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Estimates of the characteristics of nonlinear approximation of classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables in the space $B_{q,1}$

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Exact order estimates of the best m-term trigonometric approximation and the best orthogonal trigonometric approximation of functions from the Nikol'skii-Besov-type classes $B^{\Omega}_{p,\theta}$ in the Lebesgue subspaces $B_{q,1}$ for certain relations between the parameters p and q are obtained. It is shown that in the considered cases the mentioned approximation characteristics of the classes $B^{\Omega}_{p,\theta}$ in the spaces $B_{q,1}$ and L_q differ in order. In addition, it was found that for $1 < q < p < \infty$, in contrast to the case $2 \le p < q < \infty$, the obtained orders of these quantities are realized by approximation of functions from the classes $B^{\Omega}_{p,\theta}$ by their step hyperbolic Fourier sums with a corresponding number of harmonics.

Key words and phrases: Nikol'skii-Besov-type class, step hyperbolic Fourier sum, best m-term trigonometric approximation, best orthogonal trigonometric approximation.

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Introduction

In the paper, we investigate the approximation characteristics of the Nikol'skii-Besov-type classes $B^{\Omega}_{p,\theta}$ of periodic functions of many variables in the space $B_{q,1}$, $1 < q < \infty$. The norm in this space is not weaker than the L_q-norm. As indicated in the papers [4, 7, 11, 13–15, 30–33, 35, 36, 43], a motivation for considering approximation characteristics (best approximations, widhts, best m-term trigonometric approximations, etc.) of the Nikol'skii-Besov classes $B^{r}_{p,\theta}$ and their generalizations $B^{\Omega}_{p,\theta}$ in the spaces $B_{q,1}$, $q \in \{1, \infty\}$, was the fact that in certain important cases the questions on the orders of respective characteristics in the spaces L_1 and L_∞ still remain open. Later, in the works [8, 12, 18–22, 34], the approximation characteristics of some functional classes were studied already in the spaces $B_{q,1}$, $1 < q < \infty$, which was due to a similar motivation. The obtained results complement and generalize some statements from the above-mentioned works [8, 12, 18, 19].

The paper consists of three parts. In the first part, we introduce notation and define functional classes $B^{\Omega}_{p,\theta}$ and spaces $B_{q,1}$. In the second part, we define approximation characteristics under investigation and formulate auxiliary statements.

The third part of the paper is the main one. In Theotem 1 we establish exact order estimates of the best m-term trigonometric approximations of the classes $B^{\Omega}_{p,\theta}$ (denoted by $e_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$)

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in the space $B_{q,1}$, $2 \le p < q < \infty$. In Theorem 2, in addition to the mentioned approximation characteristic, we also investigate the best orthogonal trigonometric approximations $e^{\perp}_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$, $1 < q < p < \infty$. It is shown that in this situation, the quantities $e_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$ and $e^{\perp}_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$ have the same orders. We will comment on this more in remarks to the obtained results.

1 Definition of functional classes and spaces $B_{q,1}$

Let $\mathbb{R}^d$ denotes d-dimensional space. Let $(x,y) = x_1y_1 + \ldots + x_dy_d$ be a scalar product of elements $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$. By $L_p(\mathbb{T}^d)$, $\mathbb{T}^d := \prod\limits_{j=1}^{d} [0, 2\pi)$, we denote the space of functions $f(x)$ which are 2π-periodic in each variable and such that

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} := \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} := \operatorname*{ess\,sup}_{x \in \mathbb{T}^d} |f(x)| < \infty.$$

Thus, we assume that for $f \in L_p(\mathbb{T}^d)$ the condition

$$\int_0^{2\pi} f(x) dx_j = 0, \quad j = \overline{1,d},$$

is satisfied. We denote the set of such functions by $L^0_p(\mathbb{T}^d)$. Sometimes instead of $L_p(\mathbb{T}^d)$ and $L^0_p(\mathbb{T}^d)$ we use the simpler notations L_p and L^0_p, respectively.

We denote the lth difference of a function $f \in L^0_p$, $1 \le p \le \infty$, with a step h_j in the variable x_j by the formula

$$\Delta^l_{h_j} f(x) = \sum_{n=0}^{l} (-1)^{l-n} C^n_l f(x_1, \ldots, x_{j-1}, x_j + nh_j, x_{j+1}, \ldots, x_d).$$

For $f \in L^0_p, 1 \le p \le \infty, h = (h_1, \ldots, h_d)$ and $t \in \mathbb{R}^d_+$ we introduce a mixed lth difference

$$\Delta^l_h f(x) = \Delta^l_{h_1} \ldots \Delta^l_{h_d} f(x) = \Delta^l_{h_d}(\ldots(\Delta^l_{h_1} f(x)))$$

and we denote the mixed modulus of continuity of order l by

$$\Omega_l(f,t)_p := \sup_{|h_j| \le t_j,\, j=\overline{1,d}} \|\Delta^l_h f(\cdot)\|_p.$$

Let $\Omega(t) = \Omega(t_1, \ldots, t_d)$ be a given function of the type of mixed modulus of continuity of the order l. This means that the function $\Omega(t)$ satisfies the following conditions:

1) $\Omega(t) > 0$, $t_j > 0$, $j = \overline{1,d}$, and $\Omega(t) = 0$, if $\prod_{j=1}^d t_j = 0$;

2) $\Omega(t)$ is noncreasing in each variable;

3) $\Omega(m_1t_1, \ldots, m_dt_d) \le \left(\prod_{j=1}^d m_j\right)^l \Omega(t)$, $m_j \in \mathbb{N}$, $j = \overline{1,d}$;

4) $\Omega(t)$ is continuous for $t_j \ge 0$, $j = \overline{1,d}$.

Following S.N. Bernstein [5], we call the function of one variable $\varphi(\tau)$ almost increasing on $[a,b]$, if there exists a constant $C_1 > 0$, which does not depend on τ_1, τ_2, such that

$$\varphi(\tau_1) \le C_1 \varphi(\tau_2), \quad a \le \tau_1 \le \tau_2 \le b,$$

and almost decreasing on $[a,b]$, if there exists a constant $C_2 > 0$, which does not depend on τ_1, τ_2, such that

$$\varphi(\tau_1) \ge C_2 \varphi(\tau_2), \quad a \le \tau_1 \le \tau_2 \le b.$$

We assume that the function $\Omega(t)$, $t \in \mathbb{R}^d_+$, satisfies also the conditions (S^α) and (S_l), which are called the Bari-Stechkin conditions [2,42]. This means the following.

A function of one variable $\varphi(\tau) \ge 0$, $\tau \in [0,1]$, satisfies the condition (S^α) if $\varphi(\tau)/\tau^\alpha$ almost increases for some $\alpha > 0$.

A function $\varphi(\tau) \ge 0$, $\tau \in [0,1]$, satisfies the condition (S_l) if $\varphi(\tau)/\tau^\gamma$ almost decreases for some $0 < \gamma < l$, $l \in \mathbb{N}$.

In the case of $d > 1$ we say that $\Omega(t), t \in \mathbb{R}^d_+$, satisfies these conditions if $\Omega(t)$ satisfies these conditions in each variable t_j for fixed t_i, $i \neq j$.

We now define the functional classes $B^\Omega_{p,\theta}$, which were considered in the paper [46] by S. Yongshen, W. Heping.

Let $1 \le p, \theta \le \infty$ and let $\Omega(t)$ be a given function of the type of mixed modulus of continuity of the order l, which satisfies conditions 1) – 4), (S^α) and (S_l). Then the classes $B^\Omega_{p,\theta}$ are defined as follows

$$B^\Omega_{p,\theta} := \Big\{ f \in L^0_p(\mathbb{T}^d) : \|f\|_{B^\Omega_{p,\theta}} \le 1 \Big\},$$

where

$$\|f\|_{B^\Omega_{p,\theta}} := \left(\int_{\mathbb{T}^d} \left(\frac{\Omega_l(f,t)_p}{\Omega(t)} \right)^\theta \prod_{j=1}^d \frac{dt_j}{t_j} \right)^{\frac{1}{\theta}}, \quad 1 \le \theta < \infty,$$

$$\|f\|_{B^\Omega_{p,\infty}} := \sup_{t \in \mathbb{R}^d_+} \frac{\Omega_l(f,t)_p}{\Omega(t)}.$$

We note that, in the case $r = (r_1, \dots, r_d), 0 < r_j < l, j = \overline{1,d}$, and $\Omega(t) = \prod_{j=1}^d t_j^{r_j}$, the classes $B^\Omega_{p,\theta}$ are identical to analogs of the Besov classes $B^r_{p,\theta}$ which were considered in the papers [1,16]. In turn, for $\theta = \infty$ the classes $B^r_{p,\infty} = H^r_p$ are analogs of the Nikol'skii classes [17]. The classes $B^\Omega_{p,\infty} = H^\Omega_p$ were studied by N.N. Pustovoitov in [23].

In the following considerations we will use the definition of classes $B^\Omega_{p,\theta}$ in a slightly different form. To do this, we recall the definition of order relation.

For two non-negative sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ the relation (order inequality) $a_n \ll b_n$ means that there exists a constant $C_3 > 0$, which does not depend on n and such that $a_n \le C_3 b_n$. The relation $a_n \asymp b_n$ is equivalent to $a_n \ll b_n$ and $b_n \ll a_n$.

To every vector $s \in \mathbb{N}^d$ we put the set

$$\rho(s) := \Big\{ k \in \mathbb{Z}^d : 2^{s_j - 1} \le |k_j| < 2^{s_j}, j = \overline{1,d} \Big\}$$

in correspondence, and, for $f \in L^0_p, 1 < p < \infty$, we denote

$$\delta_s(f) := \delta_s(f,x) := \sum_{k \in \rho(s)} \widehat{f}(k) e^{i(k,x)},$$

where

$$\widehat{f}(k) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(t) e^{-i(k,t)} dt$$

are the Fourier coefficients of the function f.

Therefore, for $f \in B^{\Omega}_{p,\theta}$, $1 < p < \infty$, $1 \le \theta \le \infty$, where $\Omega(t)$ is a given function of the type of mixed modulus of continuity of order l, which satisfies conditions 1) – 4), (S^{α}) and (S_l) the relations

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \begin{cases} \left(\sum\limits_{s \in \mathbb{N}^d} \Omega^{-\theta}(2^{-s}) \|\delta_s(f)\|_p^{\theta} \right)^{\frac{1}{\theta}}, & 1 \le \theta < \infty, \\ \sup\limits_{s \in \mathbb{N}^d} \dfrac{\|\delta_s(f)\|_p}{\Omega(2^{-s})}, & \theta = \infty, \end{cases} \tag{1}$$

hold. Here and below, $\Omega(2^{-s}) = \Omega(2^{-s_1}, \dots, 2^{-s_d})$, $s_j \in \mathbb{N}$, $j = \overline{1,d}$.

Note that the case $1 \le \theta < \infty$ in (1) was considered in [46], and the case $\theta = \infty$ in [23].

For the norms of functions from the classes $B^{\Omega}_{p,\theta}$ for $p = 1$ and $p = \infty$ we can write relations analogous to (1) by replacing the "blocks" $\delta_s(f)$ by others. Namely, by $V_m(t), m \in \mathbb{N}, t \in \mathbb{R}$, we denote the Vall'ee-Poussin kernel

$$V_m(t) := 1 + 2\sum_{k=1}^{m} \cos kt + 2\sum_{k=m+1}^{2m-1} \left(\frac{2m-k}{m} \right) \cos kt$$

(for the correctness of the definition of $V_m(t)$, we should assume that the last sum in this formula vanishes for $m = 1$).

To every vector $s \in \mathbb{N}^d$, we put the polynomial

$$A_s(x) := \prod_{j=1}^{d} \left(V_{2^{s_j}}(x_j) - V_{2^{s_j - 1}}(x_j) \right), \quad x \in \mathbb{R}^d,$$

in correspondence, and, for $f \in L^0_p, 1 \le p \le \infty$, we set

$$A_s(f) := A_s(f, x) := (f * A_s)(x),$$

where $*$ means the convolution operation.

Then the following relations

$$\|f\|_{B^{\Omega}_{p,\theta}} \asymp \begin{cases} \left(\sum\limits_{s \in \mathbb{N}^d} \Omega^{-\theta}(2^{-s}) \|A_s(f)\|_p^{\theta} \right)^{\frac{1}{\theta}}, & 1 \le \theta < \infty, \\ \sup\limits_{s \in \mathbb{N}^d} \dfrac{\|A_s(f)\|_p}{\Omega(2^{-s})}, & \theta = \infty. \end{cases} \tag{2}$$

hold. Note that the case $1 \le \theta < \infty$ in (2) was considered in [40], and the case $\theta = \infty$ in [23].

In the following research, we consider the classes $B^{\Omega}_{p,\theta}$ defined by a function of the type of a mixed modulus of continuity of order l of the special form

$$\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j \right), \tag{3}$$

where $\omega(\tau)$ is a given function (of one variable) of the type of a modulus of continuity of order l that satisfies conditions (S^{α}) and (S_l).

It is easy to verify that the function $\Omega(t)$ of the form (3) satisfies properties 1) – 4) of a function of the type of mixed modulus of continuity of order l, and satisfies conditions (S^{α}) and (S_l). Therefore, the above mentioned relations (1), (2) for the norms of functions of the class $B^{\Omega}_{p,\theta}$ remain true.

Define the norm $\|\cdot\|_{B_{q,1}}$ in the subspaces $B_{q,1} \subset L^0_q, 1 \le q \le \infty$, by $\|t\|_{B_{q,1}} := \sum\limits_{s\in\mathbb{N}^d} \|A_s(t)\|_q$, (the sum contains a finite number of terms), where t is a trigonometric polynomial with respect to the trigonometric system $\{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$.

Similarly we define the norm for functions $f \in L^0_q$, such that the series $\sum\limits_{s\in\mathbb{N}^d} \|A_s(f)\|_q$ is convergent, namely

$$\|f\|_{B_{q,1}} := \sum_{s\in\mathbb{N}^d} \|A_s(f)\|_q, \quad 1 \le q \le \infty.$$

Note that in the case $1 < q < \infty$ the relation

$$\|f\|_{B_{q,1}} \asymp \sum_{s\in\mathbb{N}^d} \|\delta_s(f)\|_q$$

is valid.

For $f \in B_{q,1}, 1 \le q \le \infty$, the following relations

$$\|f\|_q \ll \|f\|_{B_{q,1}}; \qquad \|f\|_{B_{1,1}} \ll \|f\|_{B_{q,1}} \ll \|f\|_{B_{\infty,1}}.$$

hold.

2 Approximation characteristics and auxiliary assertions

For $n \in \mathbb{N}$ and $s \in \mathbb{N}^d$ we set

$$Q_n := \bigcup_{\|s\|_1 < n} \rho(s), \quad \text{where } \|s\|_1 := s_1 + \ldots + s_d.$$

The set Q_n is called the step hyperbolic cross. For the number of elements of the set Q_n, the following order equality $|Q_n| \asymp 2^n n^{d-1}$ holds.

Next, we will consider the sets of trigonometric polynomials

$$T(Q_n) := \left\{ t : t(x) = \sum_{k\in Q_n} c_k e^{i(k,x)},\ c_k \in \mathbb{C},\ x \in \mathbb{R}^d \right\}$$

and for $f \in L^0_1(\mathbb{T}^d)$ we put

$$S_{Q_n}(f) := S_{Q_n}(f,x) := \sum_{k\in Q_n} \widehat{f}(k) e^{i(k,x)}, \quad x \in \mathbb{R}^d.$$

Polynomials $S_{Q_n}(f)$ are called step hyperbolic Fourier sums of the function f. According to the notation considered above, $S_{Q_n}(f)$ can be expressed in the form

$$S_{Q_n}(f) := \sum_{\|s\|_1 < n} \delta_s(f) := \sum_{\|s\|_1 < n} \delta_s(f,x).$$

Taking into account the sets of trigonometric polynomials defined above, consider the following approximation characteristics.

Let $\mathcal{X}$ be a normed functional space with the norm $\|\cdot\|_{\mathcal{X}}$. For $f \in \mathcal{X}$ we define by

$$E_{Q_n}(f)_{\mathcal{X}} := \inf_{t \in T(Q_n)} \|f - t\|_{\mathcal{X}}$$

the best approximation of the function f by polynomials from the set $T(Q_n)$. Accordingly, for the functional class $F \subset \mathcal{X}$ we set

$$E_{Q_n}(F)_{\mathcal{X}} := \sup_{f \in F} E_{Q_n}(f)_{\mathcal{X}}. \tag{4}$$

In addition to the quantity (4), we will consider approximation characteristic

$$\mathcal{E}_{Q_n}(F)_{\mathcal{X}} := \sup_{f \in F} \mathcal{E}_{Q_n}(f)_{\mathcal{X}} := \sup_{f \in F} \|f - S_{Q_n}(f)\|_{\mathcal{X}}.$$

In the paper [9] it is shown that for $f \in B^{\Omega}_{p,\theta}$ the following statements hold.

Theorem A ([9]). *Let* $d \geq 2$, $1 \leq q < p \leq \infty$, $1 \leq \theta \leq \infty$, *and* $\Omega(t) = \omega\Big(\prod_{j=1}^{d} t_j\Big)$, *where* $\omega(\tau)$ *satisfies condition* (S^{α}) *with some* $\alpha > 0$ *and condition* (S_l). *Then the following relations*

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp \omega(2^{-n}) n^{(d-1)(1-\frac{1}{\theta})}$$

hold.

Theorem B ([9]). *Let* $d \geq 2$, $1 < p < q < \infty$, $1 \leq \theta \leq \infty$, *and* $\Omega(t) = \omega\left(\prod_{j=1}^{d} t_j\right)$, *where* $\omega(\tau)$ *satisfies condition* (S^{α}) *with some* $\alpha > \frac{1}{p} - \frac{1}{q}$ *and condition* (S_l). *Then the following relations*

$$\mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp E_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp \omega(2^{-n}) 2^{n\left(\frac{1}{p}-\frac{1}{q}\right)} n^{(d-1)(1-\frac{1}{\theta})}$$

hold.

Let Θ_m be a set of m arbitrary d-dimensional vectors with integer coordinates. Let us denote by

$$P(\Theta_m) := P(\Theta_m, x) := \sum_{k \in \Theta_m} c_k e^{i(k,x)}, \quad c_k \in \mathbb{C},$$

a trigonometric polynomial with "numbers" of harmonics from the set Θ_m. For $f \in \mathcal{X}$, we consider the quantity

$$e_m(f)_{\mathcal{X}} := \inf_{c_k} \inf_{\Theta_m} \|f - P(\Theta_m)\|_{\mathcal{X}},$$

which is called the best m-term trigonometric approximation of f. For a class $F \subset \mathcal{X}$, we set

$$e_m(F)_{\mathcal{X}} := \sup_{f \in F} e_m(f)_{\mathcal{X}}.$$

The quantity $e_m(f)_2 := e_m(f)_{L_2(\mathbb{T})}$ for univariate functions was introduced by S.B. Stechkin [41] in order to formulate a criterion of absolute convergence of orthogonal series in the general case of approximations by polynomials with respect to arbitrary orthogonal system in a Hilbert space.

Further, the quantities $e_m(F)_{\mathcal{X}}$ for certain functional classes and spaces $\mathcal{X} = L_q(\mathbb{T}^d)$, $d \geq 1$, as well as other normed spaces, were studied by many authors. The detailed overview can be found in the papers [3, 8, 10, 12, 15, 18, 19, 26–29, 34, 37, 38] and monographs [6, 25, 44, 45].

To formulate the auxiliary statement, we introduce the necessary notations and definitions.

Let D be a bounded set in $\mathbb{R}^d, d \in \mathbb{N}$, and $\Phi = \{\varphi_n(x)\}_{n=1}^{\infty}$ be a system of functions from $L_q(D), 1 \leq q \leq \infty$. For $f \in L_q(D)$ we set

$$e_m(f, \Phi)_{L_q(D)} := \inf_{\substack{\{n_j\}=\Lambda \subset \mathbb{Z}_+,\ |\Lambda|=m \\ \{c_j\} \in \mathbb{R}^m}} \left\| f - \sum_{j=1}^{m} c_j \varphi_{n_j} \right\|_{L_q(D)},$$

where $\mathbb{Z}_+ = \mathbb{N} \bigcup \{0\}$, and $|\Lambda|$ denotes the number of elements of the set Λ.

Further, if $F \subset L_q(D)$ is some class of functions, we define

$$e_m(F, \Phi)_{L_q(D)} := \sup_{f \in F} e_m(f, \Phi)_{L_q(D)}. \tag{5}$$

Remark 1. *The space $L_q(D)$ is defined analogically to the space $L_q(\mathbb{T}^d)$ by changing in the definition of $\|\cdot\|_{L_q(\mathbb{T}^d)}$ the multiplier $(2\pi)^{-d}$ by the multiplier $(mesD)^{-1}$ and the integration is taken over the region D.*

Remark 2. *In the case of trigonometric system $T := \{e^{i(k,x)}\}_{k \in \mathbb{Z}^d}$, we will write definition* (5) *as*

$$e_m(F, T)_{L_q(D)} = e_m\big(F, \{e^{i(k,x)}\}_{k \in \mathbb{Z}^d}\big)_{L_q(D)} := e_m(F)_q.$$

In what follows, for the vector $s \in \mathbb{N}^d$ with even numbers $s_j, j = \overline{1,d}$, we denote

$$\rho^+(s) := \left\{k \in \mathbb{N}^d : 2^{s_j - 1} \leq k_j < 2^{s_j},\ j = \overline{1,d}\right\}$$

and for $n \in \mathbb{N}$ set

$$D_n := \left\{s : (s, 1) = 2\left[\frac{n}{2}\right]\right\}, \qquad \mathcal{Y}_n := \bigcup_{s \in D_n} \rho^+(s),$$

where $[a]$ is the integer part of the number a.

Note that for the number of elements in the sets D_n and $\mathcal{Y}_n$ the following relations

$$|D_n| \asymp n^{d-1}, \qquad |\mathcal{Y}_n| \asymp 2^n n^{d-1}$$

hold.

Let $\mathcal{T}(\mathcal{Y}_n)$ be a set of polynomials of the form

$$t(x) := \sum_{|k| \in \mathcal{Y}_n} c_k e^{i(k,x)},$$

where $|k| = (|k_1|, \ldots, |k_d|)$.

If $\mathcal{X}$ is a normed space with the norm $\|\cdot\|_{\mathcal{X}}$, by $\mathcal{T}(\mathcal{Y}_n)_{\mathcal{X}}$ we denote the unit ball in the space $\mathcal{T}(\mathcal{Y}_n)$. In the introduced notation the following statement holds.

Theorem C ([15]). *There exists a constant $C_4(d) \geq 0$, such that for any set of functions $\Phi = \{\varphi_j\}_{j=1}^{l} \subset B_{1,1}, l \leq C_5|\mathcal{Y}_n|$, the following estimate*

$$e_m(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, \Phi)_{B_{1,1}} \geq C_6 n^{d-1}, \quad C_6 = C_6(d, C_5) > 0$$

holds for all $m \leq C_4(d)|\mathcal{Y}_n|$.

Remark 3. *The space* $B_{\infty,\infty}$ *is defined as follows*

$$B_{\infty,\infty} := \left\{ f \in L_\infty^0 : \|f\|_{B_{\infty,\infty}} < \infty \right\},$$

where $\|f\|_{B_{\infty,\infty}} := \sup\limits_{s \in \mathbb{N}^d} \|A_s(t)\|_\infty$.

Lemma A ([3]). *Let* $2 < q < \infty$. *Then for any trigonometric polynomial* $P(\Theta_n) := P(\Theta_n, x)$ *and any* $m < n$, *one can find a trigonometric polynomial* $\widetilde{P}(\Theta_m) := \widetilde{P}(\Theta_m, x)$ *such that*

$$\|P(\Theta_n) - \widetilde{P}(\Theta_m)\|_q \leq C_7(q) \sqrt{\frac{m}{n}} \|P(\Theta_n)\|_2,$$

and, moreover, $\Theta_m \subset \Theta_n$.

In the statement obtained in Theorem 2, in addition to the quantity $e_m(B^\Omega_{p,\theta})_{B_{q,1}}$, the following approximation characteristic is considered.

For $f \in \mathcal{X}$ we denote

$$S_{\Theta_m}(f) := S_{\Theta_m}(f, x) := \sum_{j=1}^{m} \widehat{f}(k^j) e^{i(k^j, x)}, \quad x \in \mathbb{R}^d,$$

where $\widehat{f}(k^j) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(t) e^{-i(k^j,t)} dt$ are the Fourier coefficients of the function f, which correspond to the set of vectors Θ_m.

We consider the approximation characteristic

$$e_m^\perp(f)_\mathcal{X} := \inf_{\Theta_m} \|f - S_{\Theta_m}(f)\|_\mathcal{X}$$

and for the functional class $F \subset \mathcal{X}$ we put

$$e_m^\perp(F)_\mathcal{X} := \sup_{f \in F} e_m^\perp(f)_\mathcal{X}.$$

Quantity $e_m^\perp(F)_\mathcal{X}$ is called the best orthogonal trigonometric approximation of the class F in the space $\mathcal{X}$. The quantities $e_m^\perp(F)_\mathcal{X}$ for classes of functions $B^r_{p,\theta}$ and $B^\Omega_{p,\theta}$ in spaces $L_q, 1 \leq q \leq \infty$, and $B_{\infty,1}$ were studied in the works [3,7,12,13,24,25,29,33,35].

Note that from the definitions of the quantities $e_m(F)_\mathcal{X}$ and $e_m^\perp(F)_\mathcal{X}$ we get the following relation

$$e_m(F)_\mathcal{X} \leq e_m^\perp(F)_\mathcal{X}. \tag{6}$$

3 Main results

The following statement is true.

Theorem 1. *Let* $d \geq 2$, $2 \leq p < q < \infty$, $1 \leq \theta \leq \infty$, *and* $\Omega(t) = \omega\left(\prod_{j=1}^d t_j\right)$, *where* $\omega(\tau)$ *satisfies condition* (S^α) *with some* $\alpha > \frac{1}{2}$ *and condition* (S_l). *Then for any natural numbers* m *and* n, *such that* $m \asymp 2^n n^{d-1}$, *the following relation holds*

$$e_m(B^\Omega_{p,\theta})_{B_{q,1}} \asymp \omega(2^{-n}) n^{(d-1)(1-\frac{1}{\theta})}. \tag{7}$$

Proof. First, we establish the upper estimate in (7). Note that in view of the embedding $B^{\Omega}_{p,\theta} \subset B^{\Omega}_{2,\theta}$, $p \geq 2$, it is sufficient to consider the class $B^{\Omega}_{2,\theta}$.

So, let $f \in B^{\Omega}_{2,\theta}$. We can write the function f as

$$f = \sum_{\|s\|_1 < n} \delta_s(f) + \sum_{n \leq \|s\|_1 < \beta n} \delta_s(f) + \sum_{\|s\|_1 \geq \beta n} \delta_s(f), \tag{8}$$

where $\beta > 1$ is some real number that will be chosen in the process of proving.

For a given number m, we choose the number n from the relation $m \asymp 2^n n^{d-1}$. We consider for the function f an approximation polynomial of the form

$$P(\Theta_m) := \sum_{\|s\|_1 < n} \delta_s(f) + \sum_{n \leq \|s\|_1 < \beta n} P(\Theta_{n_s}), \tag{9}$$

where $P(\Theta_{n_s})$ are the polynomials, constructed for each "block" $\delta_s(f)$ with respect to Lemma A, i.e.

$$\|\delta_s(f) - P(\Theta_{n_s})\|_q \ll \left(\frac{2^{\|s\|_1}}{n_s}\right)^{\frac{1}{2}} \|\delta_s(f)\|_2. \tag{10}$$

Suppose that the polynomial $P(\Theta_m)$ has been constructed. Then, taking into account (8) and (9), we can write

$$\begin{aligned} \|f - P(\Theta_m)\|_{B_{q,1}} &= \left\| \sum_{\|s\|_1 \geq n} \delta_s(f) - \sum_{n \leq \|s\|_1 < \beta n} P(\Theta_{n_s}) \right\|_{B_{q,1}} \\ &= \left\| \sum_{n \leq \|s\|_1 < \beta n} (\delta_s(f) - P(\Theta_{n_s})) + \sum_{\|s\|_1 \geq \beta n} \delta_s(f) \right\|_{B_{q,1}} \\ &\leq \left\| \sum_{n \leq \|s\|_1 < \beta n} (\delta_s(f) - P(\Theta_{n_s})) \right\|_{B_{q,1}} + \left\| \sum_{\|s\|_1 \geq \beta n} \delta_s(f) \right\|_{B_{q,1}} := I_1 + I_2. \end{aligned} \tag{11}$$

Next, let us estimate each term in (11). According to Theorem B for the quantity I_2 we obtain

$$I_2 = \left\| f - \sum_{\|s\|_1 < \beta n} \delta_s(f) \right\|_{B_{q,1}} \leq \sup_{f \in B^{\Omega}_{2,\theta}} \left\| f - \sum_{\|s\|_1 < \beta n} \delta_s(f) \right\|_{B_{q,1}} \ll \omega(2^{-\beta n}) 2^{n\beta(\frac{1}{2} - \frac{1}{q})} n^{(d-1)(1-\frac{1}{\theta})}. \tag{12}$$

Next, we estimate the term I_1. Using the definition of the norm $\|\cdot\|_{B_{q,1}}$, Lemma A, and the estimate (10), we get

$$\begin{aligned} I_1 &\asymp \sum_{s \in \mathbb{N}^d} \left\| \delta_s \Bigg(\sum_{n \leq \|s'\|_1 < \beta n} (\delta_{s'}(f) - P(\Theta_{n_{s'}})) \Bigg) \right\|_q \\ &= \sum_{n \leq \|s\|_1 < \beta n} \|\delta_s(f) - P(\Theta_{n_s})\|_q \ll \sum_{n \leq \|s\|_1 < \beta n} \left(\frac{2^{\|s\|_1}}{n_s} \right)^{\frac{1}{2}} \|\delta_s(f)\|_2. \end{aligned} \tag{13}$$

Substituting (13) and (12) into (11), we obtain the relation

$$\|f - P(\Theta_m)\|_{B_{q,1}} \ll \sum_{n \leq \|s\|_1 < \beta n} \frac{2^{\frac{\|s\|_1}{2}}}{n_s^{\frac{1}{2}}} \|\delta_s(f)\|_2 + \omega(2^{-\beta n}) 2^{n\beta(\frac{1}{2} - \frac{1}{q})} n^{(d-1)(1-\frac{1}{\theta})} := I_3 + I_4. \tag{14}$$

Now we choose the numbers β and n_s as follows

$$\beta := \frac{\alpha}{\alpha - \frac{1}{2} + \frac{1}{q}}, \qquad n_s := [\omega^{-1}(2^{-n})2^{\frac{n}{2}}\omega(2^{-\|s\|_1})2^{\frac{\|s\|_1}{2}}] + 1$$

and let us show that with this choice of numbers n_s the number of harmonics in the polynomials $P(\Theta_{n_s})$ does not exceed $2^n n^{d-1}$ in order, i.e. $\sum\limits_{n \le \|s\|_1 < \beta n} n_s \ll 2^n n^{d-1}$.

We have

$$\begin{aligned} \sum_{n \le \|s\|_1 < \beta n} n_s &\le \sum_{n \le \|s\|_1 < \beta n} 1 + \omega^{-1}(2^{-n})2^{\frac{n}{2}} \sum_{n \le \|s\|_1 < \beta n} \omega(2^{-\|s\|_1})2^{\frac{\|s\|_1}{2}} \\ &\ll n^d + \omega^{-1}(2^{-n})2^{\frac{n}{2}} \sum_{\|s\|_1 \ge n} \frac{\omega(2^{-\|s\|_1})}{2^{-\alpha\|s\|_1}} 2^{-(\alpha - \frac{1}{2})\|s\|_1} := I_5. \end{aligned}$$

To estimate the term I_5, we note the following. Since $\Omega(t) = \omega\Big(\prod\limits_{j=1}^{d} t_j\Big)$ satisfies the condition (S^α) with $\alpha > \frac{1}{2}$, then (see [46])

$$\frac{\omega(2^{-\|s\|_1})}{2^{-\alpha\|s\|_1}} \ll \frac{\omega(2^{-n})}{2^{-\alpha n}}, \quad \|s\|_1 \ge n. \tag{15}$$

In addition, the following relations

$$\sum_{\|s\|_1 \ge n} 2^{-\|s\|_1} = \sum_{j=n}^{\infty} \sum_{\|s\|_1 = j} 2^{-\|s\|_1} = \sum_{j=n}^{\infty} 2^{-j} \sum_{\|s\|_1 = j} 1 \asymp \sum_{j=n}^{\infty} 2^{-j} j^{d-1} \asymp 2^n n^{d-1} \tag{16}$$

hold.

Thus, taking into account (15) and (16), the estimation of the quantity I_5 can be continued as follows

$$\begin{aligned} I_5 &\ll n^d + \omega^{-1}(2^{-n})2^{\frac{n}{2}} \frac{\omega(2^{-n})}{2^{-\alpha n}} \sum_{\|s\|_1 \ge n} 2^{-(\alpha - \frac{1}{2})\|s\|_1} \\ &\asymp n^d + 2^{\frac{n}{2}} 2^{\alpha n} 2^{-n(\alpha - \frac{1}{2})} n^{d-1} = n^d + 2^n n^{d-1} \ll 2^n n^{d-1}. \end{aligned}$$

Regarding the choice of numbers n_s, we make the following remark. It is possible that for some vectors $s = (s_1, \dots, s_d)$ the inequality $n_s \ge 2^{\|s\|_1}$ holds. In this case, we take the corresponding "block" $\delta_s(f)$ as the polynomials $P(\Theta_{n_s})$, which does not affect the further estimate.

Hence, returning to estimation of the quantity I_3 and taking into account the values of n_s and β, we can write

$$\begin{aligned} I_3 &\ll \sum_{n \le \|s\|_1 < \beta n} 2^{\frac{\|s\|_1}{2}} \omega^{\frac{1}{2}}(2^{-n}) 2^{-\frac{n}{4}} \omega^{-\frac{1}{2}}(2^{-\|s\|_1}) 2^{-\frac{\|s\|_1}{4}} \|\delta_s(f)\|_2 \\ &= \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}} \sum_{n \le \|s\|_1 < \beta n} \omega^{-1}(2^{-\|s\|_1}) \|\delta_s(f)\|_2 \omega^{\frac{1}{2}}(2^{-\|s\|_1}) 2^{\frac{\|s\|_1}{4}}. \end{aligned} \tag{17}$$

Further we consider several cases, depending on the values of parameter θ.

Case 1. Let $1 < \theta < \infty$. Applying to the last sum in (17) the Hölder inequality with the exponent θ and using the relations (15) and (16), we have

$$\begin{aligned} I_3 &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\Big(\sum_{n\le\|s\|_1<\beta n}\omega^{-\theta}(2^{-\|s\|_1})\|\delta_s(f)\|_2^{\theta}\Big)^{\frac{1}{\theta}}\Big(\sum_{n\le\|s\|_1<\beta n}\Big(\omega(2^{-\|s\|_1})2^{\frac{\|s\|_1}{2}}\Big)^{\frac{\theta'}{2}}\Big)^{\frac{1}{\theta'}} \\ &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\|f\|_{B^{\Omega}_{2,\theta}}\Big(\sum_{\|s\|_1\ge n}\Big(\frac{\omega(2^{-\|s\|_1})}{2^{-\alpha\|s\|_1}}2^{-(\alpha-\frac{1}{2})\|s\|_1}\Big)^{\frac{\theta'}{2}}\Big)^{\frac{1}{\theta'}} \\ &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\omega^{\frac{1}{2}}(2^{-n})2^{\frac{\alpha n}{2}}\Big(\sum_{\|s\|_1\ge n}2^{-(\alpha-\frac{1}{2})\|s\|_1\frac{\theta'}{2}}\Big)^{\frac{1}{\theta'}} \\ &\ll \omega(2^{-n})2^{-\frac{n}{4}}2^{\frac{\alpha n}{2}}2^{-(\alpha-\frac{1}{2})\frac{n}{2}}n^{\frac{d-1}{\theta'}} = \omega(2^{-n})2^{-\frac{n}{4}}2^{\frac{\alpha n}{2}}2^{-\frac{\alpha n}{2}}2^{\frac{n}{4}}n^{\frac{d-1}{\theta'}} = \omega(2^{-n})n^{\frac{d-1}{\theta'}}, \end{aligned} \tag{18}$$

where $\frac{1}{\theta}+\frac{1}{\theta'}=1$.

Case 2. Let $\theta = 1$, then we may write

$$\begin{aligned} I_3 &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\sup_{\|s\|_1\ge n}\frac{\omega^{\frac{1}{2}}(2^{-\|s\|_1})}{2^{-\frac{\alpha}{2}\|s\|_1}}2^{-\frac{\alpha}{2}\|s\|_1}2^{\frac{\|s\|_1}{4}}\sum_{n\le\|s\|_1<\beta n}\omega^{-1}(2^{-\|s\|_1})\|\delta_s(f)\|_2 \\ &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\omega^{\frac{1}{2}}(2^{-n})2^{\frac{\alpha n}{2}}\sup_{\|s\|_1\ge n}2^{-(\alpha-\frac{1}{2})\frac{\|s\|_1}{2}}\|f\|_{B^{\Omega}_{2,1}} \\ &\ll \omega(2^{-n})2^{-\frac{n}{4}}2^{\frac{\alpha n}{2}}2^{-\frac{\alpha n}{2}}2^{\frac{n}{4}} = \omega(2^{-n}). \end{aligned} \tag{19}$$

Case 3. Let $\theta = \infty$. Using the relations (15) and (16), we get

$$\begin{aligned} I_3 &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\sum_{n\le\|s\|_1<\beta n}\frac{\|\delta_s(f)\|_2}{\omega(2^{-\|s\|_1})}2^{\frac{\|s\|_1}{4}}\omega^{\frac{1}{2}}(2^{-\|s\|_1}) \\ &\le \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\sup_{\|s\|_1\ge n}\frac{\|\delta_s(f)\|_2}{\omega(2^{-\|s\|_1})}\sum_{n\le\|s\|_1<\beta n}\frac{\omega^{\frac{1}{2}}(2^{-\|s\|_1})}{2^{-\frac{\alpha}{2}\|s\|_1}}2^{-(\alpha-\frac{1}{2})\frac{\|s\|_1}{2}} \\ &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\|f\|_{B^{\Omega}_{2,\infty}}\frac{\omega^{\frac{1}{2}}(2^{-n})}{2^{-\frac{\alpha n}{2}}}\sum_{\|s\|_1\ge n}2^{-(\alpha-\frac{1}{2})\frac{\|s\|_1}{2}} \\ &\ll \omega^{\frac{1}{2}}(2^{-n})2^{-\frac{n}{4}}\omega^{\frac{1}{2}}(2^{-n})2^{\frac{\alpha n}{2}}2^{-\frac{\alpha n}{2}}2^{\frac{n}{4}}n^{d-1} = \omega(2^{-n})n^{d-1}. \end{aligned} \tag{20}$$

So, combining the relations (18)–(20), the estimate (17) takes the form

$$I_3 \ll \omega(2^{-n})n^{(d-1)(1-\frac{1}{\theta})}, \quad 1\le\theta\le\infty. \tag{21}$$

Let us estimate further the quantity I_4. Taking into account that in our case $q > 2$ and $\beta > 1$, and using the relation (15), we obtain

$$\begin{aligned} I_4 = \frac{\omega(2^{-\beta n})}{2^{-\alpha\beta n}}2^{-\beta n\left(\alpha-\frac{1}{2}+\frac{1}{q}\right)}n^{(d-1)(1-\frac{1}{\theta})} &\ll \frac{\omega(2^{-n})}{2^{-\alpha n}}2^{-n\left(\alpha-\frac{1}{2}+\frac{1}{q}\right)\cdot\frac{\alpha}{\alpha-\frac{1}{2}+\frac{1}{q}}}n^{(d-1)(1-\frac{1}{\theta})} \\ &= \omega(2^{-n})2^{\alpha n}2^{-\alpha n}n^{(d-1)(1-\frac{1}{\theta})} = \omega(2^{-n})n^{(d-1)(1-\frac{1}{\theta})}. \end{aligned} \tag{22}$$

Finally, by substituting (21) and (22) into (14), we obtain the required upper estimate

$$e_m(B^{\Omega}_{p,\theta})_{B_{q,1}} \ll e_m(B^{\Omega}_{2,\theta})_{B_{q,1}} \ll \omega(2^{-n})n^{(d-1)(1-\frac{1}{\theta})}.$$

Let us get the respective lower estimate in (7). For a given number m, we choose the number n from the relation $m \asymp 2^n n^{d-1}$ and consider the class $B^{\Omega}_{p,\theta} \cap \mathcal{T}(\mathcal{Y}_n)$, which contains functions with $B^{\Omega}_{p,\theta}$ that are simultaneously polynomials with $\mathcal{T}(\mathcal{Y}_n)$. In this case we can write

$$e_m(B^{\Omega}_{p,\theta})_{B_{q,1}} \geq e_m(B^{\Omega}_{p,\theta} \cap \mathcal{T}(\mathcal{Y}_n))_{B_{q,1}}. \tag{23}$$

By $P_{\mathcal{Y}_n}$ we denote the operator of orthogonal projection on $\mathcal{T}(\mathcal{Y}_n)$, which to each function $f \in B_{q,1}$ put in correspondence polynomial from $\mathcal{T}(\mathcal{Y}_n)$. Then for the norm of this operator, as an operator from $B_{q,1}$ to $B_{q,1}$ $1 < q < \infty$, we have

$$\begin{aligned} \|P_{\mathcal{Y}_n}\|_{B_{q,1}\to B_{q,1}} &= \sup_{\|f\|_{B_{q,1}}\leq 1} \Big\| \sum_{|k|\in\mathcal{Y}_n} \widehat{f}(k) e^{i(k,x)} \Big\|_{B_{q,1}} \\ &\asymp \sup_{\|f\|_{B_{q,1}}\leq 1} \sum_{s\in D_n} \|\delta_s(f)\|_q \leq \sup_{\|f\|_{B_{q,1}}\leq 1} \sum_{s\in\mathbb{N}^d} \|\delta_s(f)\|_q \leq C_8, \quad C_8 > 0. \end{aligned} \tag{24}$$

Therefore, in view of (24), for an arbitrary polynomial $P(\Theta_m) \subset B_{q,1}$ and $f \in B^{\Omega}_{p,\theta} \cap \mathcal{T}(\mathcal{Y}_n)$ we can write

$$\|f - P(\Theta_m)\|_{B_{q,1}} \gg \|P_{\mathcal{Y}_n}(f - P(\Theta_m))\|_{B_{q,1}} = \|f - P_{\mathcal{Y}_n}P(\Theta_m)\|_{B_{q,1}}.$$

Then for the trigonometric system $T = \{e^{i(k,x)}\}_{k\in\mathbb{Z}^d}$ the following relation holds

$$e_m(B^{\Omega}_{p,\theta} \cap \mathcal{T}(\mathcal{Y}_n), T)_{B_{q,1}} \gg e_m(B^{\Omega}_{p,\theta} \cap \mathcal{T}(\mathcal{Y}_n), \{e^{i(k,x)}\}_{|k|\in\mathcal{Y}_n})_{B_{q,1}}. \tag{25}$$

Thus, taking into account (23) and (24), we obtain

$$e_m(B^{\Omega}_{p,\theta})_{B_{q,1}} \gg e_m(B^{\Omega}_{p,\theta} \cap \mathcal{T}(\mathcal{Y}_n), \{e^{i(k,x)}\}_{|k|\in\mathcal{Y}_n})_{B_{q,1}}. \tag{26}$$

In what follows we consider two cases.
Case 1. Let $1 \leq \theta < \infty$. Then for any polynomial $t \in \mathcal{T}(\mathcal{Y}_n)$ we get

$$\begin{aligned} \|t\|_{B^{\Omega}_{p,\theta}} &\asymp \Big(\sum_{s\in D_n} \omega^{-\theta}(2^{-\|s\|_1}) \|\delta_s(t)\|_p^{\theta} \Big)^{\frac{1}{\theta}} \asymp \omega^{-1}(2^{-n}) \Big(\sum_{s\in D_n} \|A_s(t)\|_p^{\theta} \Big)^{\frac{1}{\theta}} \\ &\leq \omega^{-1}(2^{-n}) \max_{s\in D_n} \|A_s(t)\|_p \Big(\sum_{s\in D_n} 1 \Big)^{\frac{1}{\theta}} \asymp \omega^{-1}(2^{-n}) \max_{s\in D_n} \|A_s(t)\|_p n^{\frac{d-1}{\theta}} \\ &\leq \omega^{-1}(2^{-n}) n^{\frac{d-1}{\theta}} \max_{s\in D_n} \|A_s(t)\|_{\infty} = \omega^{-1}(2^{-n}) n^{\frac{d-1}{\theta}} \|t\|_{B_{\infty,\infty}}. \end{aligned} \tag{27}$$

Case 2. Let $\theta = \infty$. Then for $t \in \mathcal{T}(\mathcal{Y}_n)$ we get

$$\begin{aligned} \|t\|_{B^{\Omega}_{p,\infty}} &\asymp \max_{s\in D_n} \frac{\|\delta_s(t)\|_p}{\omega(2^{-\|s\|_1})} \asymp \max_{s\in D_n} \frac{\|A_s(t)\|_p}{\omega(2^{-\|s\|_1})} \\ &\asymp \omega^{-1}(2^{-n}) \max_{s\in D_n} \|A_s(t)\|_p \leq \omega^{-1}(2^{-n}) \|t\|_{B_{\infty,\infty}}. \end{aligned} \tag{28}$$

From (27) and (28) we conclude that the following embeddings

$$C_9\omega(2^{-n})n^{-\frac{d-1}{\theta}}\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}} \subset B^{\Omega}_{p,\theta}\cap\mathcal{T}(\mathcal{Y}_n), \quad 1\le\theta<\infty, \ C_9>0, \tag{29}$$

$$C_{10}\omega(2^{-n})\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}} \subset B^{\Omega}_{p,\infty}\cap\mathcal{T}(\mathcal{Y}_n), \quad \theta=\infty, \ C_{10}>0. \tag{30}$$

hold. Thus, using (29) and Theorem C, for $\Phi=\left\{e^{i(k,x)}\right\}_{|k|\in\mathcal{Y}_n}$ and $l=|\mathcal{Y}_n|$, we obtain

$$\begin{aligned} e_m(B^{\Omega}_{p,\theta})_{B_{q,1}} &\ge e_m\left(B^{\Omega}_{p,\theta}\cap\mathcal{T}(\mathcal{Y}_n), \left\{e^{i(k,x)}\right\}_{|k|\in\mathcal{Y}_n}\right)_{B_{q,1}} \\ &\gg \omega(2^{-n})n^{-\frac{d-1}{\theta}}e_m\left(\mathcal{T}(\mathcal{Y}_n)_{B_{\infty,\infty}}, \left\{e^{i(k,x)}\right\}_{|k|\in\mathcal{Y}_n}\right)_{B_{1,1}} \\ &\gg \omega(2^{-n})n^{-\frac{d-1}{\theta}}n^{d-1} = \omega(2^{-n})n^{(d-1)(1-\frac{1}{\theta})}, \quad 1\le\theta<\infty. \end{aligned} \tag{31}$$

Similarly, in the case $\theta=\infty$, taking into account (30), we have

$$e_m(B^{\Omega}_{p,\infty})_{B_{q,1}} \ge \omega(2^{-n})n^{d-1}. \tag{32}$$

The lower estimate in (7) is established. Theorem 1 is proved. □

In the following statement, in addition to the quantity $e_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$ we consider the best orthogonal trigonometric approximation $e^{\perp}_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$ in the case $1<q<p<\infty$.

Theorem 2. *Let $d\ge 2$, $1<q<p<\infty$, $1\le\theta\le\infty$, and $\Omega(t)=\omega\left(\prod_{j=1}^d t_j\right)$, where $\omega(\tau)$ satisfies condition (S^{α}) with some $\alpha>0$ and condition (S_l). Then for any natural numbers m and n, such that $m\asymp 2^n n^{d-1}$, the following relations hold*

$$e_m(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp e^{\perp}_m(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp \omega(2^{-n})n^{(d-1)(1-\frac{1}{\theta})}. \tag{33}$$

Proof. According to the relation (6), it is sufficient to prove the upper estimate in (33) for the quantity $e^{\perp}_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$ and the lower estimate for $e_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$.

So, using the result of Theorem A, for $m\asymp 2^n n^{d-1}$ we can write

$$e^{\perp}_m(B^{\Omega}_{p,\theta})_{B_{q,1}} \ll \mathcal{E}_{Q_n}(B^{\Omega}_{p,\theta})_{B_{q,1}} \asymp \omega(2^{-n})n^{(d-1)(1-\frac{1}{\theta})}.$$

The lower estimate for the quantity $e_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$ can be obtained using similar considerations as in the proof of Theorem 1, namely, as in the proof of estimates (31) and (32). Theorem 2 is proved. □

At the end of the work we will make some comments on the obtained results. Comparing the estimate obtained in Theorem 1 with the estimate of the corresponding quantity $e_m(B^{\Omega}_{p,\theta})_q$ (see [39]), we observe that the best m-term trigonometric approximations of the classes $B^{\Omega}_{p,\theta}$ in the spaces $B_{q,1}$ and L_q differ in order. A similar situation is under the conditions of Theorem 2. In addition, we note that Theorem 2 considers, in particular, the case $1<q<p<2$, which for the corresponding quantities in the space L_q still remains unstudied. It is also important that the orders of approximation characteristics established in Theorem 2 are realized by approximation by trigonometric polynomials from the set $T(Q_n)$, $m\asymp 2^n n^{d-1}$, more precisely by step hyperbolic Fourier sums $S_{Q_n}(f)$, $f\in B^{\Omega}_{p,\theta}$ (see Theorem A).

Regarding Theorem 1, the situation is fundamentally different in this case. Comparing the estimates obtained in Theorem 1 and Theorem B under the condition $m\asymp 2^n n^{d-1}$, we conclude that the mentioned polynomials do not realize the orders of quantities $e_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$. In addition, the question on the orders of the best orthogonal trigonometric approximations $e^{\perp}_m(B^{\Omega}_{p,\theta})_{B_{q,1}}$ for $2\le p<q<\infty$ remains open.

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В роботі одержано точні за порядком оцінки найкращих m-членних тригонометричних наближень і найкращих ортогональних тригонометричних наближень функцій із класів типу Нікольського-Бєсова $B^{\Omega}_{p,\theta}$ у підпросторах Лебега $B_{q,1}$ для деяких співвідношень між параметрами p та q. Показано, що в розглянутих випадках згадані апроксимаційні характеристики класів $B^{\Omega}_{p,\theta}$ у просторах $B_{q,1}$ і L_q є різними за порядком. Крім цього також виявлено, що при $1 < q < p < \infty$, на противагу випадку $2 \leq p < q < \infty$, одержані порядки цих величин реалізуються за наближення функцій з класів $B^{\Omega}_{p,\theta}$ їхніми східчастими гіперболічними сумами Фур'є з відповідною кількістю гармонік.

Ключові слова і фрази: клас типу Нікольського-Бєсова, східчасто-гіперболічна сума Фур'є, найкраще m-членне тригонометричне наближення, найкраще ортогональне тригонометричне наближення.