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Continuous lattices of quotient objects in the category of compacta

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We describe approximation relations "way below" and "way above" on the poset of quotient objects of a compact Hausdorff space. Necessary and sufficient conditions are established for a quotient object to be the exact bound of quotient objects approximating it. It is shown that such objects form algebraic or dually algebraic lattices.

Key words and phrases: "way below" relation, "way above" relation, approximation relation, quotient object, compactum.

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Introduction

Subobjects and quotient objects are classical notions of category theory that are of great importance and with numerous applications [1]. In the most "convenient" topological category — the category of compact Hausdorff spaces — the subobjects of a space are represented by its closed subsets, while quotient objects correspond to closed equivalence relations on it. Both families carry obvious partial orders, which makes them natural objects of study in domain theory [3]. In particular, it is worthwhile to describe approximation relations "way below" and "way above", which proved useful, e.g., in topological algebra and denotational semantics of programming languages. These relations on posets can be "rich" or "poor" and have rather complex structure [7]. If they are "sufficiently rich" (such posets are called *continuous*), then they determine topologies (named after Scott and Lawson) [8], which often coincide with topologies defined in other reasonable ways.

For example, the dual Lawson topology on the set of non-empty subsets of a compact Hausdorff space X is the well-known Vietoris topology. The obtained topological space is compact Hausdorff as well. It is called the *hyperspace* of X and denoted $\Gamma(X)$ in [3], but the notation $\exp X$ is much more common. There are myriads of papers devoted to the hyperspace construction $\exp$, in particular, because it gives rise to a normal functor in the category of compact Hausdorff spaces [2], which simplifies application of methods from the theory of infinite-dimensional manifolds.

In this regard, posets of quotient objects of compact Hausdorff spaces have received much less attention. Topologies on these posets and their subsets have been studied in a series of

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works by K. Koporkh and M. Zarichnyi (see, e.g., [6]). O. Nykyforchyn constructed a functor of *zero-dimensional* subobjects of *zero-dimensional* compact Hausdorff spaces [9]. The presented paper clarifies why this additional requirement is essential. We describe the quotient objects that can be "well approximated" from below or from above. These two classes of objects form, respectively, continuous and dually continuous lattices, which are in fact algebraic and dually algebraic.

1 Preliminaries and notations. The poset of quotient objects of a compactum

In what follows, all mappings are assumed to be continuous. A *compactum* is a compact Hausdorff topological space, and *Comp* is the category of compacta, i.e. its objects are compacta and arrows are mappings between them.

We denote by $\mathrm{Iso}(X)$ the open set of all isolated points of a compactum X. Its complement in X consists of all limit points. The *diagonal* of X is the set $\Delta_X = \{(x,x) : x \in X\} \subset X \times X$. For a subset $A \subset X \times X$ the *equivalence closure* of A is the least equivalence relation on X that contains A and is closed in the product topology (for the sake of brevity we include closedness in the definition).

Recall the definition of a quotient object [1]. In an arbitrary category $\mathcal{C}$, an epimorphism is a right-cancellable arrow, i.e. an arrow $f : X \to Y$ such that for all arrows $g_1, g_2 : Y \to Z$ the equality $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. In the category of compacta *Comp* the epimorphisms are precisely the continuous surjections.

Epimorphisms $f : X \to Y$ and $g : X \to Y'$ are considered equivalent if there exists an isomorphism $\varphi : Y \to Y'$ such that $\varphi \circ f = g$. A *quotient object* of an object X is defined as the equivalence class of epimorphisms with domain X.

Note that, in general, in a category that is not small (as is the case for most categories arising in algebra, topology, and analysis), the epimorphisms with a fixed domain do not form a set. However, it is often possible to associate each equivalence class with a canonical representative, and these representatives do form a set. Consequently, in such cases the collection of quotient objects of a fixed object also forms a set.

For example, in the category *Comp* of compact Hausdorff spaces, a quotient object — that is, the equivalence class of an epimorphism (a continuous surjection) $f : X \to Y$ — is uniquely determined by the equivalence relation $\sim_f$ on X given by

$$x_1 \sim_f x_2 \iff f(x_1) = f(x_2).$$

This relation is called the *kernel* of f and denoted $\ker f$. By Alexandroff's well-known lemma, this provides a bijection between the quotient objects of X and the equivalence relations that are closed subsets of $X \times X$ (see [6] for details). We denote by $E(X)$ the set of all closed equivalence relations on a compactum X.

In [6], a partial order on the set $\Phi(X)$ of quotient objects of a compact space X was introduced. We write $[f] \preccurlyeq [g]$ (equivalently, $[g] \succcurlyeq [f]$) if there exists an epimorphism φ such that $\varphi \circ g = f$, or, equivalently, if $\ker f \supset \ker g$ (that is, the equivalence classes of $\ker f$ are "coarser", being unions of equivalence classes of $\ker g$). Equivalent epimorphisms have equal kernels, so ker is well defined on $\Phi(X)$ as well by defining the kernel of a class to be the kernel of an arbitrary element of this class. Then the mapping of posets $\ker : (\Phi(X), \preccurlyeq) \to (E(X), \subset)$ is an order antiisomorphism.

Thus, the least element of $\Phi(X)$ is the trivial quotient object, namely the equivalence class of a constant map, whereas the greatest element is the equivalence class of the identity map. It was proved in [6] that $(\Phi(X), \preccurlyeq)$ is a complete lattice. Least upper bounds can be easily calculated through kernels, namely the kernel of the supremum of a family of classes $\{[f_i] : i \in \mathcal{I}\}$ is the intersection of all $\ker f_i$, $i \in \mathcal{I}$, whereas the kernel of the infimum of $\{[f_i] : i \in \mathcal{I}\}$ is the *equivalence closure* of the union $\bigcup_{i\in\mathcal{I}} \ker f_i$, i.e. the intersection of all closed equivalence relations that are supersets of this union.

2 Definitions and characterizations of "way below" and "way above" relations

The results presented in this short section first appeared, in part, in the article [5], which was published in Ukrainian. For completeness and the reader's convenience, we include them here.

The definitions and notation presented below follow [3]. The relation "way below", also known as the approximation relation, is defined in an arbitrary partially ordered set $(S, \leqslant)$. We say that x_0 is *way below* x, or that x_0 *approximates* x *from below*, and write $x_0 \ll x$, if for every directed subset D, whenever $x \leqslant \sup D$ (in particular, this least upper bound exists), there is an element $d \in D$ such that $x_0 \leqslant d$.

Clearly $x_0 \ll x$ implies $x_0 \leqslant x$ (it is sufficient to put $D = \{x\}$), but the converse is false.

If, for every element $x \in S$, the set of all elements $x_0 \ll x$ is directed and has supremum equal to x, then $(S, \leqslant)$ is called *continuous*.

A partially ordered set $(S, \leqslant)$ in which every directed subset has a least upper bound is called a *directed complete partial order* (*dcpo*). A continuous dcpo is called a *domain* [3]. In particular, all *continuous lattices*, i.e. complete lattices that are continuous, are domains. Domains constitute the principal objects of study in the denotational semantics of programming languages mentioned above.

We now consider the relation "$\ll$" on the set $\Phi(X)$ of quotient objects of a compact space X. Throughout the sequel, all spaces are assumed to be compact Hausdorff.

Lemma 1. *If the image X_0 of an epimorphism $f_0 : X \to X_0$ is infinite, then the class $[f_0]$ is not way below any class $[f] \in \Phi(X)$.*

Proof. Since X_0 is infinite, we may choose a sequence $(x_n)_{n=1}^{\infty}$ of points in X whose images under f_0 are pairwise distinct. By compactness of X, the set $\{x_n : n = 1, 2, \ldots\}$ has a limit point x_0. Let $\mathcal{V}$ be the set of all closed neighborhoods of x_0, and let $q_V : X \to X/_V$ be the quotient map obtained by collapsing a neighborhood $V \in \mathcal{V}$ to a point. Then we obtain a directed set Q of quotient objects $\{[q_V] : V \in \mathcal{V}\}$. It is easy to see that its supremum is the class of the identity map, which is the greatest element of $\Phi(X)$. Hence $[f] \preccurlyeq \sup Q$. On the other hand, for every $V \in \mathcal{V}$ there are $x_i \neq x_j$ in V, so we have $q_V(x_i) = q_V(x_j)$, whereas $f_0(x_i) \neq f_0(x_j)$, and therefore $[f_0] \not\preccurlyeq [q_V]$. This proves that $[f_0] \not\ll [f]$. $\square$

Lemma 2. *If the image X_0 of an epimorphism $f_0 : X \to X_0$ is finite, then the class $[f_0]$ is way below every class $[f] \in \Phi(X)$ that succeeds it.*

Proof. Suppose that $X_0 = \{x_1, x_2, \dots, x_n\}$ is an n-point compact space. Then X is partitioned into nonempty clopen subsets $X_k = (f_0)^{-1}(x_k)$, $k = 1, 2, \dots, n$. If $f : X \to Z$ succeeds f_0, then $f(X_k) \cap f(X_l) = \varnothing$ whenever $k \neq l$.

Let $D = \{[d_i] : i \in \mathcal{I}\}$, where $d_i \to Y_i$, be a directed family of quotient objects with $\sup D = [d] \succcurlyeq [f]$, where $d \to Y$. Then the maps d_i collectively separate every pair of points separated by f. Hence, for each k, the family of closed subsets $\{d_i^{-1}(d_i(X_k)) : i \in \mathcal{I}\}$ is filtered, every member contains X_k, and their intersection is contained in the clopen subset $f^{-1}(f(X_k)) = X_k$. Therefore, by the Shura-Bura lemma, there exists an index i_k such that $d_{i_k}^{-1}(d_{i_k}(X_k)) = X_k$.

Finally, choose any quotient object $[d_j]$ in the family D that succeeds each of $[d_{i_1}]$, $[d_{i_2}]$, $\dots$, $[d_{i_n}]$. Then $d_j^{-1}(d_j(X_k)) = X_k$ for every $1 \leqslant k \leqslant n$, that is, the images $d_j(X_k)$ are pairwise disjoint. Consequently, $[f_0] \preccurlyeq [d_j]$, completing the proof that $[f_0] \ll [f]$. ◻

Combining the two lemmas, we obtain the following characterization of the "way below" relation on $\Phi(X)$.

Theorem 1. *The class $[f_0]$ is way below the class $[f]$ in $\Phi(X)$ if and only if $[f_0] \preccurlyeq [f]$ and the image of f_0 is finite.*

Corollary 1. *If the image of an epimorphism $f : X \to Y$ of compacta is connected, then the class of a constant mapping is the unique subobject of X that is way below $[f]$.*

The following definitions are dual to the preceding ones [3]. We say that x_0 is *way above x*, or that x_0 *approximates x from above*, and write $x_0 \gg x$, if for every filtered subset D, whenever $x \geq \inf D$ (in particular, this infimum exists), there exists an element $d \in D$ such that $x_0 \geq d$ (see [3]).

If, for every element $x \in S$, the set of all elements $x_0 \gg x$ is filtered and has infimum equal to x, then $(S, \leqslant)$ is called *dually continuous*.

A partially ordered set $(S, \leqslant)$ in which every filtered subset has an infimum is called *dually directed complete* (*ddcpo*), or, equivalently, *filtered complete* (*fcpo*).

The following notion is nonstandard and is introduced solely for the purposes of the present paper.

Definition 1. *The* support *of an equivalence relation "$\sim$" on a set X is the set*

$$\operatorname{supp}(\sim) = \{x \in X : \exists\, y \in X,\ y \neq x,\ y \sim x\}.$$

In other words, the support of an equivalence relation consists of all elements that are equivalent to some other elements. Let $E^f(X)$ denote the set of all equivalence relations on X having finite supports. Obviously all of them are closed. The following is an easy observation.

Lemma 3. *The equivalence closure of the union of two relations $\sim_1, \sim_2 \in E^f(X)$ and the intersection of $\sim_1$ and $\sim_2$ also have finite supports.*

By the Alexandroff lemma, every relation $\sim\, \in E^f(X)$ on a compact space X determines a quotient object. The preceding lemma therefore implies that these quotient objects form a sublattice of the lattice $(\Phi(X), \preccurlyeq)$, and hence a filtered subset $\Phi^f(X)$ of $\Phi(X)$. Its infimum is the trivial quotient object (the equivalence class of a constant map). Therefore, $\inf \Phi^f(X) \preccurlyeq [f]$ for every $[f] \in \Phi(X)$. If $[f_0] \gg [f]$, then there must exist some $[q] \in \Phi_f(X)$ such that $[q] \preccurlyeq [f_0]$. On the other hand, $\Phi^f(X)$ is an upper set, that is, every element of $\Phi(X)$ lying above an element of $\Phi^f(X)$ also belongs to $\Phi^f(X)$. Consequently, only elements of $\Phi^f(X)$ can be way above arbitrary elements of $\Phi(X)$.

Theorem 2. *A class $[f_0]$ is way above a class $[f]$ in $\Phi(X)$ if and only if $[f_0] \succcurlyeq [f]$ and the equivalence relation $\ker f_0$ has finite support, all points of $\operatorname{supp}(\ker f_0)$ being isolated in X.*

Proof. The necessity of the condition $[f_0] \in \Phi^f(X)$ has already been established. Suppose that a limit point $x \in X$ belongs to the support of $\ker f_0$. Then there exists a point $y \neq x$ such that $(x, y) \in \ker f_0$. Let $\mathcal{V}$ denote the family of all open neighborhoods of x, and let $q_V \to X/_{X \setminus V}$ be the quotient map collapsing $X \setminus V$ to a point for each $V \in \mathcal{V}$. Since $[q_{V \cap V'}]$ precedes both $[q_V]$ and $[q_{V'}]$, the family $\{[q_V] : V \in \mathcal{V}\}$ is filtered in $\Phi(X)$. Its infimum is the trivial quotient object and therefore precedes $[f]$. On the other hand, $[q_V] \not\preccurlyeq [f_0]$ for every $V \in \mathcal{V}$, showing that $[f_0] \gg [f]$ is impossible.

To prove sufficiency, assume that $[f_0] \succcurlyeq [f]$ and that $\ker f_0$ has a finite support $x_1, x_2, \ldots, x_n$ consisting entirely of isolated points. Let $D \subset \Phi(X)$ be a filtered subset whose infimum precedes $[f]$. Then, for every pair $(x_i, x_j) \in \ker f_0$, $1 \leqslant i < j \leqslant n$, we have $f(x_i) = f(x_j)$, and therefore there exists an element $[d_{ij}] \in D$ such that $d_{ij}(x_i) = d_{ij}(x_j)$. Since D is filtered, there exists an element $[d] \in D$ preceding every $[d_{ij}]$. It follows that $[d] \preccurlyeq [f_0]$, and hence $[f_0] \gg [f]$. □

3 Quotient objects that are exact bounds of approximating elements

Corollary 1 clearly shows that not every element of $\Phi(X)$ is the least upper bound of all elements way below it. A similar fact is true for the "way above" relation.

In this section, we describe the quotient objects that *can* be represented as suprema of elements way below them, or as infima of elements way above them.

Theorem 3. *A class $[f]$ in $\Phi(X)$ is the supremum of all classes $[f_0]$ way below $[f]$ if and only if the image of f is zero-dimensional.*

Proof. Consider again the sublattice $\Phi_f(X) \subset \Phi(X)$ of the classes of all epimorphisms from X to finite spaces. By Theorem 1, if $[f_0] \ll [f]$ for $f : X \to Y$, then $[f_0] \in \Phi_f(X)$. Therefore, if $[f]$ is the supremum of the set $\{[f_i] : f_i : X \to X_i,\ i \in \mathcal{I}\}$ of all classes way below it, then all X_i are finite. Let $\hat{X}$ be the image of the diagonal product $\triangle_{i \in \mathcal{I}} f_i : X \to \prod_{i \in \mathcal{I}} X_i$ of finite discrete (hence zero-dimensional) spaces. Then $\hat{X}$ is a zero-dimensional compactum as well. It is easy to see that the class $[\hat{f}]$ of the restriction $\hat{f}$ of $\triangle_{i \in \mathcal{I}} f_i$ to a mapping $X \to \hat{X}$ is also the supremum of all $[f_i]$. Hence, f and $\hat{f}$ are equivalent. Therefore, $Y \cong \hat{X}$ is zero-dimensional. This shows the necessity of zero-dimensionality.

To prove its sufficiency, consider an epimorphism f from X to a zero-dimensional compactum Y. Let $\mathcal{COP}(Y)$ be the set of all finite clopen partitions of Y. For a partition $P \in \mathcal{COP}(Y)$, let Y_P be the quotient space arising from collapsing each element of P to a point, and let $q_P : Y \to Y_P$ be the quotient map. It is obvious that the class of each composition $q_P \circ f$ is way below $[f]$, and $[f] = \sup\{[q_P \circ f] : P \in \mathcal{COP}(Y)\}$, which completes the proof. $\square$

Remark 1. *Even if a compactum X is zero-dimensional, this does not guarantee that every element $[f] \in \Phi(X)$ is the supremum of all classes $[f_0]$ way below $[f]$. For example, all metrizable compacta, including connected ones, are continuous images of the Cantor set, but only the class of a constant mapping can be way below the class of a mapping with a connected image.*

Theorem 4. *A class $[f]$ in $\Phi(X)$ is the infimum of all classes $[f_0]$ way above $[f]$ if and only if the kernel of f is the equivalence closure in $X \times X$ of some equivalence relation on the set* $\mathrm{Iso}(X)$ *of all isolated points in X.*

Proof. Let $[f]$ be the infimum of the family $\{[f_i] : i \in \mathcal{I}\}$ of classes $[f_i] \gg [f]$. The supports of all $\ker f_i$ are finite subsets of $\mathrm{Iso}(X)$, therefore all differences $\ker f_i \setminus \Delta_X$ are finite subsets of $\mathrm{Iso}(X) \times \mathrm{Iso}(X)$. For an equivalence relation $\sim\, \subset X \times X$, to contain all $\ker f_i$ is equivalent to containing the union

$$\bigcup_{i\in\mathcal{I}} \big(\ker f_i \setminus \Delta_X\big) \subset \mathrm{Iso}(X) \times \mathrm{Iso}(X).$$

There is the least equivalence relation $\sim^*$ on $\mathrm{Iso}(X)$ that contains this union, so $\ker f \subset X \times X$ is the equivalence closure of $\sim^*$, which completes the proof of necessity.

To show sufficiency, assume that $\ker f$ is the equivalence closure in $X \times X$ of an equivalence relation $\sim^*$ on $\mathrm{Iso}(X)$. Denote by $\mathcal{FI}(X)$ the set of all finite subsets of $\mathrm{Iso}(X)$. For all $F \in \mathcal{FI}(X)$ the equivalence relation $\Delta_X \cup \big(\sim^* \cap (F \times F)\big)$ has a finite support contained in $\mathrm{Iso}(X)$, therefore is the kernel of an epimorphism f_F such that $[f_F] \gg [f]$. The union $\bigcup_{F\in\mathcal{FI}(X)} \ker f_F$ is equal to $\sim^* \cup \Delta_X$, hence $[f]$ is the infimum of all $[f_F]$, $F \in \mathcal{FI}(X)$. $\square$

Remark 2. *Recall that the topological closure in $X \times X$ of an arbitrary equivalence relation on* $\mathrm{Iso}(X)$ *need not be an equivalence relation on X, nor is every equivalence relation on* $\mathrm{Iso}(X)$ *the restriction of a closed equivalence relation on X, so this theorem does not provide a bijection with the set of all equivalence relations on* $\mathrm{Iso}(X)$.

The condition of the preceding theorem is not easy to verify, so we provide an alternative formulation. We say that a function $f : A \to B$ *factors* through a function $g : A \to C$ if there is a function $h : C \to B$ such that $h \circ g = f$, or, equivalently, $g(a_1) = g(a_2)$ implies $f(a_1) = f(a_2)$ for all $a_1 = a_2$ (no continuity is taken into account here).

Proposition 1. *The kernel of an epimorphism $f : X \to Y$ of compacta is the equivalence closure in $X \times X$ of some equivalence relation $\sim^*$ on the set* $\mathrm{Iso}(X)$ *of all isolated points in X if and only if, for each continuous function φ from X to $I = [0, 1]$, if the restriction $\varphi|_{\mathrm{Iso}(X)}$ factors through f, then φ factors through f as well.*

Proof. Necessity. Observe that the restriction $\varphi|_{\mathrm{Iso}(X)}$ of a function $\varphi : X \to I$ factors through f if and only if $\ker \varphi \supset \sim^*$, which is in turn equivalent to $\ker \varphi \supset \ker f$. This implies that φ factors through f.

Sufficiency. Put $\sim^* = \ker f \cap \big(\mathrm{Iso}(X) \times \mathrm{Iso}(X)\big)$ and assume that the equivalence closure of $\sim^*$ is $\ker g$ for an epimorphism $g : X \to Z$. Embed Z into the Tychonoff cube $I^{C(Z,\mathbb{R})}$

by the diagonal product Ψ of all continuous functions $\psi_i \in C(Z, \mathbb{R})$, $i \in \mathcal{I}$. Kernels of all compositions $\psi_i \circ g : X \to I$ contain $\sim^*$, therefore by the assumption $\psi_i \circ g = \theta_i \circ f$ for continuous functions $\theta_i : Y \to I$. Then

$$\Psi \circ g = \mathop{\triangle}_{i \in \mathcal{I}} \psi_i \circ g = \mathop{\triangle}_{i \in \mathcal{I}} \theta_i \circ f.$$

This implies that the mapping

$$h = \Psi^{-1} \circ \mathop{\triangle}_{i \in \mathcal{I}} \theta_i : Y \to Z$$

satisfies $h \circ f = g$, hence $f \preccurlyeq g$. By the minimality of g, the mappings f and g are equivalent, therefore $\ker f$ is the equivalence closure of $\sim^*$ as well. $\square$

For an epimorphism $f : X \to Y$, call a subset $A \subset X$ *f-saturated* if, for all $x, x' \in X$, $f(x) = f(x')$ and $x \in A$ imply $x' \in A$. If $f(x) = f(x')$ and $x \in A$ implies $x' \in A$ for all $x, x' \in \mathrm{Iso}(X)$ only, then A is called *f-iso-saturated*.

Let us observe that $f^{-1}(f(A))$ is the least f-saturated superset of A. We call it the *f-saturation* of A. The least f-iso-saturated superset of A (its *f-iso-saturation*) is equal to $A \cup \big(f^{-1}(f(A)) \cap \mathrm{Iso}(X)\big)$.

Definition 2. *For an epimorphism $f : X \to Y$ we call a non-empty family $\mathcal{C}$ of subsets of X* f-iso-expandable *if:*

(a) all elements of $\mathcal{C}$ are closed and f-iso-saturated;

(b) for each open neighborhood U of an element $F \in \mathcal{C}$ there exists $G \in \mathcal{C}$ such that

$$F \subset \mathrm{Int}\, G \subset G \subset U.$$

Example 1. *For a continuous function φ from X to $I = [0, 1]$ such that the restriction $\varphi|_{\mathrm{Iso}(X)}$ factors through an epimorphism $f : X \to Y$, and a dense subset $S \subset (0, 1]$, the family*

$$\mathcal{C} = \{\varphi^{-1}([s, 1]) : s \in S\}$$

is f-iso-expandable.

Proposition 2. *The condition of Proposition 1 can be equivalently formulated as follows: all elements of each f-iso-expandable family $\mathcal{C}$ are f-saturated.*

Proof. Sufficiency. Use the idea of the above example. If the restriction $\varphi|_{\mathrm{Iso}(X)}$ of a continuous function $\varphi : X \to [0, 1]$ factors through an epimorphism $f : X \to Y$, then the family

$$\mathcal{C} = \{\varphi^{-1}([s, 1]) : s \in (0, 1]\}$$

is f-iso-expandable. By assumption, all preimages $\varphi^{-1}([s, 1])$ are f-saturated, hence φ factors through f.

Necessity. Let there be an f-iso-expandable family $\mathcal{C}$ such that some $F_1 \in \mathcal{C}$ is not f-saturated, i.e. there are $x_0 \notin F_1$ and $x_1 \in F_1$ such that $f(x_0) = f(x_1)$. By property *(b)* there is $F_0 \in \mathcal{C}$ such that $F_1 \subset \mathrm{Int}\, F_0 \subset F_0 \subset X \setminus \{x_0\}$. Choose $F_{\frac{1}{2}} \in \mathcal{C}$ such that

$$F_1 \subset \mathrm{Int}\, F_{\frac{1}{2}} \subset F_{\frac{1}{2}} \subset \mathrm{Int}\, F_0 \subset F_0,$$

then choose $F_{\frac{1}{4}}, F_{\frac{3}{4}} \in \mathcal{C}$ so that

$$F_1 \subset \operatorname{Int} F_{\frac{3}{4}} \subset F_{\frac{3}{4}} \subset \operatorname{Int} F_{\frac{1}{2}} \subset F_{\frac{1}{2}} \subset \operatorname{Int} F_{\frac{1}{4}} \subset F_{\frac{1}{4}} \subset \operatorname{Int} F_0 \subset F_0,$$

etc. We obtain a subfamily

$$\left\{F_q : q = \frac{k}{2^n},\ n \in \mathbb{N},\ 0 \leqslant k \leqslant 2^n\right\} \subset \mathcal{C}$$

such that $p > q$ implies $F_p \subset \operatorname{Int} F_q$. Following the usual proof of the Brouwer-Tietze-Urysohn theorem, we define a continuous function $\varphi : X \to I$ by the formula

$$\varphi(x) = \begin{cases} \sup\{q : x \in F_q,\ q = \frac{k}{2^n},\ n \in \mathbb{N},\ 0 \leqslant k \leqslant 2^n\}, & x \in F_0, \\ 0, & x \notin F_0. \end{cases}$$

Note that all F_q are f-iso-saturated. Therefore, the restriction $\varphi|_{\operatorname{Iso}(X)}$ factors through f, but $\varphi(x_0) = 0 \neq \varphi(x_1) = 1$, hence φ does not factor through f, which completes the proof. □

We now clarify the properties of epimorphisms that satisfy the requirements presented above.

Proposition 3. *If the kernel of an epimorphism $f : X \to Y$ of compacta is the equivalence closure in $X \times X$ of some equivalence relation $\sim^*$ on the set $\operatorname{Iso}(X)$ of all isolated points in X, then $f^{-1}(f(x)) = \{x\}$ for all $x \in X \setminus \operatorname{Cl}\operatorname{Iso}(X)$.*

In other words, all points outside the closure of $\operatorname{Iso}(X)$ are *points of univalence* [10].

Proof. Let $\ker f$ be the equivalence closure of an equivalence relation $\sim^*$ on $\operatorname{Iso}(X)$. Observe that $\big(\ker f \cap \big(\operatorname{Cl}\operatorname{Iso}(X) \times \operatorname{Cl}\operatorname{Iso}(X)\big)\big) \cup \Delta_X$ is a closed equivalence relation, and

$$\sim^* \subset \Big(\ker f \cap \big(\operatorname{Cl}\operatorname{Iso}(X) \times \operatorname{Cl}\operatorname{Iso}(X)\big)\Big) \cup \Delta_X \subset \ker f,$$

hence by minimality of the equivalence closure we obtain

$$\Big(\ker f \cap \big(\operatorname{Cl}\operatorname{Iso}(X) \times \operatorname{Cl}\operatorname{Iso}(X)\big)\Big) \cup \Delta_X = \ker f.$$

This means that $f(x) = f(x')$, $x \neq x'$ is possible only if $x, x' \in \operatorname{Cl}\operatorname{Iso}(X)$. □

Therefore it is sufficient to consider such epimorphisms only from *iso-dense* compacta [4], i.e. from the compact Hausdorff spaces with dense sets of isolated points.

Proposition 4. *Let an epimorphism f from an iso-dense compactum X to a compactum Y fails to satisfy the conditions of Proposition 1. Then f is not open.*

Proof. Let f be open. By assumption there is a continuous function $\varphi : X \to I$ such that the restriction $\varphi|_{\operatorname{Iso}(X)}$ factors through f, but φ does not factor through f, i.e. there exist $x_0, x_1 \in X$ such that $f(x_0) = f(x_1)$ but $\varphi(x_0) \neq \varphi(x_1)$. Without loss of generality we can assume $\varphi(x_0) = 0$, $\varphi(x_1) = 1$. The sets $A = \varphi^{-1}\big([0, \frac{1}{3})\big) \ni x_0$ and $B = \varphi^{-1}\big((\frac{2}{3}, 1]\big) \ni x_1$ are open in X and disjoint. The iso-density of X implies that, for the open intersections $A_0 = A \cap \operatorname{Iso}(X)$, $B_0 = B \cap \operatorname{Iso}(X)$, the inclusions $A \subset \operatorname{Cl} A_0$ and $B \subset \operatorname{Cl} B_0$ hold. Then, by the continuity of f, we obtain

$$f(A) \subset f(\operatorname{Cl} A_0) \subset \operatorname{Cl} f(A_0), \qquad f(B) \subset f(\operatorname{Cl} B_0) \subset \operatorname{Cl} f(B_0).$$

The open images $f(A_0)$ and $f(B_0)$ are disjoint, hence $\operatorname{Cl} f(A_0) \cap f(B_0) = \varnothing$. This implies $f(A) \cap f(B_0) = \varnothing$, therefore $f(A) \cap \operatorname{Cl} f(B_0) = \varnothing$ by the openness of $f(A)$. Finally we obtain $f(A) \cap f(B) = \varnothing$, which contradicts $f(x_0) = f(x_1) \in f(A) \cap f(B)$. □

Example 2. *Let* $Y_+ = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $Y_- = \{-1, -\frac{1}{2}, -\frac{1}{3}, \ldots\}$, *and let*

$$X = Y_+ \times \{0,1\} \sqcup Y_- \times \{0,-1\} \sqcup \{(0,-1),(0,0),(0,1)\} \subset \mathbb{R}^2$$

with the subspace topology. The space X *is iso-dense, with* $\mathrm{Iso}(X)$ *equal to*

$$X \setminus \{(0,-1),(0,0),(0,1)\}.$$

Denote by f *the restriction to* X *of the projection to the first factor. Its image is equal to* $Y = Y_+ \sqcup Y_- \sqcup \{0\}$. *The kernel of* f *is the equivalence closure of the relation* $\sim^*$ *on* $\mathrm{Iso}(X)$ *defined as* $(y_1, t_1) \sim^* (y_2, t_2) \iff y_1 = y_2$. *On the other hand,* f *is not open. Therefore openness is sufficient but not necessary for an epimorphism from an iso-dense compactum to satisfy the conditions of Theorem 4.*

This example also shows that, even if f *satisfies the conditions of Theorem 4, not every closed* f*-iso-saturated subset of* X *(not belonging to an* f*-iso-expandable family) is* f*-saturated. Namely, the set* $A = Y_+ \times \{0,1\} \sqcup \{(0,0),(0,1)\}$ *is closed,* f*-iso-saturated, but not* f*-saturated.*

4 Continuity and dual continuity of the lattice of all quotient objects of a compactum and of its subsets

We now use these characterizations to find out for what compacta X *all* elements of $\Phi(X)$ satisfy the conditions of Theorem 3 or of Theorem 4.

Theorem 5. *The lattice* $\Phi(X)$ *of all quotient objects of a compactum* X *is continuous if and only if the set* $X \setminus \mathrm{Iso}(X)$ *of all limit points of* X *is finite.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the set of all limit points of X, $f : X \to Y$ be an epimorphism of compacta. Then for all open sets $U \ni \{f(x_1), f(x_2), \ldots, f(x_n)\}$ in Y the preimage $f^{-1}(Y \setminus U)$ is compact and discrete and is therefore finite. This implies that $Y \setminus U$ is finite as well. This shows that the subspace $Y \setminus \{f(x_1), f(x_2), \ldots, f(x_n)\}$ is discrete, hence Y is a zero-dimensional compactum, i.e. the condition of Theorem 3 is satisfied.

If the set of all limit points of X is infinite, then it is an easy exercise to build an epimorphism $f : X \to I$ using the Brouwer-Tietze-Urysohn theorem. Clearly f does not satisfy the condition of Theorem 3. □

Theorem 6. *The lattice* $\Phi(X)$ *of all quotient objects of a compactum* X *is dually continuous if and only if* X *is finite.*

Proof. If X is finite, then the lattice $\Phi(X)$ is finite as well. Therefore $\Phi(X)$ is dually continuous (the relation $\gg$ coincides with $\succcurlyeq$).

Otherwise X has limit points. Let $x_1 \neq x_2$ be limit points of X. Then the quotient map $f : X \to X/_{\{x_1,x_2\}}$ with the kernel $\ker f = \Delta_X \cup \{(x_1,x_2),(x_2,x_1)\}$ clearly is an epimorphism that does not satisfy the requirements of Theorem 4.

Suppose that x_0 is the unique limit point of X and $x_1 \in X \setminus \{x_0\}$, then the epimorphism $f : X \to Y$ with the kernel $\ker f = \Delta_X \cup \{(x_0,x_1),(x_1,x_0)\}$ also fails to satisfy the conditions of Theorem 4. □

As we can see, the lattice $\Phi(X)$ is "almost never" continuous or dually continuous, so we restrict attention to suitable subposets to achieve the desired properties. The simplest way to do this is to choose all elements that are "well approximable" by Theorem 3 and Theorem 4.

Denote by $\Phi_0(X)$ the set of classes $[f]$ for all epimorphisms $f : X \to Y$ such that the image $f(X)$ is zero-dimensional.

Proposition 5. *The poset $\Phi_0(X)$ is a complete lattice.*

Proof. The idea is similar to the one used in the proof of Theorem 3. For a non-empty family $\{[f_i] : i \in \mathcal{I}\}$ of the classes of epimorphisms $f_i : X \to X_i$ onto zero-dimensional compacta, the image $\hat{X}$ of the diagonal product $\triangle_{i\in\mathcal{I}} f_i : X \to \prod_{i\in\mathcal{I}} X_i$ is a closed subset of the product of zero-dimensional spaces, therefore $\hat{X}$ is a zero-dimensional compactum as well. Hence, for each subset $\{[f_i] : i \in \mathcal{I}\} \subset \Phi_0(X)$ its supremum belongs to $\Phi_0(X)$ as well. Hence $\Phi_0(X)$ is a complete upper subsemilattice of the complete lattice $\Phi(X)$ that shares the least element, which is the class of a constant mapping. This completes the proof. □

In particular, there is the supremum $[f_0]$ of *all* classes of $\Phi_0(X)$. The corresponding epimorphism $f_0 : X \to X_0$ represents the greatest zero-dimensional quotient object of X. Obviously $\Phi_0(X_0) \cong \Phi_0(X)$. Thus, it is sufficient to consider zero-dimensional images of zero-dimensional compacta only.

Theorem 7. *The lattice $\Phi_0(X)$ is continuous.*

Proof. It has been shown that each $[f] \in \Phi_0(X)$ is the least upper bound of all $[f_i] \in \Phi_f(X) \subset \Phi_0(X)$ such that $f_i \preccurlyeq f$, and $[f_i] \ll [f]$ in $\Phi(X)$. This implies that $[f_i] \ll [f]$ in $\Phi_0(X)$. These classes $[f_i]$ form an upper subsemilattice, hence a directed set, which completes the proof. □

Denote by $\Phi^0(X)$ the set of classes $[f]$ for all epimorphisms f such that $\ker f$ is the equivalence closure in $X \times X$ of some equivalence relation on the set $\mathrm{Iso}(X)$ of all isolated points in X.

Proposition 6. *The poset $\Phi^0(X)$ is a complete lattice.*

Proof. Let $\{[f_i] : i \in \mathcal{I}\}$ be a non-empty subset of $\Phi^0(X)$, then each $\ker f_i$ is the equivalence closure of some equivalence relation $\sim_i$ on $\mathrm{Iso}(X)$. Take the least equivalence relation $\sim_0$ on $\mathrm{Iso}(X)$ that contains all $\sim_i$ (assuming $0 \notin \mathcal{I}$). Then the infimum of $\{[f_i] : i \in \mathcal{I}\}$ in $\Phi(X)$ has the kernel that is the equivalence closure of $\sim_0$, therefore belongs to $\Phi^0(X)$. We have shown that $\Phi^0(X)$ is a complete lower subsemilattice of $\Phi(X)$, and the class of the identity mapping is the greatest element of both $\Phi(X)$ and $\Phi^0(X)$. Thus, $\Phi^0(X)$ is a complete lattice as well. □

Theorem 8. *The lattice $\Phi^0(X)$ is dually continuous.*

Proof. Recall that the set $\Phi^f(X)$ of all quotient objects whose kernels have finite supports consisting of isolated points only, is a sublattice of $\Phi^0(X)$. Hence, for any $[f] \in \Phi^0(X)$ the set of all elements of $\Phi^f(X)$ that succeed $[f]$ is filtered and has infimum $[f]$. □

Remark 3. *Observe that the elements of $\Phi_f(X)$ are* compact*, i.e. way below themselves, so every element of the lattice $\Phi_0(X)$ is the supremum of compact elements way below it. Such lattices are called* algebraic *[3].*

Similarly the elements of $\Phi^f(X)$ are dually compact*, i.e. way above themselves, so every element of the lattice $\Phi^0(X)$ is the infimum of dually compact elements way above it. Hence, the lattice $\Phi^0(X)$ is* dually algebraic *[3].*

5 Conclusions and future research

We have shown that $\Phi_0(X)$ is an algebraic lattice and $\Phi^0(X)$ is a dually algebraic lattice. This implies that they carry compact Hausdorff Lawson (respectively, dual Lawson) topologies. The next step will be to describe these topologies and study the resulting topological spaces in greater detail.

We also find the conditions of Theorem 4 and Propositions 1, 2 rather difficult to verify. Hence, it would be worthwhile to simplify them if possible.

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Ми описуємо відношення апроксимації "значно нижче" та "значно вище" на частково впорядкованій множині фактороб'єктів компактного гаусдорфового простору. Встановлено необхідні і достатні умови того, що фактороб'єкт є точною гранню фактороб'єктів, які його апроксимують. Показано, що такі фактороб'єкти утворюють алгебраїчні і двоїсто алгебраїчні гратки.

Ключові слова і фрази: відношення "значно нижче", відношення "значно вище", відношення апроксимації, фактороб'єкт, компакт.