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SYMMETRIC CONTINUOUS LINEAR FUNCTIONALS ON COMPLEX SPACE $L_\infty[0,1]$

We prove that every symmetric continuous linear functional on the complex space $L_\infty[0,1]$ can be represented as a Lebesgue integral multiplied by a constant.

Key words and phrases: symmetric linear functional.

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INTRODUCTION

Let $L_\infty[0,1]$ be the space of all measurable complex-valued essentially bounded functions on $[0,1]$ with norm $\|x\| = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|$. Let Ξ be the group of all measurable transformations of $[0,1]$, which preserve measure. A functional $f : L_\infty[0,1] \to \mathbb{C}$ is called symmetric if for every $x \in L_\infty[0,1]$ and $\sigma \in \Xi$

$$f(x \circ \sigma) = f(x).$$

In [1, 2, 3, 4] symmetric polynomials are studied in ℓ_p and $L_p[0,1]$ spaces when $1 \le p < \infty$. Gonzales, Gonzalo and Jaramillo in [3] proved that every symmetric polynomial on $L_p[0,1]$ is an algebraic combination of the elementary symmetric polynomials

$$R_n(x) = \int_{[0,1]} (x(t))^n \, dt.$$

Proof of this result is based on the separability of $L_p[0,1]$ spaces. That is why the idea of this proof cannot be used in the case of symmetric polynomials on $L_\infty[0,1]$.

In this paper we restrict our attention to symmetric linear functionals as the most simple case of polynomials. Our purpose is to show that every symmetric continuous linear functional on $L_\infty[0,1]$, like in the case of $L_p[0,1]$ when $1 \le p < \infty$, is proportional to R_1.

THE MAIN RESULT

We denote by χ_A the characteristic function of the set $A \subset [0,1]$, i.e. the function

$$\chi_A(t) = \begin{cases} 1, & \text{if } t \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem. *Every symmetric continuous linear functional* $f : L_\infty[0,1] \to \mathbb{C}$ *can be represented as* $f(x) = k\int_{[0,1]} x(t)\, dt$, *where* $k = f(\chi_{[0,1]})$.

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Proof. Let A be a measurable subset of $[0,1]$. Define the function $\sigma_A : [0,1] \to [0,1]$ by

$$\sigma_A(t) = \begin{cases} \mu([0,t] \cap A), & \text{if } t \in A, \\ \mu(A) + \mu([0,t] \cap \bar{A}), & \text{if } t \in \bar{A}, \end{cases}$$

where $\bar{A} = [0,1] \setminus A$. Clearly, $\sigma_A \in \Xi$. Let $[0,b] \subset [0,1]$ and let $d \in \mathbb{R}$ be such that $[d, b+d] \subset [0,1]$. Since $\chi_{[d,b+d]} = \chi_{[0,b]} \circ \sigma_{[d,b+d]}$ and f is symmetric, it follows that

$$f(\chi_{[d,b+d]}) = f(\chi_{[0,b]}).$$

For $n \in \mathbb{N}$ we have

$$f(\chi_{[0,1]}) = f\big(\textstyle\sum_{j=1}^{n} \chi_{[\frac{j-1}{n},\frac{j}{n}]}\big) = \sum_{j=1}^{n} f(\chi_{[\frac{j-1}{n},\frac{j}{n}]}) = \sum_{j=1}^{n} f(\chi_{[0,\frac{1}{n}]}) = nf(\chi_{[0,\frac{1}{n}]}).$$

Hence, $f(\chi_{[0,\frac{1}{n}]}) = k \cdot \frac{1}{n}$. For $m \in \mathbb{Z}$, $0 \le m \le n$, we have

$$f(\chi_{[0,\frac{m}{n}]}) = f\big(\textstyle\sum_{j=1}^{m} \chi_{[\frac{j-1}{n},\frac{j}{n}]}\big) = mf(\chi_{[0,\frac{1}{n}]}) = k \cdot \frac{m}{n}.$$

Hence, for every $q \in [0,1] \cap \mathbb{Q}$

$$f(\chi_{[0,q]}) = kq. \tag{1}$$

Let $r \in [0,1]$ and let $n \in \mathbb{N}$ be such that $nr \in [0,1]$. Then

$$f(\chi_{[0,nr]}) = f\big(\textstyle\sum_{j=1}^{n} \chi_{[(j-1)r,jr]}\big) = nf(\chi_{[0,r]}). \tag{2}$$

Let us prove that for every $r \in [0,1]$

$$f(\chi_{[0,r]}) = kr.$$

Let $g : [0,1] \to \mathbb{C}$, $g(t) = f(\chi_{[0,t]})$. For $r_1, r_2 \in [0,1]$ such that $r_1 + r_2 \in [0,1]$ we have

$$\begin{aligned} g(r_1 + r_2) &= f(\chi_{[0,r_1+r_2]}) = f(\chi_{[0,r_1]}) + f(\chi_{[r_1,r_1+r_2]}) \\ &= f(\chi_{[0,r_1]}) + f(\chi_{[0,r_2]}) = g(r_1) + g(r_2). \end{aligned} \tag{3}$$

Hence, g is additive.

Suppose that there exists $\alpha \in (0,1)$ such that $g(\alpha) \neq k\alpha$. For $n \in \mathbb{N}$ choose $a_n \in (0,\alpha) \cap \mathbb{Q}$ such that $\alpha - a_n < \frac{1}{n}$ and $t_n = n(\alpha - a_n)$. By (1), (2) and (3)

$$g(t_n) = n(g(\alpha) - g(a_n)) = n(g(\alpha) - ka_n) = n(g(\alpha) - k\alpha) + nk(\alpha - a_n)$$

and

$$|g(t_n)| \ge n|g(\alpha) - k\alpha| - n|k(\alpha - a_n)| \ge n|g(\alpha) - k\alpha| - |k|.$$

So, g is unbounded. This contradicts the fact that f is continuous. Hence, for every $r \in [0,1]$

$$f(\chi_{[0,r]}) = kr.$$

Let A be the measurable subset of $[0,1]$. Since $\chi_A = \chi_{[0,\mu(A)]} \circ \sigma_A$, it follows that

$$f(\chi_A) = f(\chi_{[0,\mu(A)]}) = k\mu(A). \tag{4}$$

For every $x \in L_\infty[0,1]$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of measurable simple functions with finite range of values, which uniformly converges to x. Every x_n can be represented as

$$x_n(t) = \textstyle\sum_{j=1}^{m_n} y_{j,n}\chi_{A_{j,n}}(t),$$

where $A_{j,n}$ are the disjoint measurable subsets of $[0,1]$ and $y_{j,n} \in \mathbb{C}$. Then by (4)

$$f(x_n) = k\textstyle\sum_{j=1}^{m_n} y_{j,n}\mu(A_{j,n}) = k\displaystyle\int_{[0,1]} x_n(t)\,dt.$$

By the continuity of f

$$f(x) = \lim_{n\to\infty} f(x_n) = k \lim_{n\to\infty} \int_{[0,1]} x_n(t)\,dt = k\int_{[0,1]} x(t)\,dt.$$

□

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Василишин Т.В. *Симетричні неперервні лінійні функціонали на комплексному просторі $L_\infty[0,1]$* // Карпатські матем. публ. — 2014. — Т.6, №1. — С. 8–10.

В роботі доведено, що кожен симетричний неперервний лінійний функціонал на комплексному просторі $L_\infty[0,1]$ можна подати у вигляді інтеграла Лебега, помноженого на константу.

Ключові слова і фрази: симетричний лінійний функціонал.

Василишин Т.В. *Симметрические непрерывные линейные функционалы на комплексном пространстве $L_\infty[0,1]$* // Карпатские матем. публ. — 2014. — Т.6, №1. — С. 8–10.

В работе доказано, что каждый симметрический непрерывный линейный функционал на комплексном пространстве $L_\infty[0,1]$ можно представить в виде интеграла Лебега, умноженного на константу.

Ключевые слова и фразы: симметрический линейный функционал.