



The problem of convergence of branched continued fractions

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Using the tails of the approximants of continued fractions and their multidimensional generalizations, branched continued fractions, canonical numerators and denominators have been defined, and the conditions for the degeneracy of approximants have been investigated. Certain types of figured approximants, specifically A -, B -, and C -figured approximants and their applications, are considered. Various types of convergence for continued and branched continued fractions have been classified. By analogy with Leighton's example, a C -figured-convergent two-dimensional branched continued fraction with independent variables has been constructed; in its structure, every fourth element in each constituent continued fraction is an arbitrary complex number different from 0 and $-1/2$. Methods for investigating the convergence of branched continued fractions are analyzed.

Key words and phrases: branched continued fraction, convergence, approximant, figured convergence.

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Introduction

In the theory of continued fractions, two scientific directions have been formed: the number-theoretic and the analytic theories. The term "convergence" is interpreted differently in these fields. In the analytic theory, it refers to the existence of a limit of the approximants, whereas in the number-theoretic direction, a "convergent" is an approximant of a continued fraction. The objects of study in the latter case are regular continued fractions, which always converge in the classical sense. The primary problems in the analytic theory of continued fractions and their multidimensional generalizations — branched continued fractions — are: the construction of expansions of functions into continued fractions [30, 50]; the investigation of the convergence of continued fractions [43, 45, 48, 56]; the investigation of the convergence of numerical and various types of functional branched continued fractions with N branching branches [13, 54], with independent variables [6, 38], two-dimensional corresponding continued fractions [41, 46]; as well as the construction and investigation branched continued fraction expansions of the ratios of hypergeometric functions of several variables [2, 4, 32, 34, 35, 39].

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In this paper, we define canonical numerators and denominators using the tails of the approximants of continued fractions and branched continued fractions, investigate the conditions for the degeneracy of these approximants, and classify various types of convergence. Furthermore, we consider specific figured approximants, namely $\mathcal{A}$ -, $\mathcal{B}$ -, and $\mathcal{C}$ -figured approximants, and their applications, analyze multiple methods for investigating the convergence of branched continued fractions, and construct a $\mathcal{C}$ -figured-convergent two-dimensional branched continued fraction with independent variables by analogy with Leighton's example.

1 Convergence of continued fractions

For a compact notation of an infinite continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \dots + \cfrac{a_k}{b_k + \dots}}}}, \quad (1)$$

where $a_k, b_k \in \mathbb{C}$, $k = 1, 2, \dots$, we will use the notation

$$\prod_{k=1}^{+\infty} \frac{a_k}{b_k} \quad \text{or} \quad \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_k}{b_k} + \dots$$

The definition of a continued fraction is given by the composition of linear fractional transformations $\frac{a_k}{b_k + w}$, $k = 1, 2, \dots$ (see [45]). Therefore, it is required that all $a_k \neq 0$, $k = 1, 2, \dots$. This is sufficient to ensure that during the evaluation of each n th approximant

$$f_n = \prod_{k=1}^n \frac{a_k}{b_k}, \quad n = 1, 2, \dots, \quad (2)$$

there is no the indeterminate forms $0/0$, i.e. that f_n is well-defined.

The p th tail of the n th approximant (2) is the continued fraction

$$Q_p^{(n)} = b_p + \prod_{k=p+1}^n \frac{a_k}{b_k}, \quad p = 1, 2, \dots, n-1, \quad n \geq 2; \quad Q_n^{(n)} = b_n, \quad n = 1, 2, \dots \quad (3)$$

The canonical numerator $A_p^{(n)}$ and the canonical denominator $B_p^{(n)}$ of the tail (3) in the case $n \geq 2$ are defined by the recurrence relations

$$A_p^{(n)} = b_p A_{p+1}^{(n)} + a_{p+1} B_{p+1}^{(n)}, \quad B_p^{(n)} = A_{p+1}^{(n)}, \quad p = n-1, n-2, \dots, 1, \quad (4)$$

with initial conditions

$$A_n^{(n)} = b_n, \quad B_n^{(n)} = 1, \quad n = 1, 2, \dots \quad (5)$$

The tail $Q_p^{(n)} = A_p^{(n)} / B_p^{(n)}$ is not well-defined if $A_p^{(n)} = B_p^{(n)} = 0$.

The canonical numerator A_n and the canonical denominator B_n of the n th approximant (2) are respectively equal to

$$A_n = a_1 B_1^{(n)}, \quad B_n = A_1^{(n)}, \quad n = 1, 2, \dots \quad (6)$$

In the study of the convergence of continued fractions, the conditions $a_k \neq 0$, $k = 1, 2, \dots$, in the definition of these fractions are often removed. For example, see the following assertion.

Theorem 1 ([56, p. 26]). Let the continued fraction (1) satisfy the following conditions for some integer $m \geq 1$:

$$a_m = 0 \text{ while, if } m > 1, a_n \neq 0 \text{ for } n = 1, 2, 3, \dots, m-1.$$

Then the continued fraction converges if and only if there exists an index k such that the denominators $B_{n-1} \neq 0$ for $n \geq k$. When convergent, the value of the continued fraction is A_{m-1}/B_{m-1} .

It follows from the relations (4) that $f_n = A_n/B_n$ is well-defined if all $a_k \neq 0$, $k = 1, 2, \dots, n$. Indeed, assuming the opposite, i.e. $A_{(n)} = B_{(n)} = 0$, it follows from (4) and (6) that $A_p^{(n)} = B_p^{(n)} = 0$, $p = 1, 2, \dots, n$, which contradicts condition (5).

In the analytic theory of continued fractions, various approaches to the concept of convergence have been formulated.

Classical convergence (see [56]): a continued fraction (1) converges if there exists a finite limit $\lim_{n \rightarrow \infty} (A_n/B_n)$, and the number of canonical denominators B_k that are equal to zero is finite, or all $B_k \neq 0$, $k = 1, 2, \dots$. In published later monographs [45, 48, 49], by *convergence* of the continued fraction (1) we mean the existence of the limit $\lim_{n \rightarrow \infty} f_n$ (not necessarily finite). O. Perron [51] called such convergence “convergence in the broad sense”. To distinguish these concepts, we consider classical convergence to be convergence to a finite value.

A continued fraction (1) is called *unconditionally convergent* if the fraction itself and all its infinite tails

$$b_p + \prod_{k=p+1}^{+\infty} \frac{a_k}{b_k}, \quad p = 1, 2, \dots,$$

converge to a finite value [52]. If the fraction (1) converges to a finite value, but there exists a tail whose limit of approximants is ∞ , then (1) is called *conditionally convergent*.

A continued fraction is called *absolutely convergent* if the series $\sum_{n=1}^{\infty} |f_{n+1} - f_n|$ converges (see [56]).

Generalized convergence is also considered [44, 48]. Continued fraction is said to *converge generally* to an $f \in \hat{\mathbb{C}}$ if there exist two sequences $\{v_n\}$ and $\{w_n\}$ in the extended complex plane $\hat{\mathbb{C}}$ such that $\liminf_{n \rightarrow \infty} d(v_n, w_n) > 0$ and $\lim_{n \rightarrow \infty} S_n(v_n) = \lim_{n \rightarrow \infty} S_n(w_n) = f$, where

$$S_n(w) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n + w}, \quad d(w, z) = \frac{|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}, \quad d(w, \infty) = \frac{1}{\sqrt{1 + |w|^2}}.$$

If a continued fraction converges classically, it also converges generally with $v_n = 0$, $w_n = \infty$, $n = 1, 2, \dots$. In [44], an example of a 3-periodic continued fraction is constructed, which is divergent but converges generally for an arbitrary choice of w_n ; there exists $\lim_{n \rightarrow \infty} S_n(w_n) = f$, $w_n \neq 0$, $n = 1, 2, \dots$, where $\{w_n\}$ is bounded. This approach is used to accelerate convergence.

The problem of convergence of continued fractions is to establish conditions on the elements a_k, b_k , $k = 1, 2, \dots$, such that the continued fraction (1) converges [56]. The convergence problem for continued fractions is more complex than the analogous problem for series. An interesting example is provided by W. Leighton (see [47]), who constructed a convergent continued fraction of the form

$$\prod_{k=1}^{+\infty} \frac{a_k}{1}, \quad (7)$$

where the elements $a_{4k}, k = 1, 2, \dots$, are arbitrary complex numbers different from 0 and $-1/2$. Therefore, it is arguably difficult to formulate very general convergence criteria, as a fraction may converge even though infinitely many of its elements are arbitrary complex numbers.

For this reason, convergence criteria are most often formulated in terms of convergence sets. Here, it is convenient to consider continued fractions of the form

$$\prod_{k=1}^{+\infty} \frac{a_k}{1} \quad \text{or} \quad \prod_{k=1}^{+\infty} \frac{1}{b_k}.$$

In particular, a set $G \subset \mathbb{C}$ is called a *convergence set* of the continued fraction (7) if the condition $a_k \in G, k = 1, 2, \dots$, guarantees the convergence of this fraction. If the domain G is unbounded, an additional condition is added (divergence of an infinite series composed of the elements of the fraction). Such sets will be called *conditional convergence sets* [45]. Circular, elliptic, angular, and parabolic convergence sets are being investigated.

2 Approximants of branched continued fractions

The object of our research will be branched continued fractions (BCF) of the special form, i.e. BCFs with independent variables at fixed values of the variables [1, 6, 11]

$$\prod_{k=1}^{+\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}} = \sum_{i_1=1}^N \frac{a_{i(1)}}{b_{i(1)} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)}}{b_{i(2)} + \dots + \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)} + \dots}}}, \quad (8)$$

where $a_{i(k)} = a_{i_1, i_2, \dots, i_k}, b_{i(k)} = b_{i_1, i_2, \dots, i_k} \in \mathbb{C}, i(k) \in \mathcal{J}_k, k = 1, 2, \dots$,

$$\mathcal{J}_k = \{i(k) = (i_1, i_2, \dots, i_k) : 1 \leq i_k \leq i_{k-1} \leq \dots \leq i_0\}, \quad k \geq 1,$$

$i_0 = N$ is a fixed positive integer, also $\mathcal{J} = \bigcup_{k=1}^{\infty} \mathcal{J}_k$.

To denote the BCF (8), the following notation will also be used

$$\sum_{i_1=1}^N \frac{a_{i(1)}}{b_{i(1)}} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)}}{b_{i(2)}} + \dots + \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}} + \dots$$

Finite BCFs of the form

$$f_n = \prod_{k=1}^n \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}}, \quad n = 1, 2, \dots, \quad (9)$$

are called the n th approximants of the BCF (8).

2.1 Conditions for the existence of approximants

BCFs (8) are defined by means of a composition of linear fractional and multidimensional linear fractional transformations. Naturally, as in the one-dimensional case, it is required that all $a_{i(k)} \neq 0, i(k) \in \mathcal{J}_k, k = 1, 2, \dots$. However, unlike continued fractions, this condition does not guarantee that no indeterminate form $0/0$ will arise during the evaluation of the approximants, i.e. that the corresponding approximants will be well-defined.

A BCF of the form

$$Q_{i(p)}^{(n)} = b_{i(p)} + \sum_{i_{p+1}=1}^{i_p} \frac{a_{i(p+1)}}{b_{i(p+1)} + \dots + \sum_{i=1}^{i_{n-1}} \frac{a_{i(n)}}{b_{i(n)}}} = b_{i(p)} + \prod_{k=p+1}^n \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}}, \quad (10)$$

where $i(p) \in \mathcal{J}_p$, $p = \overline{1, n-1}$, $n \geq 2$, is called the $i(p)$ th tail of the n th approximant (9), where $Q_{i(n)}^{(n)} = b_{i(n)}$, $i(n) \in \mathcal{J}_n$, $n = 1, 2, \dots$.

Formally, we have the following recurrence relations between the tails

$$Q_{i(p)}^{(n)} = b_{i(p)} + \sum_{i_{p+1}=1}^{i_p} \frac{a_{i(p+1)}}{Q_{i(p+1)}^{(n)}}, \quad i(p) \in \mathcal{J}_p, \quad p = \overline{1, n-1}, \quad n \geq 2.$$

By analogy with continued fractions, we define the canonical numerator $A_{i(p)}^{(n)}$ and the canonical denominator $B_{i(p)}^{(n)}$ of the tail (10) of the n th approximant (9) as follows

$$A_{i(p)}^{(n)} = b_{i(p)} \prod_{i_{p+1}=1}^{i_p} A_{i(p+1)}^{(n)} + \sum_{i_{p+1}=1}^{i_p} a_{i(p+1)} B_{i(p+1)}^{(n)} \prod_{j=1, j \neq i_{p+1}}^{i_p} A_{i(p),j}^{(n)}, \quad B_{i(p)}^{(n)} = \prod_{i_{p+1}=1}^{i_p} A_{i(p+1)}^{(n)}, \quad (11)$$

where $i(p) \in \mathcal{J}_p$, $p = n-1, n-2, \dots, 1$, $n \geq 2$, under the initial conditions

$$A_{i(n)}^{(n)} = b_{i(n)}, \quad B_{i(n)}^{(n)} = 1, \quad i(n) \in \mathcal{J}_n, \quad n = 1, 2, \dots,$$

moreover, if (11) $i_p = 1$, then $\prod_{j=1, j \neq i_{p+1}}^{i_p} A_{i(p),j}^{(n)} = 1$.

The BCF (10) is not well-defined if $A_{i(p)}^{(n)} = B_{i(p)}^{(n)} = 0$.

The canonical numerator A_n and the canonical denominator B_n of the n th approximant (10) $f_n = A_n/B_n$ are defined by the formulas

$$A_n = \sum_{i_1=1}^N a_{i_1} B_{i_1}^{(n)} \prod_{j=1, j \neq i_1}^N A_j^{(n)}, \quad B_n = \prod_{i_1=1}^N A_{i_1}^{(n)}, \quad n = 1, 2, \dots \quad (12)$$

If $A_n = B_n = 0$, then the n th approximant (9) is not well-defined.

From the relations (12) the next theorem follows.

Theorem 2. *The n th approximant (9) of the BCF (8), in which $N > 1$ and $a_{i_1} \neq 0$, $i_1 = \overline{1, N}$, is not well-defined if and only if there exists an index i_1 , $1 \leq i_1 \leq N$, such that the tail $Q_{i_1}^{(n)}$ is not well-defined, or there exist indices r and s , such that $r, s \in \{1, 2, \dots, N\}$, $r \neq s$, and $A_r^{(n)} = A_s^{(n)} = 0$.*

Proof. Let $A_n = B_n = 0$. It follows from the second equality of (12) that there exists an index i_1 , $1 \leq i_1 \leq N$, such that $A_{i_1}^{(n)} = 0$. Then the first equality from (12) can be written in the form

$$A_n = a_{i_1} B_{i_1}^{(n)} \prod_{j=1, j \neq i_1}^N A_j^{(n)} = 0.$$

Hence, either $B_{i_1}^{(n)} = 0$, or there exists $j \neq i_1$ such that $A_j^{(n)} = 0$.

Conversely, if there exists an index i_1 , such that $A_{i_1}^{(n)} = B_{i_1}^{(n)} = 0$, or there exist indices r and s such that $r, s \in \{1, 2, \dots, N\}$, $r \neq s$, for which $B_{i_1}^{(n)} \neq 0$ and $A_r^{(n)} = A_s^{(n)} = 0$, then $A_n = B_n = 0$. $\square$

Corollary 1. *If there exists a tail $Q_{i(p)}^{(n)}$ of the n th approximant (9) which is not well-defined, then the n th approximant (9) is not well-defined.*

Proof. If there exists a tail $Q_{i(p+1)}^{(n)}$, which is not well-defined, then it follows from the formulas (11) that $A_{i(p)}^{(n)} = B_{i(p)}^{(n)} = 0$, and so on. Consequently, $Q_{i_1}^{(n)}$ is not well-defined, and based on Theorem 2, we conclude that $A_n = B_n = 0$. $\square$

Corollary 2. *The following statements hold:*

- a) *the tail $Q_{i(n-1)}^{(n)}$ of the n th approximant (9), in which all $a_{i(n)} \neq 0$, $i(n) \in \mathcal{J}_n$ and $i_n > 1$ is not well-defined if and only if there exist indices r and s such that $r, s \in \{1, 2, \dots, N\}$, $r \neq s$, for which $b_{i(n-1),r} = b_{i(n-1),s} = 0$;*
- b) *the tail $Q_{i(p)}^{(n)}$, $i(p) \in \mathcal{J}_p$, $1 \leq p \leq n-2$, $n > 2$, is not well-defined if and only if there exists a tail $Q_{i(p+1)}^{(n)}$, which is not well-defined, or in the case when $i_{p+1} > 1$, there exist indices r and s such that $r, s \in \{1, 2, \dots, N\}$, $r \neq s$, and $A_{i(p),r}^{(n)} = B_{i(p),s}^{(n)} = 0$.*

The proof of Corollary 2 is conducted similarly to the proof of Theorem 2.

3 Figured approximants

In solving certain problems, other constructions of approximants arise that differ from (9). Such approximants are called “figured” approximants. To distinguish between them, we will use the terminology: $\mathcal{A}$ -, $\mathcal{B}$ -, $\mathcal{C}$ -, ... figured approximants.

The continued fraction

$$\frac{a_{i(1)}}{b_{i(1)}} + \frac{a_{i(2)}}{b_{i(2)}} + \dots + \frac{a_{i(n)}}{b_{i(n)}} + \dots, \quad (13)$$

where $i(n) \in \mathcal{J}$, $n = 1, 2, \dots$, is a fixed set of multi-indexes, is called a $(i_1, i_2, \dots, i_n, \dots)$ branch of the BCF (8). If (13) is a finite continued fraction, then the number of its levels is called the length of this branch.

We define a figured approximant of the BCF (8) as a branched continued fraction that is a part of (8) and contains at least two branches of different lengths or at least one branch of infinite length. Moreover, as the index of the figured approximant increases, all elements of the BCF are exhausted; that is, the minimum branch length of the figured approximant $\tilde{f}_n$ tends to ∞ as $n \rightarrow +\infty$.

3.1 $\mathcal{A}$ -figured approximants

In an $\mathcal{A}$ -figured approximant, each subsequent approximant is formed by adding the next partial quotient in the natural representation of the BCF:

$$\begin{aligned} \tilde{f}_1 &= \frac{a_1}{b_1}, \quad \tilde{f}_2 = \frac{a_1}{b_1} + \frac{a_2}{b_2}, \quad \dots, \quad \tilde{f}_N = \sum_{i=1}^N \frac{a_i}{b_i}, \\ \tilde{f}_{N+1} &= \frac{a_1}{b_1 + \frac{a_{1,1}}{b_{1,1}}} + \sum_{i=2}^N \frac{a_i}{b_i}, \quad \dots, \quad \tilde{f}_{2N} = \sum_{i=1}^N \frac{a_i}{b_i + \frac{a_{i,1}}{b_{i,1}}}, \dots \end{aligned}$$

Let us introduce an order relation $\prec$ on the set $\mathcal{J} = \bigcup_{k=1}^{\infty} \mathcal{J}_k$ of all multiindices of the elements of the BCF (8). Let $j(q), j(p) \in \mathcal{J}$, where $j(s) = (j_1, j_2, \dots, j_s)$, $s = 1, 2, \dots$. Then

- 1) $i(p) \prec j(q)$, if $p < q$,
- 2) $i(p) \prec j(p)$, if $i_1 < j_1$,
- 3) $i(p) \prec j(p)$, if there exists l , $1 \leq l < p$, such that $i_k = j_k$, $k = \overline{1, l}$, $i_{k+1} < j_{k+1}$.

In [6], it is proved that the n th $\mathcal{A}$ -figured approximant $\tilde{f}_n$ of the BCF (8) has the form

$$\tilde{f}_n = \sum_{i_1=1}^N \frac{a_{i(1)}}{b_{i(1)}} + \dots + \sum_{i_{s-1}=1}^{i_{s-2}} \frac{a_{i(s-1)}}{b_{i(s-1)}} + \sum_{i_s=1}^{i_{s-1}} \delta\left(\frac{a_{i(s)}}{b_{i(s)}}\right), \quad (15)$$

where

$$\delta\left(\frac{a_{i(s)}}{b_{i(s)}}\right) = \begin{cases} a_{i(s)}/b_{i(s)}, & \text{if } i(s) \preceq r(s), \\ 0/1, & \text{if } r(s) \prec i(s), \end{cases}$$

the number s and the indices $r_1, r_2, \dots, r_s$ of the multi-index $r(s) \in \mathcal{J}_s$ are uniquely determined by the following decomposition of the number n :

$$n = \binom{N+s-1}{s-1} + \sum_{q=1}^s \binom{r_q+s-q-1}{s+1-q}. \quad (16)$$

Example 1. Let $n = 100$, $N = 5$. Let us find the indices $r_1, r_2, \dots, r_s$ of the multi-index $r(s)$.

The maximum possible value of $\binom{5+s-1}{s-1}$ is reached at $s = 4$. Then $\binom{8}{3} = 56$. We successively choose the maximum possible values for the indices r_1, r_2, r_3, r_4 . The maximum possible value of $\binom{r_1+2}{4}$ for $s = 4$ is reached at $r_1 = 5$, where $\binom{7}{4} = 35$. Similarly, the maximum possible values for $\binom{r_2+1}{3}$ with $s = 4, r_1 = 5$ are $r_2 = 3$ and $\binom{4}{3} = 4$. In addition, $\max \binom{r_3}{2} = \binom{3}{2} = 3$, $r_3 = 3$; $\max \binom{r_4-1}{1} = \binom{2}{1} = 2$, $r_4 = 3$.

Thus, $s = 4$, $r_1 = 5$, $r_2 = r_3 = r_4 = 3$. The decomposition (16) takes the following form: $100 = 56 + 35 + 4 + 3 + 2$. Here, the BCF (15) with $n = 100$, $N = 5$, $\tilde{f}_{100}$ is the fourth approximant in which all partial quotients $\frac{a_{i(4)}}{b_{i(4)}}$ that follow $\frac{a_{5,3,3,3}}{b_{5,3,3,3}}$ are replaced by $0/1$.

Along with the BCF (8), let us consider the BCF

$$\prod_{k=1}^{+\infty} \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}^*}{b_{i(k)}^*}, \quad (17)$$

where $a_{i(k)}^*, b_{i(k)}^* \in \mathbb{C}$, $i(k) \in \mathcal{J}_k$, $k = 1, 2, \dots$. Let $\tilde{f}_n^*$ denote its n th $\mathcal{A}$ -figured approximant.

We call the BCFs (8) and (17) equivalent if

$$\tilde{f}_n = \tilde{f}_n^*, \quad n = 1, 2, \dots$$

From the above condition, it follows in particular that

$$f_n = f_n^*, \quad n = 1, 2, \dots \quad (18)$$

Theorem 3 ([6]). The BCFs (8) and (17), in which all $a_{i(k)} \neq 0$, $a_{i(k)}^* \neq 0$, $i(k) \in \mathcal{J}_k$, $k = 1, 2, \dots$, are equivalent if and only if there exist non-zero constants $\rho_{i(k)}$, $i(k) \in \mathcal{J}_k$, $k = 1, 2, \dots$, such that

$$a_{i(k)}^* = \rho_{i(k)} \rho_{i(k-1)} a_{i(k)}, \quad b_{i(k)}^* = \rho_{i(k)} b_{i(k)}, \quad i(k) \in \mathcal{J}_k, \quad k = 1, 2, \dots, \quad (19)$$

where $\rho_{i(0)} = 1$.

If the definition of equivalence for BCFs requires only the equality of the approximants (18), then conditions (19) are sufficient for the equivalence of the BCFs (8) and (17), but they are not necessary.

For example, the BCFs (8) and (17), in which $N = 2$, $a \in \mathbb{C} \setminus \{0\}$ and

$$\begin{aligned} a_{2[k]} &= \frac{2}{3}a, \quad a_{2[k],1} = \frac{1}{3}a, \quad a_{2[k],1[p]} = a, \quad k = 1, 2, \dots, \quad p = 2, 3, \dots, \\ a_{2[k]}^* &= \frac{3}{4}a, \quad a_{2[k],1}^* = \frac{1}{4}a, \quad a_{2[k],1[p]}^* = a, \quad k = 1, 2, \dots, \quad p = 2, 3, \dots, \end{aligned}$$

where $2[k] = \underbrace{2, 2, \dots, 2}_k$, $1[p] = \underbrace{1, 1, \dots, 1}_p$ have equal n th approximants f_n and f_n^* , because they are equal to the n th approximant of the periodic continued fraction

$$\frac{a}{1} + \frac{a}{1} + \frac{a}{1} + \dots$$

However, there do not exist numbers $\rho_{i(k)}, i(k) \in \mathcal{J}_k, k = 1, 2, \dots$, such that conditions (19) are satisfied; that is, these conditions are not necessary.

3.2 $\mathcal{B}$ -figured approximants

We define the n th $\mathcal{B}$ -figured approximant $\tilde{f}_n$ of the BCF (8) as a finite BCF that is a part of (8) and contains all those elements $a_{i(k)}, b_{i(k)}, i(k) \in \mathcal{J}_k, k = \overline{1, n}$, for which the sum of the indices in the multi-index does not exceed n , $i_1 + i_2 + \dots + i_k \leq n$, i.e.

$$\tilde{f}_n = \sum_{i_1=1}^N \delta_n \left(\frac{a_{i(1)}}{b_{i(1)}} \right) + \dots + \sum_{i_{n-1}=1}^{i_{n-1}} \delta_n \left(\frac{a_{i(n)}}{b_{i(n)}} \right), \quad (20)$$

where

$$\delta_n \left(\frac{a_{i(k)}}{b_{i(k)}} \right) = \begin{cases} a_{i(k)}/b_{i(k)}, & \text{if } i_1 + i_2 + \dots + i_k \leq n, \\ 0/1, & \text{if } i_1 + i_2 + \dots + i_k > n, \end{cases}$$

$i(k) \in \mathcal{J}_k, k = \overline{1, n}, n \geq 1$ (see [10]).

Figured approximants similar to (20) arise in the problem of correspondence between double power series and BCFs of a special structure (two-dimensional corresponding continued fractions) [31, 41, 46]:

$$\Phi_0(z_1, z_2) + \prod_{i=1}^{+\infty} \frac{a_{i,i} z_1 z_2}{\Phi_i(z_1, z_2)}, \quad (21)$$

where

$$\Phi_i(z_1, z_2) = 1 + \prod_{j=1}^{+\infty} \frac{a_{i+j,i} z_1}{1} + \prod_{j=1}^{+\infty} \frac{a_{i,i+j} z_2}{1},$$

$a_{i,j} \in \mathbb{C}, i, j = 1, 2, \dots$.

Let $\tilde{f}_n$ be the n th $\mathcal{B}$ -figured approximant of the BCF (21), which is a part of (21) and contains only those elements for which the sum of the indices in their coefficients does not exceed n .

Let a formal double power series $\sum_{i,j=0}^{+\infty} c_{i,j} z_1^i z_2^j$ be given, where $c_{0,0} = 1$. The fraction (21) is said to be corresponding to this series if the expansions of each n th $\mathcal{B}$ -figured approximant $\tilde{f}_n$ into a formal double power series coincide with the given series for all homogeneous polynomials up to the n th order inclusive.

Such conditions allow us to determine the coefficients $a_{i,j}$ of the BCF (21). With this specific choice of approximants, the number of unknown coefficients $a_{i,j}$ in $\tilde{f}_n$ coincides with the number of terms in the corresponding partial sum of the given double power series.

This problem concerning the choice of the figured approximants does not arise for BCFs with independent variables. In this case, the n th approximant satisfies the conditions mentioned above. This has allowed us to construct corresponding and associated branched continued fractions with independent variables for multiple formal power series [21, 33, 36, 40].

$\mathcal{B}$ -figured approximants also arise when representing the ratio of two linearly independent solutions of a third-order linear recurrence relation

$$y_n = c_n y_{n-1} + b_n y_{n-2} + a_n y_{n-3}, \quad n = 2, 3, \dots, \quad (22)$$

with the initial conditions $y_{-1} = 1, y_0 = 0, y_1 = b_1$ and $y_{-1} = 0, y_0 = 1, y_1 = c_1$, respectively, in the form of a BCF [10].

3.3 $\mathcal{C}$ -figured approximants

A BCF (8), where $i_0 = N$, is called N -dimensional. Let all the continued fractions within the structure of (8), i.e.

$$b_0 + \prod_{k=1}^{+\infty} \frac{a_{1[k]}}{b_{1[k]}}, \quad b_{i(s)} + \prod_{k=1}^{+\infty} \frac{a_{i(s),1[k]}}{b_{i(s),1[k]}},$$

converge, and let $b_0^{(1)}, b_{i(s)}^{(1)}$ be their respective values, where $i(s) \in \mathcal{J}^{(2)}$,

$$\mathcal{J}^{(2)} = \{i(n) = (i_1, i_2, \dots, i_n), 2 \leq i_n \leq i_{n-1} \leq \dots \leq i_0 = N, n \geq 1\}.$$

Then the finite $(N-1)$ -dimensional BCF, which is a part of the BCF (8) and contains some of its infinite branches, is of the form

$$\tilde{f}_n = b_0^{(1)} + \prod_{k=1}^n \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}^{(1)}}, \quad n = 1, 2, \dots, i_0 = N-1, \quad (23)$$

and is called the n th $\mathcal{C}_1$ -figured approximant of the BCF (8) [18].

Let all $(N-1)$ -dimensional BCFs (8) that are part of the BCF (8), i.e.

$$b_0 + \prod_{\ell=1}^{+\infty} \sum_{i_\ell=1}^{i_{\ell-1}} \frac{a_{i(\ell)}}{b_{i(\ell)}}, \quad b_{N[s]} + \prod_{\ell=1}^{+\infty} \sum_{i_\ell=1}^{i_{\ell-1}} \frac{a_{N[s],i(\ell)}}{b_{N[s],i(\ell)}},$$

converge, and let $b_0^{(N-1)}, b_{N[s]}^{(N-1)}$ be their respective values, where $i_0 = N-1, s = \overline{1, n}$. Then the finite continued fraction

$$\tilde{f}_n = b_0^{(N-1)} + \prod_{s=1}^n \frac{a_{N[s]}}{b_{N[s]}^{(N-1)}}, \quad n = 1, 2, \dots, \quad (24)$$

is called the $\mathcal{C}_2$ -figured approximant of the BCF (8) [18].

$\mathcal{C}_1$ - and $\mathcal{C}_2$ -figured approximants are used in the study of the convergence of BCF (8) and in estimating the approximation errors of its approximants [18]. In this process, well-known convergence criteria for continued fractions and error estimates for their approximation by approximants are applied.

It is impossible to define $\mathcal{C}_1$ - and $\mathcal{C}_2$ -figured approximants (similar to (23) and (24)) for the following BCFs with N branches of branching

$$b_0 + \prod_{k=1}^{+\infty} \sum_{i_k=1}^N \frac{a_{i(k)}}{b_{i(k)}}, \quad (25)$$

where $a_{i(k)}, b_{i(k)} \in \mathbb{C}, i_p = \overline{1, N}, p = \overline{1, k}, k = 1, 2, \dots$.

4 Convergence of branched continued fractions

The BCF (8) *converges to a finite value* if there exists a finite limit of its approximants (9), i.e.

$$\lim_{n \rightarrow \infty} f_n = f, \quad f \in \mathbb{C}, \quad f_n \in \mathbb{C}, \quad n = 1, 2, \dots,$$

provided that either only a finite number of f_n are undefined, or a finite number of canonical denominators B_n are equal to zero.

The BCF (8) *converges* if there exists a limit

$$\lim_{n \rightarrow \infty} f_n = f, \quad f \in \bar{\mathbb{C}},$$

of its approximants (9), where $\bar{\mathbb{C}}$ is the extended complex plane, $f_n \in \bar{\mathbb{C}}, n = 1, 2, \dots$, and provided that only a finite number of f_n are undefined.

The BCF (8) *converges absolutely* if the following series

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n|$$

converges.

The BCF (8) *converges unconditionally* if the BCF (8) converges and all of its infinite tails converge to finite values.

In a similar manner, one can define figured convergence, or specifically $\mathcal{A}$ -, $\mathcal{B}$ -, and $\mathcal{C}$ -figured convergence, by replacing f_n in the previous definitions with the corresponding $\tilde{f}_n$. It is only required that the minimum length of the branches of the figured approximant tends to infinity as $n \rightarrow \infty$ in order to completely exhaust the entire BCF. By analogy with the corresponding results for the BCF (25) (see [13]), the following results can be established.

Theorem 4. *If a BCF (8) with positive elements converges, then it converges unconditionally.*

Theorem 5. *If a BCF (8), in which all $a_{i(k)}, b_{i(k)}, i(k) \in \mathcal{J}_k, k = 1, 2, \dots$, are positive real numbers, converges, then the BCF (8) converges figuredly for any choice of figured approximants.*

In the analytic theory of continued fractions, linear recurrence relations for the canonical numerators and denominators of the approximants are essentially used when investigating convergence [45]

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \quad B_n = b_n B_{n-1} + a_n B_{n-2}, \quad n = 1, 2, \dots, \quad (26)$$

with the initial conditions $A_{-1} = 1, A_0 = 0, B_{-1} = 0, B_0 = 1$. In the first multidimensional generalizations of continued fractions, higher-order linear difference equations were considered instead of (26) (see [14, 25, 26]). Such generalizations were primarily used in the number-theoretic direction. It was shown in [10] that the ratio of two linearly independent solutions of the recurrence relation (22) can be expanded into a BCF with two branching lines with specific coefficients. However, an arbitrary BCF with two branching lines cannot be reduced to this form. Consequently, there are no difference equations of a higher order than (26) for the canonical numerators and canonical denominators of an arbitrary BCF.

As a result, the first problem arose: how to investigate the convergence of BCFs.

In the early publications dedicated to BCFs, formulas for canonical numerators and denominators with “thinning out” were used [29, 53, 55].

Let us order the last terminal nodes of the n th approximant $\frac{a_{i(n)}}{b_{i(n)}}$ of a BCF with N branches of branching (25) with complex elements, taking into account (14)

$$\frac{a_{1,1,\dots,1,1}}{b_{1,1,\dots,1,1}}, \frac{a_{1,1,\dots,1,2}}{b_{1,1,\dots,1,2}}, \dots, \frac{a_{1,1,\dots,1,N}}{b_{1,1,\dots,1,N}}, \frac{a_{1,1,\dots,2,1}}{b_{1,1,\dots,2,1}}, \dots, \frac{a_{N,N,\dots,N,N}}{b_{N,N,\dots,N,N}}. \quad (27)$$

Let r , $0 \leq r \leq N^n$, be an arbitrary integer, and let $1 \leq j_1 < j_2 < \dots < j_r \leq N^n$ be an arbitrary set of indices, if $r \geq 1$. In the n th approximant f_n of the BCF (25), let us replace the links that have the ordinal numbers $j_1, j_2, \dots, j_r$ in (27) with $\frac{0}{1}$ and evaluate the resulting figured approximant. We denote its canonical numerator and canonical denominator as

$$A_n \left(\begin{smallmatrix} 0 \\ j_1, j_2, \dots, j_r \end{smallmatrix} \right), B_n \left(\begin{smallmatrix} 0 \\ j_1, j_2, \dots, j_r \end{smallmatrix} \right),$$

respectively. For the canonical numerator and denominator of the $(n+1)$ th approximant of (25), the following recurrence relations

$$A_{n+1} = \sum_{j(r)} A_n \left(\begin{smallmatrix} 0 \\ j_1, j_2, \dots, j_r \end{smallmatrix} \right) \prod_{i(n)} c_{i(n)}, \quad B_{n+1} = \sum_{j(r)} B_n \left(\begin{smallmatrix} 0 \\ j_1, j_2, \dots, j_r \end{smallmatrix} \right) \prod_{i(n)} c_{i(n)}, \quad n = 1, 2, \dots, \quad (28)$$

hold, where the products are taken over all possible sets of indices $i_1, i_2, \dots, i_n$, and the summations are performed over all possible r , $0 \leq r \leq N^n$, and $j_1, j_2, \dots, j_r$ such that $1 \leq j_1 < j_2 < \dots < j_r \leq N^n$, if $r \geq 1$. Here $c_{i(n)} = a'_{i(n)}$ if $i(n)$ has one of the ordinal numbers $j_1, j_2, \dots, j_r$ in (27), and $c_{i(n)} = b'_{i(n)}$ otherwise, where

$$a'_{i(n)} = b_{i(n)} \prod_{i_{n+1}=1}^N b_{i(n+1)} + \sum_{i_{n+1}=1}^N a_{i(n+1)} \prod_{s=1, s \neq i_{n+1}}^N b_{i(n),s}, \quad b'_{i(n)} = \prod_{i_{n+1}=1}^N b_{i(n+1)}.$$

Using these formulas, V.Ya. Skorobogatko [53] established the convergence conditions for the BCF

$$b_0 + \bigvee_{k=1}^{+\infty} \sum_{i_k=1}^N \frac{1}{b_{i(k)}}. \quad (29)$$

Theorem 6. The BCF (29), where $b_{i(k)} > 0$, $i_k = \overline{1, N}$, $k = 1, 2, \dots$, converges if

$$\prod_{i(n)} b'_{i(n)} > 1 + \frac{B_{n+1}}{B_n}.$$

The formulas (28) are quite cumbersome for repeated use. As it is stated in [29, pp. 51–55], I.F. Kliuinyk and I.P. Pustomelnykov established formulas for calculating canonical numerators and denominators using spatial matrices. Although these formulas did not find wide application, their derivation marked the first time that abbreviated notation for representing BCFs with N branches of branching – similar to the right-hand side of (8) – was proposed.

In the works [22, 23], a formula for the difference between the approximants of the BCF (25) was established for the first time, using the tails of these fractions

$$f_n - f_m = (-1)^m \sum_{i_1, i_2, \dots, i_{m+1}=1}^N \frac{\prod_{r=1}^{m+1} a_{i(r)}}{\prod_{r=1}^m Q_{i(r)}^{(m)} \prod_{r=1}^{m+1} Q_{i(r)}^{(n)}} \quad (30)$$

under the assumption that $n > m$ and $Q_{i(r)}^{(s)} \neq 0$, $s = n, m$, $i_p = \overline{1, N}$, $p = \overline{1, r}$, $r = \overline{1, s}$.

The first theorem whose proof employed this formula, specifically for $n = 2k + 1$ and $m = 2k$, was the following result (see [23]).

Theorem 7. The BCF (29), where $b_{i(k)} > 0$, $i_p = \overline{1, N}$, $p = \overline{1, k}$, $k = 1, 2, \dots$, converges if the series

$$\sum_{k=1}^{\infty} \min \{b_{i(k)} \cdot b_{i(k+1)} : i_p = \overline{1, N}, p = \overline{1, k+1}\}$$

diverges.

This formula was modified for other types of BCFs: BCFs with independent variables [6], two-dimensional corresponding continued fractions [46], and BCFs into which ratios of hypergeometric functions of several variables were expanded [4, 39, 50]. Analogs of this formula were also established for $\mathcal{A}$ - and $\mathcal{B}$ -figured approximants. The application of this formula made it possible to establish multidimensional analogs of many well-known convergence criteria for numerical and various types of functional BCFs. In particular, the convergence of DCFs of a special form in angular, parabolic [15–17], and circular domains [7, 8] was investigated. The objects of research were functional BCFs with independent variables [20, 27, 28, 37], the BCF expansions of the ratios of hypergeometric functions of several variables [4, 39, 50]. Truncation error bounds for BCFs were obtained in [3, 9, 19]. In some cases, new criteria were obtained that differ from the known convergence criteria for continued fractions themselves [6, 38].

As in the analytic theory of continued fractions, the technique of element regions and value regions works effectively for estimating $|a_{i(r)}|$ and $|Q_{i(r)}^{(s)}|$ in formula (30) to obtain an estimate for $|f_n - f_m|$.

By analogy with Leighton's example [47], we construct a $\mathcal{C}_1$ -figured convergent two-dimensional BCF of the special form, in which every fourth element of each continued fraction within its structure is an arbitrary complex number distinct from 0 and $-\frac{1}{2}$.

Let us consider Euler's continued fractions

$$\begin{aligned}\Gamma_1 &= \frac{a_1/2}{1} - \frac{1/2}{1} - \frac{\alpha_3}{1} - \frac{\alpha_4}{1} - \dots - \frac{\alpha_k}{1} - \dots, \\ \Gamma_n &= \frac{a_{2[n-1]}/2}{1} - \frac{1/2}{1} - \frac{\alpha_{n,3}}{1} - \frac{\alpha_{n,4}}{1} - \dots - \frac{\alpha_{n,k}}{1} - \dots, \quad n \geq 2,\end{aligned}$$

where

$$\begin{aligned}\alpha_{2k-1} &= \frac{-a_{1[2k-1]}}{2(1 + a_{1[2k-1]})}, & \alpha_{2k} &= \frac{-1}{2(1 + a_{1[2k-1]})}, & \alpha_{n,2k-1} &= \frac{-a_{2[n-1],1[2k-2]}}{2(1 + a_{2[n-1],1[2k-2]})}, \\ \alpha_{n,2k} &= \frac{-1}{2(1 + a_{2[n-1],1[2k-2]})}, & n \geq 2, k \geq 2, i[p] &= \underbrace{i, i, \dots, i}_p, i = \overline{1, 2}.\end{aligned}$$

Let $\{c_{j,4k}\}$, $j, k = 1, 2, \dots$, be an arbitrary sequence of complex numbers such that $c_{j,4k} \neq 0$, and $c_{j,4k} \neq -\frac{1}{2}$, $j, k = 1, 2, \dots$. We choose the elements $a_{1[4k-1]}$, $k = 1, 2, \dots$, and $a_{2[n-1],1[4k-2]}$, $n \geq 2, k = 1, 2, \dots$, as follows

$$a_{1[4k-1]} = -\frac{1 + 2c_{1,4k}}{2c_{1,4k}}, \quad a_{2[n-1],1[4k-2]} = -\frac{1 + 2c_{n,4k}}{2c_{n,4k}}, \quad k = 1, 2, \dots \text{ and } n \geq 2.$$

Then $\alpha_{4k} = c_{1,4k}$, $\alpha_{n,4k} = c_{n,4k}$ for $k = 1, 2, \dots$ and $n \geq 2$. The elements $a_{1[4k+1]}$ and $a_{2[n-1],1[4k]}$, $k = 1, 2, \dots$ and $n \geq 2$, are chosen such that the series

$$\begin{aligned}\gamma_1 &= a_1 \left(1 + \sum_{k=2}^{\infty} a_{1[3]} a_{1[5]} \dots a_{1[2k-1]} \right), \\ \gamma_1 &= a_{2[n-1]} \left(1 + \sum_{k=2}^{\infty} a_{2[n-1],1[2]} a_{2[n-1],1[4]} \dots a_{2[n-1],1[2k-2]} \right), \quad k \geq 2,\end{aligned}$$

converge, and their sums γ_p , $p = 1, 2, \dots$, are non-zero. For instance, one can set

$$a_{1[4k+1]} = -r_{1,4k+1} \frac{2c_{1,4k}}{1 + 2c_{1,4k}}, \quad a_{2[n-1],1[4k]} = -r_{n,4k} \frac{2c_{n,4k}}{1 + 2c_{n,4k}}, \quad k = 1, 2, \dots, n \geq 2,$$

given an appropriate choice of the constants $r_{1,4k+1}$ and $r_{n,4k}$. The continued fractions Γ_k , $k \geq 1$, converge, and their values are equal to γ_k , $k \geq 1$, respectively. Let us consider a two-dimensional BCF of a special form

$$\frac{1}{1 + \Gamma_1} - \frac{\gamma_2}{1 + \Gamma_2} - \dots - \frac{\gamma_n}{1 + \Gamma_n} - \dots$$

This fraction is $\mathcal{C}_1$ -figurally convergent, and its value is $\eta = \gamma_1(1 - 1/\gamma)$, where

$$\gamma = 1 + \sum_{k=1}^{\infty} \gamma_1 \gamma_2 \dots \gamma_k.$$

The numbers $a_1, a_{2[k]}$, $k = 1, 2, \dots$, are chosen such that the series converges and its sum equals zero.

Let us formulate certain methods for investigating the convergence of BCFs.

Use of inequalities for harmonic means

This is used when investigating the convergence of BCFs with positive elements and N branches of branching. Let $x^{(r)} = (x_{1,r}, x_{2,r}, \dots, x_{n,r})$, $r = \overline{1, k}$, where $x_{j,r} > 0$, $j = \overline{1, n}$, $r = \overline{1, k}$ are real numbers.

If $x_{cp}^{(r)} = n(x_{1,r}^{-1} + x_{2,r}^{-1} + \dots + x_{n,r}^{-1})^{-1}$ is the harmonic mean, then

$$\sum_{r=1}^k x_{cp}^{(r)} \leq \left(\sum_{r=1}^k x^{(r)} \right)_{cp}.$$

These inequalities are used in the investigation of the convergence of BCFs with positive elements, where $x_{i,j}$ represent ratio of the tails of the approximants of BCF (29) [13, 24, 42].

Method of majorants

A BCF of the form

$$\prod_{k=1}^{+\infty} \sum_{i_k=1}^{i_{k-1}} \frac{c_{i(k)}}{d_{i(k)}} \quad (31)$$

with complex elements, where $i_0 = N_1$, $N_1 \in \mathbb{N}$ (often $N_1 = 1$), is called a majorant of the BCF (8) if there exist $n_0 \in \mathbb{N}$ and a real constant $M > 0$ such that for all $n, m \geq n_0$ we have

$$|f_n - f_m| \leq M|g_n - g_m|,$$

where g_k , $k = 1, 2, \dots$, is the k th approximant of the BCF (31).

For example, if for the BCF (8) the conditions

$$|b_{i(k)}| \geq |a_{i(k)}| + 1, \quad i(k) \in \mathcal{J}_k, \quad k = 1, 2, \dots,$$

or

$$|b_{i(k)}| \geq i_{k-1}|a_{i(k)}| + i_k, \quad i(k) \in \mathcal{J}_k, \quad k = 1, 2, \dots,$$

are satisfied, then a majorant for the BCF (8) is the BCF

$$\prod_{k=1}^{+\infty} \sum_{i_k=1}^{i_{k-1}} \frac{|a_{i(k)}|}{|b_{i(k)}|},$$

where $N_1 = N$ (see [6]).

The proof of convergence is based on the statement that the convergence of the majorant implies the convergence of the BCF [13].

Method of fundamental inequalities

The method of fundamental inequalities is used in the analytic theory of continued fractions to reduce the convergence of a continued fraction to the convergence of a series.

For the continued fraction

$$\frac{1}{1 + \frac{a_2}{1 + \frac{a_3}{1 + \cdots + \frac{a_n}{1 + \cdots}}}} \quad (32)$$

fundamental inequalities are satisfied if there exist non-negative real numbers r_p , $p = 1, 2, \dots$, such that

$$r_p |1 + a_p + a_{p+1}| \geq r_p r_{p-2} |a_p| + |a_{p+1}|, \quad p = 1, 2, \dots,$$

where $a_1 = 0$, $r_0 = r_{-1} = 0$.

Let $\rho_p = -\frac{a_{p+1} B_{p-1}}{B_{p+1}}$, $p = 1, 2, \dots$, then

$$\frac{A_n}{B_n} = 1 + \sum_{k=1}^{n-1} \rho_1 \rho_2 \dots \rho_k, \quad n = 2, 3, \dots$$

It follows from the fundamental inequalities that $|\rho_p| \leq r_p$, $p = 1, 2, \dots$. Thus, the continued fraction (32) converges if the series $\sum_{p=1}^{\infty} r_1 r_2 \dots r_p$ converges [56].

In particular, we have

$$\left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} \right| \leq r_1 r_2 \dots r_n.$$

A certain analogy with this last estimate can be obtained based on formula (30).

For the BCF (25), fundamental inequalities are satisfied if $Q_{i(k)}^{(s)} \neq 0$, $i_p = \overline{1, N}$, $p = \overline{1, k}$, $k = \overline{1, s}$, $s = 1, 2, \dots$, and there exist positive real numbers r_p , $p = 1, 2, \dots$, such that

$$\sum_{i_1=1}^N \left| \frac{a_{i_1}}{Q_{i_1}^{(s)}} \right| \leq r_1, \quad s = 1, 2, \dots, \quad (33)$$

$$\sum_{i_{k+1}=1}^N \left| \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(s)}} \right| \leq r_{k+1} \left| b_{i(k)} + \sum_{i_{k+1}=1}^N \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(s)}} \right|, \quad i_p = \overline{1, N}, \quad p = \overline{1, s}, \quad s = 1, 2, \dots \quad (34)$$

It follows from conditions (33), (34), and formula (30) that

$$|f_n - f_m| \leq \prod_{k=1}^{m+1} r_k, \quad n, m = 1, 2, \dots, \quad n > m.$$

Thus, a BCF (25) with complex elements for which the fundamental inequalities (33), (34) are satisfied converges if $\prod_{k=1}^{\infty} r_k = 0$.

This approach was used, in particular, in the works [5, 13].

Expansion of the convergence domain of a sequence of holomorphic functions

In the investigation of the convergence of the continued fraction (1), we replace its numerical elements with functions $a_k(z)$ and $b_k(z)$, $k = 1, 2, \dots$, $z \in D$. We choose a domain D such that, within this domain, the approximants of the functional continued fraction are holomorphic functions. There exists $z_0 \in D$ such that $a_k(z_0) = a_k$ and $b_k(z_0) = b_k$, $k = 1, 2, \dots$. To investigate the convergence of the sequence of holomorphic functions $\{f_n(z)\}$, $z \in D$, the Stieltjes-Vitali theorem is used [45].

As such a compact set, we take the point z_0 , such that $a_k(z_0) = a_k$ and $b_k(z_0) = b_k$. Thus, we obtain the convergence of the continued fraction with numerical elements. Such an approach somewhat narrows the domain of convergence of the continued fraction and does not provide an estimate for the rate of convergence.

For the BCF (8), we use the classical Stieltjes-Vitali theorem when $a_{i(k)}(z)$, $b_{i(k)}(z)$, $z \in D \subset \mathbb{C}$, or the following theorem when $a_{i(k)}(z)$, $b_{i(k)}(z)$, $z \in D \subset \mathbb{C}^N$, $i(k) \in \mathcal{J}$.

Theorem 8 ([13]). *Let $\{f_n(z)\}$ be a sequence of functions of several variables, holomorphic in a domain D and uniformly bounded on every compact subset of this domain. If $\{f_n(z)\}$ converges at every point of a set $\Delta \subset D$, which is a neighborhood of a point (either a real or an imaginary neighborhood of this point), then the sequence converges uniformly on compact subsets of D to a function holomorphic in D .*

This theorem was used in the proof of a multidimensional analogue of the parabolic theorem, specifically Van Vleck's theorem on angular regions of convergence.

There are many unsolved problems in the analytic theory of BCFs, including: establishing necessary conditions for the convergence of BCFs with complex elements; determining estimates for the rate of convergence of BCFs, particularly in parabolic regions; and investigating the convergence of BCFs into which ratios of hypergeometric functions of several variables are expanded, where existing convergence criteria for BCFs cannot always be applied, among others [12].

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Боднар Д.І., Біланик І.Б., Боднар О.С., Саламаха О.Б., Шарин С.В., Кондур О.С. *Проблема збіжності гіллястих ланцюгових дробів* // Карпатські матем. публ. — 2026. — Т.18, №2. — С. 447–464.

Використовуючи залишки підхідних дробів неперервних дробів та їх багатовимірних узагальнень — гіллястих ланцюгових дробів, означено канонічні чисельники і знаменники та досліджено умови виродженості підхідних дробів. Розглянуто деякі типи фігурних підхідних дробів, зокрема *A*-, *B*- та *C*-фігурні підхідні дробі та їхні застосування. Класифіковано різні види збіжності неперервних та гіллястих ланцюгових дробів. За аналогією з прикладом Лейтона побудовано *C*-фігурно збіжний двовимірний гіллястий ланцюговий дріб з нерівнозначними змінними, у якому в кожному неперервному дробі, що входить у його структуру, кожний четвертий елемент є довільним комплексним числом відмінним від 0 і $-1/2$. Проаналізовано методи дослідження збіжності гіллястих ланцюгових дробів.

Ключові слова і фрази: гіллястий ланцюговий дріб, збіжність, підхідний дріб, фігурна збіжність.