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ON RIGID DERIVATIONS IN RINGS

We prove that in a ring R with an identity there exists an element $a \in R$ and a nonzero derivation $d \in \operatorname{Der} R$ such that $ad(a) \neq 0$. A ring R is said to be a d-rigid ring for some derivation $d \in \operatorname{Der} R$ if $d(a) = 0$ or $ad(a) \neq 0$ for all $a \in R$. We study rings with rigid derivations and establish that a commutative Artinian ring R either has a non-rigid derivation or $R = R_1 \oplus \cdots \oplus R_n$ is a ring direct sum of rings $R_1, \ldots, R_n$ every of which is a field or a differentially trivial v-ring. The proof of this result is based on the fact that in a local ring R with the nonzero Jacobson radical $J(R)$, for any derivation $d \in \operatorname{Der} R$ such that $d(J(R)) = 0$, it follows that $d = 0_R$ if and only if the quotient ring $R/J(R)$ is differentially trivial field.

Key words and phrases: derivation, semiprime ring, Artinian ring.

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INTRODUCTION

Throughout, let R be an associative ring with 1 and $\operatorname{Der} R$ the set of all derivations of R. Recall that a map $\delta : R \to R$ is called *a derivation* of R if $\delta(x + y) = \delta(x) + \delta(y)$ and $\delta(xy) = \delta(x)y + x\delta(y)$ for any $x, y \in R$. We prove the following

Proposition 1. *Let R be a ring. Then the following conditions hold:*

(1) *if d is a nonzero derivation of a commutative ring R, then $ad(a) \neq 0$ for some $a \in R$,*

(2) *there exists an element $a \in R$ and a nonzero derivation $d \in \operatorname{Der} R$ such that $ad(a) \neq 0$.*

Different aspects of rigidity of derivations are studied in [4,6,15]. J. Krempa has introduced the concept of a σ-rigid ring [12]. Namely, R is said to be *a σ-rigid ring* for some ring endomorphism $\sigma \in \operatorname{End} R$ if $a\sigma(a) \neq 0$ for all nonzero $a \in R$. By analogy with this and in view of Proposition 1, we say that R is *a d-rigid ring* (or a derivation *d is rigid*), where $d \in \operatorname{Der} R$, if for any $a \in R$ it holds $d(a) = 0$ or $ad(a) \neq 0$. Clearly, the zero derivation 0_R of R is rigid. Every derivation of an integral domain is rigid.

M. Brešar [5], T.-K. Lee and J.-S. Lin [13] have investigated when, for a semiprime ring R, the condition $ad(R)^n = 0$, where n is fixed integer, $a \in R$, $d \in \operatorname{Der} R$, implies that $ad(R) = 0$. By Proposition 1 and results from [13, p.1688] and [8], we obtain the next

Corollary 1. *Let R be a semiprime ring with the derivation d and $a \in R$. If $ad(R)^n = 0$, where n is a fixed integer, then $d = 0_R$.*

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This corollary is an extension of some results from [11] and [8]. We prove the our next

Proposition 2. *Let R be a 2-torsion-free semiprime ring. Then all derivations of R are rigid if and only if R is reduced (that is without nonzero nilpotent elements).*

Recall [2] that a ring R is called *differentially trivial* if $\operatorname{Der} R = \{0_R\}$. Commutative Artinian rings with derivations to be rigid are characterized in the following

Theorem 1. *Let R be a commutative Artinian ring. Then one of the following holds:*

(1) *R has a non-rigid derivation,*

(2) *$R = R_1 \oplus \cdots \oplus R_n$ is a ring direct sum of rings $R_1, \ldots, R_n$ every of which is a field or a differentially trivial v-ring.*

For any ring R, $\partial_x : R \to R$ is its inner derivation generated by $x \in R$ that is $\partial_x(r) = xr - rx$ for every $r \in R$, $[R, R] = \{\partial_x(r) \mid x, r \in R\}$, $C(R)$ is the commutator ideal of R that is the ideal generated by $\partial_x(r)$ for all $x, r \in R$, $J(R)$ is its Jacobson radical, $N(R)$ is the set of all nilpotent elements of R, $U(R)$ is the unit group of R, $Z(R)$ is the center of R, $\operatorname{ann}_r a = \{x \in R \mid ax = 0\}$ is the right annihilator of $a \in R$, $\operatorname{ann}_l X = \{a \in R \mid aX = 0\}$ is the right annihilator of $X \subseteq R$. Any unexplained terminology is standard as in [3] and [10].

1 RINGS WITH PROPERTY $ad(a) = 0$

For the proof of Proposition 1, we need some preliminary lemmas.

Lemma 1. *Let R be a ring. Then the following properties hold:*

(1) *if $a\partial_x(a) = 0$ and $x\partial_a(x) = 0$ for some a, x of R, then $\partial_x(a)^2 = 0$,*

(2) *if $a\partial_x(a) = 0$ for any $a, x \in R$, then $C(R) \subseteq N(R)$,*

(3) *$d(C(R)) \subseteq C(R)$ for each $d \in \operatorname{Der} R$.*

Proof. (1) From $0 = a\partial_x(a) = a(xa - ax)$ and $0 = x\partial_a(x) = x(ax - xa)$ it follows that $axa = a^2x$ and $xax = x^2a$. This gives that

$$\partial_x(a)^2 = (xa - ax)(xa - ax) = xaxa - xa^2x - ax^2a + axax = 0.$$

(2) In view of (1), we see that $\partial_x(a)^2 = 0$, and therefore $C(R) \subseteq N(R)$.

(3) Since $d(r[a, x]t) = d(r)[a, x]t + r[d(a), x]t + r[a, d(x)]t + r[a, x]d(t)$ for any $a, x, r, t \in R$, we have $d(C(R)) \subseteq C(R)$. □

Lemma 2. *Let d be a nonzero derivation of R such that $ad(a) = 0$ for any $a \in R$. Then:*

(1) *R is non-commutative,*

(2) *$d(U(R)) = 0$ (in particular $d(J(R)) = 0$).*

(3) *if I is an ideal of a commutative ring R, then $d(R) \subseteq I$.*

Proof. (1) Indeed, if R is commutative, then $0 = (a+b)d(a+b) = ad(b) + bd(a) = d(ab)$ for any $a, b \in R$, and so $d(R^2) = 0$. But this means that $d = 0_R$, a contradiction.

(2) Let $u \in U(R)$. Then $ud(u) = 0$ and $u \in \operatorname{Ker} d$. Since $1 + J(R) \subseteq U(R)$, we see that $d(J(R)) = 0$.

(3) Let $a, b \in R$. Inasmuch as $ad(a) \in I$ for all $a \in R$ and

$$d(ab) = (a+b)d(a+b) - ad(a) - bd(b),$$

we deduce that $d(R) \subseteq I$. □

Proof of Proposition 1. (1) It follows from Lemma 2 (1).

(2) By contrary, assume that $ad(a) = 0$ for any $a \in R$ and $d \in \operatorname{Der} R$. By Lemma 1 (2) and Lemma 2 (2), $C(R) \subseteq Z(R)$. Let $\overline{R}$ denote $R/C(R)$ and, for $a \in R$, $\overline{a}$ denote the coset $a + C(R)$. The rule $D(\overline{a}) = d(a) + C(R)$ determines a derivation D of the quotient ring $\overline{R}$ such that

$$\overline{a}D(\overline{a}) = \overline{0}_{\overline{R}}.$$

By (1), $D = \overline{0}$, and so $d(a) \in Z(R)$. Then $0 = (a+b)d(a+b) = d(ab)$ and consequently $d(R^2) = 0$. This shows that $d = 0_R$. □

Now we establish some properties of rigid derivations.

Lemma 3. *Let R be a reduced ring, $a \in R$ and $d \in \operatorname{Der} R$. Then:*

(1) $ad(a) = 0$ *if and only if* $d(a)a = 0$,

(2) d *is a rigid derivation.*

Proof. (1) Straightforward.

(2) Assume, by contrary, that there is $a \in R$ such that $d(a) \neq 0$ and $ad(a) = 0$. Then, by item (1), we have that $d(a)a = 0$. Moreover, $0 = d(ad(a)) = d(a)d(a) + ad^2(a)$ and from this, by multiplication from the left by $d(a)$, we obtain that

$$0 = (d(a))^3 + d(a)ad^2(a) = (d(a))^3.$$

This yields that $d(a) = 0$, a contradiction. □

Let p be a prime and

$$F_p(R) = \{x \in R \mid p^k x = 0 \text{ for some positive integer } k\}.$$

Recall that a ring R is called 2-*torsion-free* if the implication

$$2x = 0 \Longrightarrow x = 0$$

is true for any $x \in R$. A ring R is 2-torsion-free if and only if $F_2(R) = 0$.

Lemma 4. *If all derivations in R are rigid and* $\exp F_2(R)$ *is finite, then in $R/F_2(R)$ also.*

Proof. If, by contrary,

$$\delta : R/F_2(R) \ni r + F_2(R) \mapsto t_r + F_2(R) \in R/F_2(R) \tag{1}$$

is a derivation such that

$$t_u \notin F_2(R) \qquad \text{and} \qquad ut_u \in F_2(R)$$

for some $u \in R$, then $d : R \ni r \mapsto 2^s t_r$, with $\exp F_2(R) = 2^s$ and t_r as in (1), is a derivation which is not rigid. □

Lemma 5. *Let R be a 2-torsion-free ring and $d \in \operatorname{Der} R$. If R is d-rigid and $\partial_{d(a)}$-rigid for any $a \in N(R)$, then $d(N(R)) = 0$.*

Proof. We prove by induction on the nilpotency index n of nil-elements in R. Let $a \in N(R)$ and $a^2 = 0$. Left multiplying of $0 = d(a^2) = ad(a) + d(a)a$ by a, we obtain that $ad(a)a = 0$. Since $\partial_{d(a)}$ is rigid and

$$a\partial_{d(a)}(a) = ad(a)a - a^2 d(a) = 0,$$

we deduce that $\partial_{d(a)}(a) = 0$ that is $ad(a) = d(a)a$. Hence $0 = d(a^2) = 2ad(a)$. In view of the rigidity of d and the condition $F_2(R) = 0$, we have $d(a) = 0$.

Now suppose that $a \in N(R)$ and $a^3 = 0$. Then $(a^2)^2 = 0$ and, by the above, $d(a^2) = 0$. Since

$$0 = d(a^3) = d(a)a^2 + ad(a^2) = d(a)a^2 \quad \text{and} \quad 0 = d(a^3) = d(a^2)a + a^2 d(a) = a^2 d(a),$$

the assertion holds by using the same argument as for $n = 2$.

Assume that assertion is true for all positive integer $k < n$ that is if $b^k = 0$ with $b \in N(R)$, then $d(b) = 0$. Let us $a \in N(R)$ and $a^n = 0$. Then there exist positive integer k_1, k_2 such that $k_1, k_2 < n$ but $2k_1 > n$ and $3k_2 > n$ and $(a^2)^{k_1} = 0$ and $(a^3)^{k_2} = 0$ and, by assumption, $d(a^2) = d(a^3) = 0$ and the result follows by using the same argument. □

Proof of Proposition 2. ($\Leftarrow$) It follows from Lemma 3.
($\Rightarrow$) By Lemma 5, we have $N(R) \subseteq Z(R)$ and hence $N(R) = 0$. □

Corollary 2. *If a derivation d of a 2-torsion-free commutative ring R is rigid, then $d(N(R)) = 0$ and $N(R)d(R) = 0$.*

Proof. Indeed, if $a \in R$ and $b \in N(R)$, then $ab \in N(R)$ and therefore, By Lemma 5,

$$0 = d(ab) = d(a)b.$$

□

Example 1. *The condition $F_2(R) = 0$ is essential in Corollary 2.*

In fact, the quotient ring $R = \mathbb{Z}_2[X]/(X^2 + 1)$ of the polynomial ring $\mathbb{Z}_2[X]$ by the ideal $(X^2 + 1)$ contains elements $0, 1, x, x + 1$, where $x(x + 1) = x + 1$. Then a mapping $d : R \to R$ such that $d(0) = d(1) = 0$ and $d(x) = d(x + 1) = 1$ is a derivation of R. But then R is a d-rigid ring with $(x + 1)^2 = 0$ and $d(x + 1) \neq 0$.

Corollary 3. *If d is a rigid derivation of a ring R, then $d(\operatorname{ann}_l d(R)) = 0$.*

Proof. Since $\operatorname{ann}_l d(R) \cdot d(\operatorname{ann}_l d(R)) = 0$, we deduce that $d(\operatorname{ann}_l d(R)) = 0$. □

2 CONSTANTS IN LEFT PERFECT RINGS

D. F. Anderson and P. S. Livingston [1] (see also S. B. Mulay [14]) have shown that any automorphism f of a commutative finite ring R that is not a field such that $f(x) = x$ for all zero divisors $x \in R$, is the identity automorphism. Since any commutative finite ring is a finite ring direct sum of local rings, it is clear that the statement needs a proof only when a ring is local. In view of this, P. K. Sharma [16] proved that if a commutative finite local ring R which is not a field, then for any $f \in \operatorname{Aut} R$ with $f(x) = x$ for all $x \in J(R)$, $f = \mathrm{id}_R$ if and only if the residue field is differentially trivial. We extended this result in the next

Proposition 3. *Let R be a local ring with the nonzero Jacobson radical $J(R)$. Then the following statements are equivalent.*

(1) *For any derivation $d \in \operatorname{Der} R$ such that $d(J(R)) = 0$ it follows that $d = 0_R$.*

(2) *The quotient ring $R/J(R)$ is a differentially trivial field.*

(3) *Every automorphism $f \in \operatorname{Aut} R$ such that $f(x) = x$ for any $x \in J(R)$ is trivial, i.e. $f = \mathrm{id}_R$.*

Lemma 6. *Let R be a ring with an ideal I and $d \in \operatorname{Der} R$. If $d(I) = 0$, then $d(R) \subseteq \operatorname{ann} I$.*

Proof. Indeed, for any $r \in R$, $j \in I$ we observe that $0 = d(jr) = jd(r)$ and $0 = d(rj) = d(r)j$. □

Corollary 4. *Let R be a ring with an ideal I, $d \in \operatorname{Der} R$, $f \in \operatorname{Aut} R$ and $\operatorname{ann} I \subseteq I$.*

(i) *If $d(I) = 0$, then $d^2(R) = 0$ and $(d(R))^2 = 0$.*

(ii) *If $f(x) = x$ for any $x \in I$, then $f - \mathrm{id}_R \in \operatorname{Der} R$.*

Proof. (i) By Lemma 6, $d(R) \subseteq \operatorname{ann} I$ and therefore

$$d^2(R) \subseteq d(\operatorname{ann} I) \subseteq d(I) = 0 \qquad \text{and} \qquad (d(R))^2 \subseteq (\operatorname{ann} I)I = 0.$$

(ii) Let $x \in I$ and $a, b, r \in R$. Then $xr, rx \in I$,

$$xf(r) = f(x)f(r) = f(xr) = xr, \quad f(r)x = f(r)f(x) = f(rx) = rx$$

and so $x(f(r) - r) = 0 = (f(r) - r)x$. Hence $f(r) - r \in \operatorname{ann} I$. In view of this, we see that

$$\begin{aligned}(f - \mathrm{id}_R)(a + b) &= f(a + b) - \mathrm{id}_R(a + b) \\ &= (f(a) - \mathrm{id}_R(a)) + (f(b) - \mathrm{id}_R(b)) = (f - \mathrm{id}_R)(a) + (f - \mathrm{id}_R)(b)\end{aligned}$$

and

$$\begin{aligned}(f - \mathrm{id}_R)(a)b + a(f - \mathrm{id}_R)(b) &= f(a)b - ab + af(b) - ab \\ &= f(a)f(b) + (f(a) - a)(b - f(b)) - ab = f(ab) - ab = (f - \mathrm{id}_R)(ab).\end{aligned}$$

This means that $f - \mathrm{id}_R \in \operatorname{Der} R$. □

Corollary 5. *Let R be a local ring that is not a skew field, I its ideal and $0_R \neq d \in \operatorname{Der} R$. If the left annihilator $\operatorname{ann}_l I = \{a \in R \mid aI = 0\}$ (respectively the right annihilator $\operatorname{ann}_r I = \{a \in R \mid Ia = 0\}$) is zero, then $d(I) \neq 0$.*

Proof. If $d(I) = 0$, then, by Lemma 6, $d(R) \subseteq \operatorname{ann} I \subseteq \operatorname{ann}_l I = 0$, a contradiction. □

Lemma 7. *Let R be a ring, I an ideal with $\operatorname{ann} I \subseteq I$. Then the following statements are equivalent.*

(i) *For every $f \in \operatorname{Aut} R$ such that $f(x) = x$ for any $x \in I$ it follows that $f = \mathrm{id}_R$.*

(ii) *Every derivation $d \in \operatorname{Der} R$ such that $d(I) = 0$ is zero.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $d \in \operatorname{Der} R$ and $d(I) = 0$. Then, for any $a, b \in R$, we see that

$$(d + \mathrm{id}_R)(a + b) = d(a + b) + \mathrm{id}_R(a + b)$$
$$= (d(a) + \mathrm{id}_R(a)) + (d(b) + i_R(b)) = (d + \mathrm{id}_R)(a) + (d + \mathrm{id}_R)(b)$$

and, in view of Corollary 4,

$$(d + \mathrm{id}_R)(a) \cdot (d + \mathrm{id}_R)(b) = (d(a) + a)(d(b) + b) = d(a)d(b) + d(a)b + ad(b) + ab$$
$$= d(a)b + ad(b) + ab = (d + \mathrm{id}_R)(ab) \text{ and } (d + \mathrm{id}_R)(1) = 1.$$

So $d + \mathrm{id}_R$ is a ring endomorphism of R. Moreover,

$$(d + \mathrm{id}_R)(\mathrm{id}_R - d) = \mathrm{id}_R = (\mathrm{id}_R - d)(d + \mathrm{id}_R)$$

and therefore $d + \mathrm{id}_R \in \operatorname{Aut} R$. Since $(d + \mathrm{id}_R)(x) = d(x) + x = x = \mathrm{id}_R(x)$ for any $x \in I$, we conclude that $d = 0_R$.

$(ii) \Rightarrow (i)$ Let $f \in \operatorname{Aut} R$ and $f(x) = x$ for all $x \in I$. Then, in view of Corollary 4, we have that $f - \mathrm{id}_R \in \operatorname{Der} R$. Inasmuch as $(f - \mathrm{id}_R)(I) = 0$, we conclude $f = \mathrm{id}_R$. □

Lemma 8. *Let R be a local ring, $d \in \operatorname{Der} R$. Then the following hold:*

(1) *if $d(J(R)) = 0$, then $d = 0_R$ or $\operatorname{ann} J(R) \neq 0$,*

(2) *if $\operatorname{ann} J(R) = 0$, then $d = 0_R$ or $d(J(R)) \neq 0$.*

Lemma 9. *Let R be a ring and let I be a nonzero ideal such that, for $d \in \operatorname{Der} R$, $d(I) = 0$ implies $d = 0_R$. Then*

$$\operatorname{ann} I \subseteq C_R(I) \subseteq Z(R),$$

where the centralizer $C_R(I) = \{z \in R \mid zj = jz \text{ for all } j \in I\}$.

Proof. Clearly, $\operatorname{ann} I \subseteq C_R(I)$. If $a \in C_R(I)$, then $\partial_a(I) = 0$ and therefore $\partial_a(R) = 0$. Hence $a \in Z(R)$. □

Corollary 6. *Let R be a ring and let I be a nonzero ideal with $\operatorname{ann} I \subseteq I$. If $I \subseteq Z(R)$, then R/I is commutative.*

Proof. For any element $x \in R$ we have that $\partial_x(I) = 0$, and so, by Lemma 6, we deduce that $\partial_x(R) \subseteq \operatorname{ann} I \subseteq I$. This yields that R/I is commutative. □

Proof of Proposition 3. $(1) \Rightarrow (2)$ Since R is local, $\operatorname{ann} J(R) \subseteq J(R)$. Suppose that $\theta : R/J(R) \to R/J(R)$ is a nonzero derivation and, for every element $t \in R$, there exists such $w_t \in R$ that

$$\theta(t + J(R)) = w_t + J(R)$$

with $w_{t_0} \notin J(R)$ for some $t_0 \in R$. The left T-nilpotent ideal $J(R)$ has a nonzero annihilator. If $0 \neq u \in \operatorname{ann} J(R)$, then the rule $\mu_u(t) = uw_t$ $(t \in R)$ determines a nonzero derivation μ_u of R for some u. Indeed, if $uw_t = 0$ $(t \in R)$ for all $u \in \operatorname{ann} J(R)$, then $w_{t_0} \in J(R)$, a contradiction. Thus μ_u is nonzero. Inasmuch $\mu_u(J(R)) = 0$, we conclude that $\mu_u(R) = 0$, which gives a contradiction. Hence the quotient ring $R/J(R)$ is differentially trivial.

$(2) \Rightarrow (1)$ Suppose that $R/J(R)$ is a differentially trivial ring. Then every inner derivation of $\overline{R}$ is zero and so $\overline{R}$ is commutative. As a consequence,

$$R/J(R) = F$$

is a differentially trivial field. Assume that d is a nonzero derivation of R such that $d(J(R)) = 0$. Then the rule

$$D : \overline{R} \ni \overline{r} \mapsto d(r) \in A \cap J(R) \ (r \in R)$$

determines a nonzero map D. Since $A \cap J(R)$ is a left F-linear space, there exists such field $F = F_{i_0}$ $(1 \leq i_0 \leq n)$ that a map

$$\theta : F \ni \overline{a} \mapsto d(a) \in A \cap J(R)$$

is nonzero. If char $F = p$ is a prime, then, by Proposition 1.3 of [2], we have $\overline{a} = \overline{b}^p$ for some $\overline{b} \in F$ and therefore

$$\theta(\overline{a}) = \theta(\overline{b}^p) = p\overline{b}^{p-1}d(b) = 0.$$

Assume that char $F = 0$. By Proposition 1.2 of [2], F is algebraic over the rational number field $\mathbb{Q}$ and so for every $\overline{a} \in F$ there exists its minimal polynomial

$$m_{\overline{a}} = X^n + c_1X^{n-1} + \cdots + c_{n-1}X + c_n \in \mathbb{Q}[X].$$

Then

$$0 = \theta(m_{\overline{a}}(\overline{a})) = (n\overline{a}^{n-1} + (n-1)c_1\overline{a}^{n-2} + \cdots + c_{n-1}\overline{1})d(a)$$

and, consequently, $d(a) = 0$. Hence θ is zero, a contradiction.

(1) and (3) are equivalent in view of Lemma 7. □

3 ARTINIAN d-RIGID RINGS

Recall that in a commutative local ring R can be introduced a topology by taking ideals

$$J(R), J(R)^2, \ldots, J(R)^n, \ldots$$

to be neighborhoods of zero. This generate the $J(R)$-adic topology. If, for any natural m,

$$a_k - a_l \in J(R)^m,$$

with k, l sufficiently large, then the sequence $\{a_n\}$ is called *regular*. A commutative local ring R is called *complete* if every regular sequence of R has a limit in R. Each commutative Artinian ring is complete. A *v-ring* is unramified complete regular local Noetherian domain of dimension one whose characteristic is different from that of its residue field [7, p.88].

Remark 1. *If the residue field W/pW of a v-ring W has a nonzero derivation d, then, by Proposition 2 of [9], there exists a nonzero derivation $D : W \to W$ such that*

$$D(a + pW) = d(a) + pW$$

for every $a \in W$ and $D(W) \nsubseteq pW$. Consequently,

$$D(p^{k-1}W) \nsubseteq p^kW$$

for any positive integer k.

Below we study the structure of a commutative Artinian ring with rigid derivations.

Lemma 10. *Let R be a local left Artinian ring. If in R all derivations are rigid, then in $R/ \operatorname{ann} J(R)$ also.*

Proof. If, by contrary,

$$\mu : R/\operatorname{ann} J(R) \ni r + \operatorname{ann} J(R) \mapsto v_r + \operatorname{ann} J(R) \in R/\operatorname{ann} J(R) \qquad (2)$$

is a derivation such that $v_u \notin \operatorname{ann} J(R)$ and $uv_u \in \operatorname{ann} J(R)$ for some $0 \neq u \in R$, then the rule

$$\delta : R \ni r \mapsto j_0 v_r \ (r \in R),$$

with $j_0 v_u \neq 0$ and v_r as in (2), determines a derivation δ of R which is not rigid. □

If R is a ring of prime power characteristic p^n $(n \geq 2)$, then

$$\Omega_k = \Omega_k(R) = \{x \in R \mid p^k x = 0\} \ (1 \leq k \leq n).$$

Obviously, Ω_k is an ideal of R.

Remark 2. *Let R be a local ring and $d \in \operatorname{Der} R$. If $J(R) = 0$ or $d(J(R)) = 0$, then a derivation d is rigid.*

In fact, $R = J(R) \cup U(R)$. If $u \in U(R)$ (respectively $j \in J(R)$), then $d(u) = 0$ or $ud(u) \neq 0$ (respectively $d(j) = 0$). Hence R is d-rigid.

Lemma 11. *Let R be a local left Artinian ring. If in R all derivations are rigid and $J(R)^2 = 0$, then one of the following holds:*

(1) *R is a commutative ring,*

(2) *$d(J(R)) = 0$ and $d(R)J(R) = 0$ for any $d \in \operatorname{Der} R$,*

(3) *$C(R) = R$ and $J(R) \cap Z(R) = 0$.*

If R is a 2-torsion-free, then R is a skew field or $C(R) \neq R$.

Proof. Suppose that R is non-commutative (that is $C(R) \neq 0$) and $d \in \operatorname{Der} R$. Then $d(C(R)) \subseteq C(R)$. If $0 \neq c \in J(R) \cap Z(R)$, then $cd \in \operatorname{Der} R$ and

$$J(R) \cdot cd(J(R)) = 0,$$

and so $cd(J(R)) = 0$. This gives that $d(J(R)) \subseteq J(R)$. Since $J(R)d(J(R)) = 0$, we conclude that $d(J(R)) = 0$. Then

$$0 = d(RJ(R)) = d(R)J(R).$$

Assume that $J(R) \cap Z(R) = 0$. If $C(R) \subseteq J(R)$, then

$$C(R)d(C(R)) = 0$$

and consequently $C(R) \subseteq Z(R) \cap J(R)$, a contradiction with the assumption. Hence $C(R) \nsubseteq J(R)$ and and therefore $C(R) = R$. □

Proof of Theorem 1. By Lemma 10 we can assume that $J(R)^2 = 0$. We have two cases.

1) Let $\operatorname{char}(R) = \operatorname{char}(R/J(R))$. By Theorem 9 of [7], the ring

$$R = J(R) + T$$

is a group direct sum, where T is a subfield of R. Then, for every element $r \in R$, there exist unique elements $j \in J(R)$ and $t \in T$ such that

$$r = j + t. \qquad (3)$$

The rule

$$\delta(r) = j \ (r \in R),$$

with j as in (3), determines a derivation δ of R which is not rigid. Hence $J(R) = 0$.

2) Let $\mathrm{char}(R) = p^2$. By Theorem 11 of [7], the ring

$$R = J(R) + C$$

is a group sum, where C is a coefficient ring such that $C \cong W/p^2W$ for some v-ring W and $J(R) \cap C = pC$. Clearly, $\Omega_1 \leq J(R)$. If

$$\mu : R/\Omega_1 \ni r + \Omega_1 \mapsto a_r + \Omega_1 \in R/\Omega_1 \tag{4}$$

is a non-rigid derivation, then there exists an element $v \in R$ such that $a_v \notin \Omega_1$ and $va_v \in \Omega_1$. Then the rule

$$\delta(r) = pa_r \ (r \in R),$$

with a_r as in (4), determines a nonzero derivation δ of R, where $\delta(v) \neq 0$ and $v\delta(v) = vpa_v = 0$, a contradiction. Hence in the quotient ring $\overline{R} = R/\Omega_1$ all derivations are rigid. From the part 1) it follows that $\overline{R}$ is a field and $J(R) = \Omega_1$. Since $\Omega_1 d(\Omega_1) = 0$ for all $d \in \mathrm{Der}\, R$, we see that $d(J(R)) = d(\Omega_1) = 0$. Obviously that $J(R) = J_1 \oplus pC$ is a group direct sum, where $J_1 \leq J(R)$ is some subgroup. Then

$$J_1C = J_1 \oplus (pC \bigcap J_1C)$$

is a group direct sum. If

$$0 \neq pc_0 \in J_1C \bigcap pC$$

for some $c_0 \in C$, then $c_0 \in U(R)$ and $Cc_0 = C$. Then $pc_0 \in J_1Cc_0$ and $pc_0 = j_1c_1c_0$ for some $j_1 \in J_1$ and $c_1 \in C$. From this it holds that

$$(p - j_1c_1)c_0 = 0,$$

and, hence, $j_1 = pc_1^{-1} \in J_1 \cap pC = 0$, a contradiction. This yields that $J = J_1C \oplus pC$ and $R = J_1C \oplus C$ is a group direct sum. Then, for very element $r \in R$, there are unique elements $j \in J_1C$ and $c \in C$ such that

$$r = j + c. \tag{5}$$

The rule $\gamma(r) = j \ (r \in R)$, with j as in (5), determines a nonzero derivation of R, where $\gamma(J(R)) \neq 0$, a contradiction. Thus $R = C$. If the residue field C/pC has a nonzero derivation d, then, in view of Remark 1, the ring C has a nonzero derivation D such that

$$D(C) \nsubseteq pC,$$

a contradiction. Hence, C/pC (and, by Proposition 3, the ring R) is differentially trivial.

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Доведено, що в кільці R з одиницею існує елемент $a \in R$ та ненульове диференціювання $d \in \operatorname{Der} R$ такі, що $ad(a) \neq 0$. Кажуть, що R — d-жорстке кільце для деякого диференціювання $d \in \operatorname{Der} R$, якщо $d(a) = 0$ або $ad(a) \neq 0$ для усіх $a \in R$. Досліджено кільця із жорсткими диференціюваннями та встановлено, що комутативне артінове кільце R або має нежорстке диференціювання, або $R = R_1 \oplus \cdots \oplus R_n$ — пряма сума кілець $R_1, \ldots, R_n$, кожне з яких є полем або диференціально тривіальним v-кільцем. Доведення цього результату базується на тому факті, що в лівому досконалому кільці R з ненульовим радикалом Джекобсона $J(R)$ для будь-якого диференціювання $d \in \operatorname{Der} R$ такого, що $d(J(R)) = 0$, випливає, що $d = 0_R$ тоді і тільки тоді, коли фактор-кільце $R/J(R)$ — диференціально тривіальне поле.

Ключові слова і фрази: диференціювання, напівпервинне кільце, артінове кільце, досконале кільце.

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Доказано, что в кольце R с единицей существует элемент $a \in R$ и ненулевое дифференцирование $d \in \operatorname{Der} R$ такие, что $ad(a) \neq 0$. Кольцо R называется d-жестким кольцом для дифференцирования $d \in \operatorname{Der} R$, если $d(a) = 0$ или $ad(a) \neq 0$ для всех $a \in R$. Исследуются кольца с жесткими дифференцированиями и установлено, что коммутативное артиново кольцо R либо имеет нежесткое дифференцирование, либо $R = R_1 \oplus \cdots \oplus R_n$ — прямая сумма колец $R_1, \ldots, R_n$ каждое из которых является полем или дифференциально тривиальным v-кольцом. Доказательство этого результата основано на том, что в локальном кольце R с ненулевым радикалом Джекобсона $J(R)$ для любого дифференцирования $d \in \operatorname{Der} R$ такого, что $d(J(R)) = 0$, следует, что $d = 0_R$ тогда и только тогда, когда фактор-кольцо $R/J(R)$ — дифференциально тривиальное поле.

Ключевые слова и фразы: дифференцирование, полупервичное кольцо, артиново кольцо.