

$$\frac{t}{t^2 + y^2} = \frac{1}{2i} \left(\frac{1}{y - it} - \frac{1}{y + it} \right)$$

by applying the integral representations

$$\frac{1}{y \pm it} = \int_0^{+\infty} e^{-\omega y} e^{\mp i\omega t} d\omega,$$

then we can establish the parameterization

$$\frac{t}{t^2 + y^2} = \int_0^{+\infty} e^{-\omega y} \sin\,(\omega t)\, d\omega.$$

Next, it is necessary to multiply the above equality by $\dfrac{\sin\,(qt)}{t^2}$ and substitute the obtained result into the right-hand side of (20). Then it turns out that

$$\mathcal{I}\,(q, y) = \int_0^{+\infty} e^{-\omega y} \left(\int_0^{+\infty} \frac{\sin\,(qt) \sin\,(\omega t)}{t^2}\, dt \right) d\omega. \tag{21}$$

Using the trigonometric identity $2 \sin\,(qt) \sin\,(\omega t) = \cos\,\{(q - \omega)\, t\} - \cos\,\{(q + \omega)\, t\}$ and the formula for the cosine of a double angle, we can write out the relationship

$$\sin\,(qt) \sin\,(\omega t) = \left\{ \sin \left[(q + \omega)\, \frac{t}{2} \right] \right\}^2 - \left\{ \sin \left[(q - \omega)\, \frac{t}{2} \right] \right\}^2.$$

Therefore, the inner integral in (21) has the form of the expression

$$\int_0^{+\infty} \frac{\sin(qt) \sin(\omega t)}{t^2}\, dt = \int_0^{+\infty} \frac{\left\{ \sin \left[(q + \omega)\, \frac{t}{2} \right] \right\}^2}{t^2}\, dt - \int_0^{+\infty} \frac{\left\{ \sin \left[(q - \omega)\, \frac{t}{2} \right] \right\}^2}{t^2}\, dt.$$

The integrals on the right-hand side of the last equality can be transformed by using a new integration variable $\tau = t/2$. Then we obtain the formula

$$\int_0^{+\infty} \frac{\sin\,(qt) \sin\,(\omega t)}{t^2}\, dt = \frac{1}{2} \big\{ \mathcal{I}_1(q + \omega) - \mathcal{I}_1(q - \omega) \big\}, \tag{22}$$

where
$$\mathcal{I}_1(\alpha) = \int_0^{+\infty} \frac{\{\sin(\alpha\tau)\}^2}{\tau^2}\, d\tau.$$

It can be seen that $\mathcal{I}_1(0) = 0$. Considering the cases $\alpha < 0$ and $\alpha > 0$, it is advisable to introduce new integration variables $\lambda = -2\alpha\tau$ and $\lambda = 2\alpha\tau$, respectively. So,

$$\mathcal{I}_1(\alpha) = |\alpha| \int_0^{+\infty} \frac{1-\cos\lambda}{\lambda^2}\, d\lambda. \tag{23}$$

In order to evaluate the integral (23), let us write down the parameterization

$$\frac{1}{\lambda} = \int_0^{+\infty} e^{-k\lambda} dk. \tag{24}$$

If we integrate the identity
$$\lambda k e^{-k\lambda} = e^{-k\lambda} - \frac{\partial}{\partial k}\left(k e^{-k\lambda}\right)$$
with respect to the variable k from 0 to $+\infty$ and take into account the integral representation (24), we can arrive at the parameterization

$$\frac{1}{\lambda^2} = \int_0^{+\infty} k e^{-k\lambda}\, dk.$$

On the basis of this result, we obtain

$$\int_0^{+\infty} \frac{1-\cos\lambda}{\lambda^2}\, d\lambda = \int_0^{+\infty} k \left\{ \int_0^{+\infty} e^{-k\lambda} d\lambda - \int_0^{+\infty} e^{-k\lambda} \cos(\lambda)\, d\lambda \right\} dk. \tag{25}$$

Let us consider the inner integrals on the right-hand side of the equality (25). The first integral can be obtained from the integral (24) by replacement $k \longleftrightarrow \lambda$. The second integral can be evaluated by applying the identities

$$k e^{-k\lambda} \cos\lambda = -e^{-k\lambda} \sin\lambda - \frac{\partial}{\partial\lambda}\left(e^{-k\lambda}\cos\lambda\right) \tag{26}$$

and
$$e^{-k\lambda}\cos\lambda = k e^{-k\lambda}\sin\lambda + \frac{\partial}{\partial\lambda}\left(e^{-k\lambda}\sin\lambda\right). \tag{27}$$

We need to multiply the equality (26) by k and add the obtained result to the equality (27). Then we derive the identity

$$(1+k^2) e^{-k\lambda}\cos\lambda = \frac{\partial}{\partial\lambda}\left\{e^{-k\lambda}(\sin\lambda - k\cos\lambda)\right\}$$

that enables us to find the integral

$$\int_0^{+\infty} e^{-k\lambda}\cos(\lambda)\, d\lambda = \frac{k}{1+k^2}.$$

Then the identity (25) with the transformed right-hand side makes it possible to reduce the function (23) into the following form

$$\mathcal{I}_1(\alpha) = |\alpha| \int_0^{+\infty} \frac{dk}{1+k^2}.$$

After evaluating the above integral, we get

$$\mathcal{I}_1(\alpha) = \frac{\pi|\alpha|}{2},$$

that transforms the formula (22) into the following one

$$\int_0^{+\infty} \frac{\sin(qt)\sin(\omega t)}{t^2}\,dt = \frac{\pi}{4}\big(|q+\omega| - |q-\omega|\big). \tag{28}$$

Next, we need to substitute the integral (28) into (21). Since $|q+\omega| = q+\omega$, we obtain

$$\mathcal{I}(q,y) = \frac{\pi q}{4}\int_0^{+\infty} e^{-\omega y}\,d\omega + \frac{\pi}{4}\int_0^{+\infty} \omega e^{-\omega y}\,d\omega - \frac{\pi}{4}\int_0^{+\infty} |q-\omega|\, e^{-\omega y}\,d\omega. \tag{29}$$

Let us consider the third integral on the right-hand side of the equality (29). Obviously, it can be calculated by dividing the integration interval by the point $\omega = q$. Then we can write

$$\int_0^{+\infty} |q-\omega| e^{-\omega y}\,d\omega = \int_q^{+\infty} \omega e^{-\omega y}\,d\omega - q\int_q^{+\infty} e^{-\omega y}\,d\omega + q\int_0^q e^{-\omega y}\,d\omega - \int_0^q \omega e^{-\omega y}\,d\omega. \tag{30}$$

Let us consider the second and fourth integrals on the right-hand side of the identity (30). Obviously, they can be written in the following forms

$$\int_0^q e^{-\omega y}\,d\omega = \int_0^{+\infty} e^{-\omega y}\,d\omega - \int_q^{+\infty} e^{-\omega y}\,d\omega$$

and

$$\int_q^{+\infty} \omega e^{-\omega y}\,d\omega = \int_0^{+\infty} \omega e^{-\omega y}\,d\omega - \int_0^q \omega e^{-\omega y}\,d\omega.$$

When substituting the last two representations into the right-hand side of (30), it is quite convenient to move the integrals with integration from 0 to $+\infty$ into the left-hand side. Then we obtain

$$q\int_0^{+\infty} e^{-\omega y}\,d\omega + \int_0^{+\infty} \omega e^{-\omega y}\,d\omega - \int_0^{+\infty} |q-\omega| e^{-\omega y}\,d\omega = 2\int_0^q \omega e^{-\omega y}\,d\omega + 2q\int_q^{+\infty} e^{-\omega y}\,d\omega,$$

that makes it possible to transform the two-dimensional function (29) into the following one

$$\mathcal{I}(q,y) = \frac{\pi}{2}\int_0^q \omega e^{-\omega y} d\omega + \frac{\pi q}{2}\int_q^{+\infty} e^{-\omega y} d\omega.$$

After evaluating the integrals on the right-hand side of the last equality, we get

$$\mathcal{I}(q,y) = \frac{\pi}{2y^2}\left(1 - e^{-qy}\right),$$

which must then be substituted into the integral equation (19). Then it turns out that

$$\frac{1}{2y}\int_0^{+\infty} \mathcal{B}(q)\left(1 - e^{-qy}\right) dq = 1. \tag{31}$$

The right-hand side of the equality (31) indicates that its left-hand side does not actually depend on the variable y. Therefore, we can consider the integral equation (31) for the asymptotic case when $y \to +0$. On the basis of the integral representation

$$\frac{1 - e^{-qy}}{y} = \int_0^q e^{-py}\,dp$$

we find the limit

$$\lim_{y\to+0}\frac{1-e^{-qy}}{y}=q,$$

that makes the integral equation (31) look like this

$$\frac{1}{2}\int_0^{+\infty} q\mathcal{B}\left(q\right)dq=1. \tag{32}$$

It should be noted that the integral equation (32) can be solved within the framework of the theory of generalized functions. It is known that the Dirac delta function $\delta\left(q\right)$ satisfies the condition

$$2\int_0^{+\infty}\delta\left(q\right)dq=1. \tag{33}$$

After comparing the integrals (32) and (33), we can finally write down the solution

$$\mathcal{B}\left(q\right)=\frac{4}{q}\,\delta\left(q\right). \tag{34}$$

We will show that the generalized function (34) satisfies not only the integral equation (32), but also the integral equation (31). To do this, we need to substitute the solution (34) into the integral of the formula (31). Then we can write that

$$\int_0^{+\infty}\mathcal{B}(q)\left(1-e^{-qy}\right)dq=4\int_0^{+\infty}\delta(q)\frac{1-e^{-qy}}{q}\,dq. \tag{35}$$

By applying the integral representation

$$\delta\left(q\right)=\frac{1}{\pi}\int_0^{+\infty}\cos\left(qz\right)dz \tag{36}$$

for the Dirac delta function, it is possible to reduce the equality (35) into the following form

$$\int_0^{+\infty}\mathcal{B}\left(q\right)\left(1-e^{-qy}\right)dq=\frac{4}{\pi}\int_0^{+\infty}\left(\int_0^{+\infty}\frac{1-e^{-qy}}{q}\cos\left(qz\right)dq\right)dz. \tag{37}$$

The parameterization

$$\frac{1-e^{-qy}}{q}=\int_0^{y}e^{-q\lambda}\,d\lambda$$

transforms the inner integral on the right-hand side of the identity (37) into a double integral. Therefore,

$$\int_0^{+\infty}\frac{1-e^{-qy}}{q}\cos\left(qz\right)dq=\int_0^{y}\left(\int_0^{+\infty}e^{-q\lambda}\cos\left(qz\right)dq\right)d\lambda. \tag{38}$$

Using the integral

$$\int_0^{+\infty}e^{-q\lambda}\cos\left(qz\right)dq=\frac{\lambda}{\lambda^2+z^2}\,,$$

we can rewrite the equality (38) as follows

$$\int_0^{+\infty}\frac{1-e^{-qy}}{q}\cos\left(qz\right)dq=\int_0^{y}\frac{\lambda}{\lambda^2+z^2}\,d\lambda.$$

The calculation of the integral on the right-hand side of the above identity yields the result

$$\int_0^{+\infty} \frac{1-e^{-qy}}{q} \cos(qz)\, dq = \frac{1}{2} \ln\left(1+\frac{y^2}{z^2}\right),$$

which needs to be substituted into the right-hand side of the equality (37). As a result, we get

$$\int_0^{+\infty} \mathcal{B}(q)\left(1-e^{-qy}\right) dq = \frac{2}{\pi} \int_0^{+\infty} \ln\left(1+\frac{y^2}{z^2}\right) dz. \tag{39}$$

In order to evaluate the integral on the right-hand side of (39), it is convenient to introduce a new integration variable φ by means of the relationship $z = y \tan(\varphi)$. Then, instead of the formula (39), we will have the following one

$$\int_0^{+\infty} \mathcal{B}(q)\left(1-e^{-qy}\right) dq = -\frac{4y}{\pi} \int_0^{\pi/2} \frac{\ln(\sin\varphi)}{(\cos\varphi)^2}\, d\varphi. \tag{40}$$

The integral on the right-hand side of the identity (40) can be evaluated by constructing an integral representation for the subintegral function. Using the properties of logarithms, we can first consider an elementary transformation

$$\ln|\sin\varphi| = \frac{1}{2}\ln(\sin\varphi)^2.$$

The basic trigonometric identity enables us to reduce this formula into the following form

$$\ln|\sin\varphi| = \frac{1}{2}\ln\left\{1-(\cos\varphi)^2\right\}. \tag{41}$$

By introducing a two-dimensional function

$$\mathcal{F}(\varepsilon,\varphi) = \ln\left\{1-\varepsilon(\cos\varphi)^2\right\}, \tag{42}$$

it is possible to arrive at the following result

$$\ln\left\{1-(\cos\varphi)^2\right\} = \mathcal{F}(1,\varphi) - \mathcal{F}(0,\varphi).$$

The difference on the right-hand side of the above equality can be considered as a difference between the antiderivatives in the Newton-Leibniz formula. Such a circumstance generates an integral representation

$$\ln\left\{1-(\cos\varphi)^2\right\} = \int_0^1 \frac{\partial\mathcal{F}(\varepsilon,\varphi)}{\partial\varepsilon}\, d\varepsilon,$$

which allows us to replace the formula (41) by the following one

$$\ln|\sin\varphi| = \frac{1}{2}\int_0^1 \frac{\partial\mathcal{F}(\varepsilon,\varphi)}{\partial\varepsilon}\, d\varepsilon.$$

Taking into account the two-dimensional function (42), we can transform the last identity into a parametrization

$$\frac{\ln|\sin\varphi|}{(\cos\varphi)^2} = -\frac{1}{2}\int_0^1 \frac{d\varepsilon}{1-\varepsilon(\cos\varphi)^2}. \tag{43}$$

The integral on the right-hand side of the equality (43) can be presented in a slightly different form. This can be performed by using the power reduction formula in order to write down the identity

$$2\left\{1-\varepsilon(\cos\varphi)^2\right\} = 2-\varepsilon-\varepsilon\cos(2\varphi)\,. \tag{44}$$

Next, we will perform the transition from the integration variable ε to the new integration variable ρ by using the relationship

$$\frac{\varepsilon}{2-\varepsilon} = \frac{2\rho}{1+\rho^2}\,. \tag{45}$$

Directly from the equation (45), we find the formula

$$\varepsilon = \frac{4\rho}{(1+\rho)^2} \tag{46}$$

for the variable ε and the formula

$$d\varepsilon = \frac{4\,(1-\rho)\,d\rho}{(1+\rho)^3} \tag{47}$$

for its differential.

Substituting the representation (46) into the right-hand side of the equality (44), we obtain the result

$$\frac{1}{1-\varepsilon\,(\cos\varphi)^2} = \frac{(1+\rho)^2}{1-2\rho\cos(2\varphi)+\rho^2}\,. \tag{48}$$

The transition from the integration variable ε to the integration variable ρ involves changing the integration limits in the integral (43). This means that we first need to solve the equation (45) with respect to the variable ρ. This equation is quadratic and has the following form

$$\rho^2 + 2\left(1-\frac{2}{\varepsilon}\right)\rho + 1 = 0.$$

Completing the square on the left-hand side of this identity, we obtain

$$\left(\rho + 1 - \frac{2}{\varepsilon}\right)^2 - \frac{4\,(1-\varepsilon)}{\varepsilon^2} = 0.$$

This quadratic equation has the solution

$$\rho = \frac{1}{\varepsilon}\left(1-\sqrt{1-\varepsilon}\right)^2. \tag{49}$$

However, the formula (49) turns out to be inconvenient for determining the new integration limits. Indeed, if $\varepsilon = 0$, then an indeterminacy of the type $\left(\frac{0}{0}\right)$ arises on the right-hand side of the equality (49). It is possible to resolve such an indeterminacy after applying the identity

$$1-\sqrt{1-\varepsilon} = \frac{\varepsilon}{1+\sqrt{1-\varepsilon}}\,.$$

Then the formula (49) can be replaced by the following one:

$$\rho = \frac{\varepsilon}{\left(1+\sqrt{1-\varepsilon}\right)^2}\,. \tag{50}$$

In the presence of the solution (50), we can consider the difference $1-\rho$ and arrive at the expression

$$\rho = 1 - \frac{2\sqrt{1-\varepsilon}}{1+\sqrt{1-\varepsilon}}\,. \tag{51}$$

The results (50) and (51) show that the integration limits in the integral (43) remain unchanged even after the transition from the integration variable ε to the integration variable ρ. Then the transformations (47) and (48) allow us to transform the integral representation (43) into a parameterization

$$\frac{\ln|\sin\varphi|}{(\cos\varphi)^2} = -2\int_0^1 \frac{1-\rho}{(1+\rho)\left(1-2\rho\cos(2\varphi)+\rho^2\right)}\,d\rho. \tag{52}$$

The validity of the equality (52) can be verified by direct evaluation of the integral on its right-hand side. However, the integration is only possible after applying the expansion

$$\frac{1-\rho}{(1+\rho)\left(1-2\rho\cos(2\varphi)+\rho^2\right)} = \frac{A_1}{1+\rho} + \frac{A_2\rho + A_3}{1-2\rho\cos(2\varphi)+\rho^2}\,, \tag{53}$$

which establishes a representation of the subintegral function via elementary fractions.

From the formula (53), we obtain the identity

$$(A_1+A_2)\,\rho^2 + \big(A_2+A_3-2A_1\cos(2\varphi)\big)\rho + A_1 + A_3 = 1-\rho,$$

which generates a closed system of three linear equations

$$\begin{cases} A_1 + A_2 = 0, \\ A_2 + A_3 - 2A_1\cos(2\varphi) = -1, \\ A_1 + A_3 = 1. \end{cases}$$

This system has the following solution

$$A_1 = \frac{1}{2\,(\cos\varphi)^2}\,, \qquad A_2 = -\frac{1}{2\,(\cos\varphi)^2}\,, \qquad A_3 = \frac{\cos(2\varphi)}{2\,(\cos\varphi)^2}\,.$$

These coefficients make it possible to reduce the expansion (53) into the following form

$$\frac{4\,(\cos\varphi)^2\,(1-\rho)}{(1+\rho)\left(1-2\rho\cos(2\varphi)+\rho^2\right)} = \frac{2}{1+\rho} - \frac{2\rho - 2\cos(2\varphi)}{1-2\rho\cos(2\varphi)+\rho^2}\,.$$

The integration of the above equality with respect to the variable ρ from 0 to 1 leads to the formula

$$\int_0^1 \frac{4(\cos\varphi)^2(1-\rho)}{(1+\rho)\left(1-2\rho\cos(2\varphi)+\rho^2\right)}\,d\rho = \ln\left\{\frac{(1+\rho)^2}{1-2\rho\cos(2\varphi)+\rho^2}\right\}\Bigg|_0^1.$$

After simplifications on the right-hand side of the last identity, we arrive at the parameterization (52). It should be noted that the subintegral function in the integral (52) contains the Poisson kernel

$$\frac{1-\rho^2}{2\left(1-2\rho\cos(2\varphi)+\rho^2\right)} = \frac{1}{2} + \sum_{k=1}^{+\infty} \rho^k \cos(2k\varphi)\,.$$

Then, instead of the equality (52), we can write down the following one

$$\frac{\ln|\sin\varphi|}{(\cos\varphi)^2} = -\,4\int_0^1 \frac{d\rho}{(1+\rho)^2}\left\{\frac{1}{2} + \sum_{k=1}^{+\infty}\rho^k\cos(2k\varphi)\right\}.$$

On the basis of this integral representation, we obtain

$$\int_0^{\pi/2}\frac{\ln(\sin\varphi)}{(\cos\varphi)^2}\,d\varphi = -\pi\int_0^1\frac{d\rho}{(1+\rho)^2}\,.$$

After calculating the integral on the right-hand side of the last identity, we obtain

$$\int_0^{\pi/2}\frac{\ln(\sin\varphi)}{(\cos\varphi)^2}d\varphi = -\,\frac{\pi}{2}\,.$$

The found above integral transforms the equality (40) into the formula (31). Such a circumstance indicates that the generalized function (34) is the solution to both integral equations (31) and (32).

The final step in finding the extremal function will be to substitute the generalized function (34) into the right-hand side of the identity (18). Then we can write that

$$f(t) - f(-t) = 4\int_0^{+\infty}\delta(q)\,\frac{\sin(qt)}{q}\,dq. \tag{54}$$

In order to evaluate the integral in the formula (54), it is more convenient to rewrite the integral representation (36) for the Dirac delta function in the form of a limit

$$\delta(q) = \frac{1}{\pi}\lim_{L\to+\infty}\left(\int_0^L\cos(qz)\,dz\right).$$

After calculating the above integral, we obtain the representation

$$\delta(q) = \lim_{L\to+\infty}\frac{\sin(qL)}{\pi q},$$

that must then be substituted into the right-hand side of the equality (54). As a result, we obtain

$$f(t) - f(-t) = \frac{4}{\pi}\lim_{L\to+\infty}\left(\int_0^{+\infty}\frac{\sin(qL)\sin(qt)}{q^2}\,dq\right). \tag{55}$$

The integral in the formula (55) is completely equivalent to the integral (28). Therefore, instead of the identity (55), we will have the following one

$$f(t) - f(-t) = 2t. \tag{56}$$

The functional equation (56) shows that the extremal function $f(t)$ can be expressed as a linear function $f(t) = kt$. Substituting it into the left-hand side of the identity (56), we find the angular coefficient $k = 1$. Then it turns out that $f(t) = t$.

The extremal function $f(t) = t$ satisfies the condition (3) for the functions of the Hölder class H^1 and transforms the inequality (14) into the equality. Such a circumstance indicates the validity of the result (4). The theorem is proved. □

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На даний час мало дослідженими є апроксимаційні властивості гармонійних інтегралів Пуассона для верхньої півплощини, які є розв'язками рівняння Лапласа в декартовій системі координат. У даній роботі вивчаються апроксимаційні властивості спряжених гармонійних інтегралів Пуассона для верхньої півплощини на класах функцій Гельдера. Знайдено точну рівність для верхньої межі відхилення спряженого гармонійного інтеграла Пуассона від його граничного значення у рівномірній метриці.

Ключові слова і фрази: рівняння Лапласа, спряжений гармонійний інтеграл Пуассона для верхньої півплощини, клас функцій Гельдера, аналітична функція.