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ASYMPTOTIC BEHAVIOR OF ENTIRE FUNCTIONS OF IMPROVED REGULAR GROWTH IN THE METRIC OF $L^p[0,2\pi]$

We describe an asymptotic behavior of entire functions of improved regular growth with zeros on a finite system of rays in the metric of $L^p[0,2\pi]$.

Key words and phrases: entire function of improved regular growth, finite system of rays, Fourier coefficients, Hausdorff-Young theorem.

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1 INTRODUCTION AND MAIN RESULT

The asymptotic behavior of entire and meromorphic functions of positive order of completely regular growth (for details, see [2, 3, 5, 11]) in the metric of $L^p[0,2\pi]$ was described in [6, 11, 12, 13]. Similar results for entire functions of zero order whose zero-counting functions are slowly increasing were obtained in [1]. In particular, from [11, Theorem 7.2, p. 78] it follows the following statement.

Theorem A. *If an entire function f of order $\rho \in (0,+\infty)$ with the indicator $h(\theta, f)$ is of completely regular growth, then for any $p \in [1,+\infty)$*

$$\lim_{r\to+\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\log|f(re^{i\theta})|}{r^\rho}-h(\theta,f)\right|^p d\theta\right\}^{1/p}=0.$$

Conversely, if for some entire function f, $f(0)=1$, there exist $p \in [1,+\infty)$ and $\widetilde{h} \in L^p[0,2\pi]$ such that

$$\lim_{r\to+\infty}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\log|f(re^{i\theta})|}{r^\rho}-\widetilde{h}(\theta)\right|^p d\theta\right\}^{1/p}=0,$$

then f is of completely regular growth and $\widetilde{h}(\theta)=h(\theta,f)$ for almost all $\theta\in[0,2\pi]$.

The aim of the present paper is obtain an analog of Theorem A for entire functions of improved regular growth with zeros on a finite system of rays (see [4, 7, 8, 9, 10, 14]).

An entire function f is called a function of *improved regular growth* (see [7, 8, 9, 10, 14]) if for some $\rho\in(0,+\infty)$ and $\rho_1\in(0,\rho)$, and a 2π-periodic ρ-trigonometrically convex function $h(\varphi)\not\equiv-\infty$ there exists the set $U\subset\mathbb{C}$ contained in the union of disks with finite sum of radii and such that $\log|f(z)|=|z|^\rho h(\varphi)+o(|z|^{\rho_1})$, $U\not\ni z=re^{i\varphi}\to\infty$.

If an entire function f is a function of improved regular growth, then it has the order ρ and indicator h [14]. Our main result is the following theorem.

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Theorem 1. *An entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is a function of improved regular growth if and only if for some $\rho_2 \in (0,\rho)$ and any $p \in [1,+\infty)$, one has*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\log|f(re^{i\varphi})|}{r^\rho} - h(\varphi) \right|^p d\varphi \right\}^{1/p} = o(r^{\rho_2-\rho}), \quad r \to +\infty. \tag{1}$$

2 Preliminaries

Let f be an entire function with $f(0) = 1$, let $(\lambda_n)_{n\in\mathbb{N}}$ be the sequence of its zeros, let p be the smallest integer for which $\sum_{n=1}^{\infty} |\lambda_n|^{-p-1} < +\infty$, let Q_ρ be the coefficient of z^ρ in the exponential factor in the Hadamard-Borel representation [5, p. 38] of an entire function of finite order, and let $c_k(r,\log|f|)$ be the Fourier coefficients of $\log|f|$, i.e.

$$c_k(r,\log|f|) := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi}\log|f(re^{i\varphi})|\,d\varphi, \quad k \in \mathbb{Z}, \quad r > 0.$$

Further, let f be an entire function of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$. Furthermore, if ρ is noninteger and f is a function of improved regular growth, then an indicator h of f has the form ([8, 10, 14])

$$h(\varphi) = \sum_{j=1}^{m} h_j(\varphi),$$

where $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j, \psi_j + 2\pi)$

$$h_j(\varphi) = \frac{\pi\Delta_j}{\sin\pi\rho}\cos\rho(\varphi - \psi_j - \pi), \quad \Delta_j \in [0,+\infty).$$

In the case $\rho \in \mathbb{N}$, the indicator h is defined by the formula ([8, 9, 10])

$$h(\varphi) = \begin{cases} \tau_f \cos(\rho\varphi + \theta_f) + \sum_{j=1}^{m} h_j(\varphi), & p = \rho, \\ Q_\rho \cos\rho\varphi, & p = \rho - 1, \end{cases}$$

where $\delta_f \in \mathbb{C}$, $\tau_f = |\delta_f/\rho + Q_\rho|$, $\theta_f = \arg(\delta_f/\rho + Q_\rho)$ and $h_j(\varphi)$ is a 2π-periodic function such that on $[\psi_j, \psi_j + 2\pi)$ we have

$$h_j(\varphi) = \Delta_j(\pi - \varphi + \psi_j)\sin\rho(\varphi - \psi_j) - \frac{\Delta_j}{\rho}\cos\rho(\varphi - \psi_j).$$

Lemma 1 ([8]). *If an entire function f of order $\rho \in (0,+\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1,\dots,m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is of improved regular growth, then for some $\rho_3 \in (0,\rho)$*

$$c_k(r,\log|f|) = c_k r^\rho + \frac{o(r^{\rho_3})}{k^2+1}, \quad r \to +\infty, \tag{2}$$

holds uniformly in $k \in \mathbb{Z}$, where

$$c_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\varphi} h(\varphi)\,d\varphi = \frac{\rho}{\rho^2 - k^2}\sum_{j=1}^{m} \Delta_j e^{-ik\psi_j}, \quad \Delta_j \in [0,+\infty),$$

for a noninteger ρ, and

$$c_k = \begin{cases} \dfrac{\rho}{\rho^2 - k^2} \sum\limits_{j=1}^{m} \Delta_j e^{-ik\psi_j}, \quad |k| \neq \rho = p, \\ \dfrac{\tau_f e^{i\theta_f}}{2} - \dfrac{1}{4\rho} \sum\limits_{j=1}^{m} \Delta_j e^{-i\rho\psi_j}, \quad k = \rho = p, \\ 0, \quad |k| \neq \rho = p+1, \\ \dfrac{Q_\rho}{2}, \quad k = \rho = p+1, \end{cases}$$

if ρ is an integer.

Remark that, using Lemma 1 and the Riesz-Fischer theorem [11, p. 5], we get that there exists an indicator $h \in L^2[0, 2\pi]$ defined by the equality $h(\varphi) := \sum\limits_{k\in\mathbb{Z}} c_k e^{ik\varphi}$ (see [11, Definition 7.2, p. 77]).

3 Proof of Theorem 1

Necessity. If an entire function f of order $\rho \in (0, +\infty)$ with zeros on a finite system of rays $\{z : \arg z = \psi_j\}$, $j \in \{1, \dots, m\}$, $0 \le \psi_1 < \psi_2 < \dots < \psi_m < 2\pi$, is of improved regular growth, then by (2), we have

$$\left| \frac{c_k(r, \log |f|)}{r^\rho} - c_k \right| \le \frac{C}{k^2 + 1}, \quad k \in \mathbb{Z}, \tag{3}$$

for some constant $C > 0$ and all $r > 0$. In view of this, the sequence $(r^{-\rho} c_k(r, \log |f|) - c_k)_{k\in\mathbb{Z}}$ belongs to the space l_q for all $q > 1$ and $r > 0$. Moreover, applying the Hausdorff-Young theorem [11, p. 5], for $p \ge 2$, $p^{-1} + q^{-1} = 1$, we get

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\log |f(re^{i\varphi})|}{r^\rho} - h(\varphi) \right|^p d\varphi \right\}^{1/p} \le \left\{ \sum_{k\in\mathbb{Z}} \left| \frac{c_k(r, \log |f|)}{r^\rho} - c_k \right|^q \right\}^{1/q}.$$

According to (3), the last series is uniformly convergent with respect to r. Therefore, using Lemma 1, we obtain (1) for $p \ge 2$. From this and Hölder's inequality it follows that (1) holds for $1 \le p < 2$.

Sufficiency. Let (1) holds. Then for some $\rho_2 \in (0, \rho)$ and each $k \in \mathbb{Z}$

$$\begin{aligned} \left| \frac{c_k(r, \log |f|)}{r^\rho} - c_k \right| &\le \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\log |f(re^{i\varphi})|}{r^\rho} - h(\varphi) \right| d\varphi \\ &\le \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{\log |f(re^{i\varphi})|}{r^\rho} - h(\varphi) \right|^p d\varphi \right\}^{1/p} = o(r^{\rho_2 - \rho}), \quad r \to +\infty. \end{aligned}$$

Hence, for some $\rho_2 \in (0, \rho)$ and each $k \in \mathbb{Z}$ we get $c_k(r, \log |f|) = c_k r^\rho + o(r^{\rho_2})$, $r \to +\infty$. Then by [10, Theorem 1, p. 1718] an entire function f is a function of improved regular growth. This concludes the proof of the theorem.

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Хаць Р.В. *Асимптотична поведінка цілих функцій покращеного регулярного зростання в* $L^p[0,2\pi]$*-метриці* // Карпатські математичні публікації. — 2013. — Т.5, №2. — C. 341–344.

Описано асимптотичну поведінку цілих функцій покращеного регулярного зростання з нулями на скінченній системі променів в $L^p[0,2\pi]$-метриці.

Ключові слова і фрази: ціла функція покращеного регулярного зростання, скінченна система променів, коефіцієнти Фур'є, теорема Гаусдорфа-Юнга.

Хаць Р.В. *Асимптотическое поведение целых функций улучшенного регулярного роста в* $L^p[0,2\pi]$*-метрике* // Карпатские математические публикации. — 2013. — Т.5, №2. — C. 341–344.

Описано асимптотическое поведение целых функций улучшенного регулярного роста с нулями на конечной системе лучей в $L^p[0,2\pi]$-метрике.

Ключевые слова и фразы: целая функция улучшенного регулярного роста, конечная система лучей, коэффициенты Фурье, теорема Гаусдорфа-Юнга.