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ON φ-SYMMETRIC τ-CURVATURE TENSOR IN $N(k)$-CONTACT METRIC MANIFOLD

In this paper we study τ-curvature tensor in $N(k)$-contact metric manifold. We study τ-φ-recurrent, τ-φ-symmetric and globally τ-φ-symmetric $N(k)$-contact metric manifold.

Key words and phrases: contact metric manifold, symmetry.

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INTRODUCTION

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. In the context of contact geometry the notion of φ-symmetry is introduced and studied by Boeckx E., Buecken P. and Vanhecke L. [3] with several examples. As a weaker version of local symmetry, Takahashi T. [13] introduced the notion of locally φ-symmetry on a Sasakian manifold. Generalizing the notion of φ-symmetry, the authors De U.C., Shaikh A.A. and Sudipta Biswas [4] introduced the notion of φ-recurrent Sasakian manifolds. This notion has been studied by many authors for different types of contact manifolds like Venkatesha and Bagewadi C.S. [14, 15], De U.C. and Abdul Kalam Gazi [5], Nagaraja H.G. [9] etc.

In [12] Tanno S. introduced the notion of k-nullitiy distribution of a contact metric manifold as a distribution such that the characteristic vector field ξ of the contact metric manifold belongs to the distribution. The contact metric manifold with ξ belonging to the k-nullity distribution is called $N(k)$-contact metric manifold such a manifold is also studied by various authors. Generalizing this notion in 1995, Blair D.E., Koufogiorgos T. and Papantoniou B.J. [2] introduced the notion of a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution, where k and μ are real constants. In particular, if $\mu = 0$ then the notion of (k,μ)-nullity distribution reduces to the notion of k-nullity distribution.

In [6] Mukut Mani Tripathi and et.al. introduced the τ-curvature tensor which is a particular cases of known curvatures like conformal, concircular, projective, M-projective, W_i-curvature tensor$(i = 0,\dots,9)$ and W_j^*-curvature tensor$(j = 0,1)$. Further, in [7,8] Mukut Mani Tripathi and et.al. studied τ-curvature tensor in K-contact, Sasakian and Semi-Riemannian manifolds. Later in [10] the authors studied some properties of τ-curvature tensor and they obtained some interesting results.

Motivated by all these work in this paper we studied the φ-symmetric τ-curvature tensor in $N(k)$-contact metric manifold.

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1 PRELIMINARIES

A $(2n+1)$-dimensional differential manifold M is said to have an almost contact structure (φ, ξ, η) if it carries a tensor field φ of type $(1,1)$, a vector field ξ and 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0. \tag{1}$$

Let g be a compatible Riemannian metric with almost contact structure (φ, ξ, η) such that,

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$g(\varphi X, Y) = -g(X, \varphi Y), \qquad g(X, \xi) = \eta(X). \tag{2}$$

where X, Y are vector fields defined on M. Then the structure (φ, ξ, η, g) on M is said to have an almost contact metric structure and the manifold M equipped with this structure is called an almost contact metric manifold [1]. An almost contact metric structure (φ, ξ, η, g) becomes a contact metric structure if for all vector fields X, Y on M we have $d\eta(X, Y) = g(X, \varphi Y)$.

Given a contact metric manifold $(M, \varphi, \xi, \eta, g)$, we define a $(1,1)$ tensor field h by $h = \frac{1}{2}L\varphi$, where L denotes the Lie differentiation. Then h is symmetric and satisfies $h\varphi = -\varphi h$. Also we have $Tr.h = Tr.\varphi h = 0$ and $h\xi = 0$. Moreover, if ∇ denotes the Riemannian connection on M, then the following relation holds

$$\nabla_X \xi = -\varphi X - \varphi h X. \tag{3}$$

For a contact metric manifold $M(\varphi, \xi, \eta, g)$ the (k, μ)-nullity distribution is

$$\begin{aligned} p \longrightarrow & N_p(k, \mu) \\ & = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] + \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned}$$

for all vector fields $X, Y \in T_pM$, where k, μ are real numbers and R is the curvature tensor. Hence if the characteristic vector field ξ belongs to the (k, μ)-nullity distribution, then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{4}$$

Thus a contact metric manifold satisfying (4) is called a (k, μ)-contact metric manifold. In particular, if $\mu = 0$, then the notion of (k, μ)-nullity distribution reduces to the notion of k-nullity distribution introduced by Tanno S. [12]. In a (k, μ)-contact metric manifold [11] the following relations hold:

$$\begin{aligned} h^2 &= (k-1)\varphi^2, \qquad k \le 1, \\ (\nabla_X \varphi)Y &= g(X + hX, Y)\xi - \eta(Y)[X + hX], \\ R(\xi, X)Y &= k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \\ \eta(R(X,Y)Z) &= k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + \mu[g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)], \\ S(X, \xi) &= 2nk\eta(X), \\ S(X,Y) &= [2(n-1) - n\mu]g(X,Y) + [2(n-1) + \mu]g(hX,Y) \\ &\quad + [2(1-n) + n(2k+\mu)]\eta(X)\eta(Y), \qquad n \ge 1, \\ r &= 2n[2n - 2 + k - n\mu], \\ S(\varphi X, \varphi Y) &= S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n - 2 + \mu)g(hX, Y), \end{aligned}$$

where S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator, that is, $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold. From (2), it follows that

$$(\nabla_X \eta)Y = g(X + hX, \varphi Y).$$

The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by

$$p \longrightarrow N_p(k) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$-contact metric manifold. In $N(k)$-contact metric manifold the following relations hold [5]:

$$h^2 = (k-1)\varphi^2, \qquad k \leq 1,$$

$$(\nabla_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)[X+hX],$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X],$$

$$S(X,\xi) = 2nk\eta(X), \tag{5}$$

$$S(X,Y) = 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) + [2(1-n)+2nk]\eta(X)\eta(Y), \quad n \geq 1, \tag{6}$$

$$QX = 2(n-1)X + 2(n-1)hX + [2(1-n)+2nk]\eta(X)\xi, \qquad n \geq 1, \tag{7}$$

$$r = 2n[2n-2+k],$$

$$S(\varphi X,\varphi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y), \tag{8}$$

$$(\nabla_X\eta)Y = g(X+hX,\varphi Y).$$

Definition 1. *An $N(k)$-contact metric manifold M is said to be locally φ-symmetric if*

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0$$

for all vector fields X, Y, Z, W, which are orthogonal to ξ.

This notion was introduced by Takahashi T. [13] for Sasakian manifolds.

Definition 2. *An $N(k)$-contact metric manifold M is said to be φ-symmetric if*

$$\varphi^2((\nabla_W R)(X,Y)Z) = 0$$

for all arbitrary vector fields X, Y, Z, W.

Definition 3. *An $N(k)$-contact metric manifold M is said to be locally τ-φ-symmetric if*

$$\varphi^2((\nabla_W \tau)(X,Y)Z) = 0$$

for all vector fields X, Y, Z, W, which are orthogonal to ξ.

Definition 4. *An $N(k)$-contact metric manifold M is said to be φ-recurrent if and only if there exists a non zero 1- form A such that*

$$\varphi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z$$

for all vector fields X, Y, Z, W. Here X, Y, Z, W are arbitrary vector fields which are not necessarily orthogonal to ξ.

If the 1-form A vanishes identically, then the manifold is said to be a locally φ-symmetric manifold.

Definition 5. *An $N(k)$-contact metric manifold M is said to be τ-φ-recurrent if and only if there exists a non zero 1-form A such that*

$$\varphi^2((\nabla_W \tau)(X,Y)Z) = A(W)\tau(X,Y)Z,$$

for all vector fields X, Y, Z, W. Here X, Y, Z, W are arbitrary vector fields which are not necessarily orthogonal to ξ.

The τ-curvature tensor [7] is given by

$$\begin{aligned}\tau(X,Y)Z = a_0R(X,Y)Z + a_1S(Y,Z)X + a_2S(X,Z)Y + a_3S(X,Y)Z + a_4g(Y,Z)QX \\ + a_5g(X,Z)QY + a_6g(X,Y)QZ + a_7[g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{9}$$

where $a_0, \dots, a_7$ are some smooth functions on M. For different values of $a_0, \dots, a_7$ the τ- curvature tensor reduces to the curvature tensor R, quasi-conformal curvature tensor, conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor, pseudo-projective curvature tensor, projective curvature tensor, M-projective curvature tensor, W_i-curvature tensors $(i = 0, \dots, 9)$, W_j^*-curvature tensors $(j = 0, 1)$.

2 τ-φ-RECURRENT $N(k)$-CONTACT METRIC MANIFOLD

In this section, we define τ-φ-recurrent $N(k)$-contact metric manifold by

$$\varphi^2((\nabla_W\tau)(X,Y)Z) = A(W)\tau(X,Y)Z$$

for all vector fields X, Y, Z, W. By virtue of (1), we have

$$-(\nabla_W\tau)(X,Y)Z + \eta((\nabla_W\tau)(X,Y)Z)\xi = A(W)\tau(X,Y)Z. \tag{10}$$

By taking an innerproduct with U, then we get

$$-g((\nabla_W\tau)(X,Y)Z,U) + \eta((\nabla_W\tau)(X,Y)Z)g(\xi,U) = A(W)g(\tau(X,Y)Z,U). \tag{11}$$

Let $\{e_i : i = 1, 2, \dots, 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$ in (11) and taking summation over i, $1 \le i \le 2n+1$, by virtue of (11), we obtain

$$\begin{aligned}&- [a_0 + (2n+1)a_1 + a_2 + a_3](\nabla_W S)(Y,Z) - [a_4 + 2na_7](\nabla_W r)g(Y,Z) \\ &- a_5g((\nabla_W Q)Y,Z) - a_6g((\nabla_W Q)Z,Y) + a_0\eta((\nabla_W R)(\xi,Y)Z) + a_1(\nabla_W S)(Y,Z) \\ &+ a_2(\nabla_W S)(\xi,Z)\eta(Y) + a_3(\nabla_W S)(Y,\xi)\eta(Z) + a_4g(Y,Z)\eta((\nabla_W Q)\xi) \\ &+ a_5\eta(Z)\eta((\nabla_W Q)Y) + a_6\eta(Y)\eta((\nabla_W Q)Z) + a_7(\nabla_W r)[g(Y,Z) - \eta(Y)\eta(Z)] \\ &\quad = A(W)[[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]S(Y,Z) + [a_4 + 2na_7]rg(Y,Z)].\end{aligned} \tag{12}$$

Putting $Z = \xi$ in (12) and simplifying, we get

$$\begin{aligned}&- [a_0 + 2na_1 + a_2 + a_3](\nabla_W S)(Y,\xi) - [a_4 + 2na_7](\nabla_W r)\eta(Y) \\ &- a_6g((\nabla_W Q)\xi,Y) + a_3(\nabla_W S)(Y,\xi) \\ &\quad = A(W)\eta(Y)[[a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]2nk + [a_4 + 2na_7]r].\end{aligned} \tag{13}$$

We know that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W\xi). \tag{14}$$

By using (3), (5) in (14), we have

$$(\nabla_W S)(Y,\xi) = S(Y,\varphi W) + S(Y,\varphi hW) - 2nkg(Y,\varphi W) - 2nkg(Y,\varphi hW). \tag{15}$$

Substituting (15) in (13), we obtain

$$\begin{aligned}&- [a_0 + 2na_1 + a_2 + a_6]\{S(Y,\varphi W) + S(Y,\varphi hW) - 2nkg(Y,\varphi W) - 2nkg(Y,\varphi hW)\} \\ &- [a_4 + 2na_7](\nabla_W r)\eta(Y) \\ &\quad = [a_0 + (2n+1)a_1 + a_2 + a_3 + a_5 + a_6]2nkA(W)\eta(Y) + [a_4 + 2na_7]rA(W)\eta(Y).\end{aligned} \tag{16}$$

Replacing Y by φY in (16), we have

$$\begin{aligned}&-[a_0+2na_1+a_2+a_6]\{S(\varphi Y,\varphi W)+S(\varphi Y,\varphi hW)\\&\quad-2nk[g(\varphi Y,\varphi W)+g(\varphi Y,\varphi hW)]\}=0.\end{aligned}\tag{17}$$

If $[a_0+2na_1+a_2+a_6]\neq 0$, then by virtue of (1) and (8) in (17), we obtain

$$\begin{aligned}S(Y,W)&=[2nk-2(n-1)(k-1)]g(Y,W)+2(n-1)(k-1)\eta(Y)\eta(W)\\&\quad+[2nk+2(n-1)(k-1)]g(Y,hW)-[2nk+2(n-1)(k-1)]\eta(Y)\eta(hW).\end{aligned}\tag{18}$$

Replacing in place W as hW in (18), we get

$$g(Y,hW)=n(k-1)g(Y,W)-n(k-1)\eta(Y)\eta(W).\tag{19}$$

By substituting (19) in (18), we get

$$\begin{aligned}S(Y,W)&=[2nk+2(n-1)^2(k-1)+2n^2k(k-1)]g(Y,W)\\&\quad+[-2(n-1)^2(k-1)-2n^2k(k-1)]\eta(Y)\eta(W).\end{aligned}$$

Hence we state the following

Theorem 1. *A τ-φ-recurrent $N(k)$-contact metric manifold is an η-Einstein manifold with* $-[a_0+2na_1+a_2+a_6]\neq 0$.

Now from (10), we have

$$(\nabla_W\tau)(X,Y)Z=\eta((\nabla_W\tau)(X,Y)Z)\xi-A(W)\tau(X,Y)Z,\tag{20}$$

from (20) and the second Bianchi identity, we get

$$\begin{aligned}&(\nabla_W\tau)(X,Y)Z+(\nabla_X\tau)(Y,W)Z+(\nabla_Y\tau)(W,X)Z\\&\quad=\eta((\nabla_W\tau)(X,Y)Z)\xi+\eta((\nabla_X\tau)(Y,W)Z)\xi+\eta((\nabla_Y\tau)(W,X)Z)\xi\\&\qquad-\{A(W)\tau(X,Y)Z+A(X)\tau(Y,W)Z+A(Y)\tau(W,X)Z\}.\end{aligned}\tag{21}$$

From (21), we get

$$A(W)\eta(\tau(X,Y)Z)+A(X)\eta(\tau(Y,W)Z)+A(Y)\eta(\tau(W,X)Z)=0.\tag{22}$$

By using (9) in (22), we obtain

$$\begin{aligned}&A(W)\{a_0k[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]+a_1\eta(X)[2(n-1)\{g(Y,Z)+g(hY,Z)\}\\&\quad+[2(1-n)+2nk]\eta(Y)\eta(Z)]+a_2\eta(Y)[2(n-1)g(X,Z)+2(n-1)g(hX,Z)\\&\quad+[2(1-n)+2nk]\eta(X)\eta(Z)]+a_3\eta(Z)[2(n-1)g(X,Y)+2(n-1)g(hX,Y)\\&\quad+[2(1-n)+2nk]\eta(X)\eta(Y)]+2nka_4g(Y,Z)\eta(X)+2nka_5g(X,Z)\eta(Y)\\&\quad+2nka_6g(X,Y)\eta(Z)+a_7[g(Y,Z)\eta(X)-g(X,Z)\eta(Y)]\}+A(X)\{a_0k[g(W,Z)\eta(Y)\\&\quad-g(Y,Z)\eta(W)]+a_1\eta(Y)[2(n-1)g(W,Z)+2(n-1)g(hW,Z)\\&\quad+[2(1-n)+2nk]\eta(W)\eta(Z)]+a_2\eta(W)[2(n-1)g(Y,Z)+2(n-1)g(hY,Z)\\&\quad+[2(1-n)+2nk]\eta(Y)\eta(Z)]+a_3\eta(Z)[2(n-1)g(Y,W)+2(n-1)g(hY,W)\\&\quad+[2(1-n)+2nk]\eta(Y)\eta(W)]+2nka_4g(W,Z)\eta(Y)+2nka_5g(Y,Z)\eta(W)\\&\quad+2nka_6g(Y,W)\eta(Z)+a_7[g(W,Z)\eta(Y)-g(Y,Z)\eta(W)]\}+A(Y)\{a_0k[g(X,Z)\eta(W)\\&\quad-g(W,Z)\eta(X)]+a_1\eta(W)[2(n-1)g(X,Z)+2(n-1)g(hX,Z)\\&\quad+[2(1-n)+2nk]\eta(X)\eta(Z)]+a_2\eta(X)[2(n-1)g(W,Z)+2(n-1)g(hW,Z)\\&\quad+[2(1-n)+2nk]\eta(W)\eta(Z)]+a_3\eta(Z)[2(n-1)g(W,X)+2(n-1)g(hW,X)\\&\quad+[2(1-n)+2nk]\eta(W)\eta(X)]+2nka_4g(X,Z)\eta(W)+2nka_5g(W,Z)\eta(X)\\&\quad+2nka_6g(W,X)\eta(Z)+a_7[g(X,Z)\eta(W)-g(W,Z)\eta(X)]\}=0.\end{aligned}\tag{23}$$

Putting $Y = Z = e_i$ in (23), we get

$$\begin{aligned} &A(W)\eta(X)[(2n-1)(a_0k+ra_7)+2na_1[2(n-1)+k]+[2nk+2(n-1)]a_2 \\ &\quad +2nk[a_3+(2n+1)a_4+2a_5+a_6]]+A(X)\eta(W)[-(2n-1)(a_0k+ra_7) \\ &\quad +a_1[2(n-1)+k]+2na_2[k+2(n-1)]+2nk[a_3+2a_4+(2n+1)a_5+a_6]] \\ &\quad +[a_1+a_2+a_3][2(1-n)+2nk]A(\xi)\eta(X)\eta(W)+2(n-1)a_1A(hX)\eta(W) \\ &\quad +2(n-1)a_2A(hW)\eta(X)+[2(n-1)a_3+2nka_6]A(\xi)g(W,X) \\ &\quad +2(n-1)a_3A(\xi)g(hW,X)=0. \end{aligned} \tag{24}$$

Putting $X = \xi$ in (24) and simplifying, we obtain

$$\begin{aligned} &A(W)[(2n-1)(a_0k+ra_7)+2na_1[2(n-1)+k]+a_2[2nk+2(n-1)] \\ &\quad +2nk[a_3+(2n+1)a_4+2a_5+a_6]]+A(\xi)\eta(W)[-(2n-1)(a_0k+ra_7) \\ &\quad +a_2[2(n-1)(2n-1)+4nk]+4nk[a_1+a_3+a_4+a_6]+2nk(2n+1)a_5] \\ &\quad +2(n-1)a_2A(hW)=0. \end{aligned} \tag{25}$$

Replacing W by hW in (25), we obtain

$$A(hW) = \frac{2(n-1)a_2(k-1)}{L}[A(W)-A(\xi)\eta(W)], \tag{26}$$

where

$$\begin{aligned} L = (2n-1)(a_0k+ra_7)+2na_1[2(n-1)+k]+a_2[2nk+2(n-1)] \\ +2nk[a_3+(2n+1)a_4+2a_5+a_6]. \end{aligned}$$

Substituting (26) in (25), we get

$$A(W) = \frac{-[ML-E]}{L^2+E}A(\xi)\eta(W), \tag{27}$$

where

$$\begin{aligned} M &= -(2n-1)(a_0k+ra_7)+a_2[2(n-1)(2n-1)+4nk] \\ &\quad +2nk[2a_1+2a_3+2a_4+(2n+1)a_5+2a_6], \\ E &= 4(n-1)^2a_2^2(k-1). \end{aligned}$$

Here $A(\xi) = g(\xi,\rho)$, ρ being the vector field associated to the 1-form A, that is, $g(X,\rho) = A(X)$. Hence we state the following

Theorem 2. *In a τ-φ-recurrent $N(k)$-contact metric manifold the characteristic vector field ξ and the vector field ρ associated to the 1-form A are codirectional and the 1-form A is given in* (27).

3 τ-φ-SYMMETRIC $N(k)$-CONTACT METRIC MANIFOLD

In this section we define τ-φ-symmetric $N(k)$-contact metric manifold by

$$\varphi^2((\nabla_W\tau)(X,Y)Z) = 0$$

for all vector fields X, Y, Z, W, which are orthogonal to ξ. By using (6), (7) in (9), we get

$$
\begin{aligned}
\tau(X,Y)Z = {} & a_0R(X,Y)Z + a_1[2(n-1)g(Y,Z)X + [2(1-n)+2nk]\eta(Y)\eta(Z)X \\
& + 2(n-1)g(hY,Z)X] + a_2[2(n-1)g(X,Z)Y + 2(n-1)g(hX,Z)Y \\
& + [2(1-n)+2nk]\eta(X)\eta(Z)Y] + a_3[2(n-1)g(X,Y)Z + 2(n-1)g(hX,Y)Z \\
& + [2(1-n)+2nk]\eta(X)\eta(Y)Z] + a_4g(Y,Z)[2(n-1)X + 2(n-1)hX \\
& + [2(1-n)+2nk]\eta(X)\xi] + a_5g(X,Z)[2(n-1)Y + 2(n-1)hY \\
& + [2(1-n)+2nk]\eta(Y)\xi] + a_6g(X,Y)[2(n-1)Z + 2(n-1)hZ \\
& + [2(1-n)+2nk]\eta(Z)\xi] + a_7r[g(Y,Z)X - g(X,Z)Y].
\end{aligned} \tag{28}
$$

Differentiating (28) with respect to W, we obtain

$$
\begin{aligned}
(\nabla_W\tau)(X,Y)Z = {} & a_0(\nabla_WR)(X,Y)Z + a_1[[2(1-n)+2nk]\{g(Y,\nabla_W\xi)\eta(Z)X \\
& + g(Z,\nabla_W\xi)\eta(Y)X\} + 2(n-1)g((\nabla_Wh)Y,Z)X] + a_2[2(n-1)g((\nabla_Wh)X,Z)Y \\
& + [2(1-n)+2nk]\{g(X,\nabla_W\xi)\eta(Z)Y + g(Z,\nabla_W\xi)\eta(X)Y\}] \\
& + a_3[2(n-1)g((\nabla_Wh)X,Y) + [2(1-n)+2nk]\{g(X,\nabla_W\xi)\eta(Y) + g(Y,\nabla_W\xi)\eta(X)\}]Z \\
& + a_4g(Y,Z)[2(n-1)(\nabla_Wh)X + [2(1-n)+2nk]\{g(X,\nabla_W\xi)\xi + \eta(X)\nabla_W\xi\}] \\
& + a_5g(X,Z)[2(n-1)(\nabla_Wh)Y + [2(1-n)+2nk]\{g(Y,\nabla_W\xi)\xi + \eta(Y)\nabla_W\xi\}] \\
& + a_6g(X,Y)[2(n-1)(\nabla_Wh)Z + [2(1-n)+2nk]\{g(Z,\nabla_W\xi)\xi + \eta(Z)\nabla_W\xi\}] \\
& + a_7(\nabla_Wr)[g(Y,Z)X - g(X,Z)Y].
\end{aligned} \tag{29}
$$

We assume that all vector fields X, Y, Z, W are orthogonal to ξ, then we have

$$
\begin{aligned}
(\nabla_W\tau)(X,Y)Z = {} & a_0(\nabla_WR)(X,Y)Z + a_4g(Y,Z)[2(n-1)\{(1-k)g(W,\varphi X) \\
& + g(W,h\varphi X)\}\xi + [2(1-n)+2nk]\{-g(X,\varphi W) - g(X,\varphi hW)\}\xi] \\
& + a_5g(X,Z)[2(n-1)\{(1-k)g(W,\varphi Y) + g(W,h\varphi Y)\}\xi + [2(1-n)+2nk]\{-g(Y,\varphi W) \\
& - g(Y,\varphi hW)\}\xi] + a_6g(X,Y)[2(n-1)\{(1-k)g(W,\varphi Z) + g(W,h\varphi Z)\}\xi \\
& + [2(1-n)+2nk]\{-g(Z,\varphi W) - g(Z,\varphi hW)\}\xi] + a_7(\nabla_Wr)[g(Y,Z)X - g(X,Z)Y].
\end{aligned}
$$

Applying φ^2 on both sides of the above equation, we have

$$
\varphi^2((\nabla_W\tau)(X,Y)Z) = a_0\varphi^2((\nabla_WR)(X,Y)Z) + a_7(\nabla_Wr)[g(Y,Z)\varphi^2X - g(X,Z)\varphi^2Y].
$$

Hence we state the following

Theorem 3. *Let M be an $N(k)$-contact metric manifold. If any two of the following statements holds, then the remaining statement holds*

1) *M is locally τ-φ-symmetric,*
2) *M is locally φ-symmetric,*
3) *either $a_7 = 0$ or r is constant.*

4 GLOBALLY τ-φ-SYMMETRIC $N(k)$-CONTACT METRIC MANIFOLD

In this section, we define globally τ-φ-symmetric $N(k)$-contact metric manifold by

$$
\varphi^2((\nabla_W\tau)(X,Y)Z) = 0 \tag{30}
$$

for all vector fields X, Y, Z, W, which are arbitrary vector fields. By (1) and (30), we obtain

$$-((\nabla_W \tau)(X,Y)Z) + \eta((\nabla_W \tau)(X,Y)Z)\xi = 0. \tag{31}$$

By taking an innerproduct with U in (31), we have

$$-g(((\nabla_W \tau)(X,Y)Z), U) + \eta((\nabla_W \tau)(X,Y)Z)g(\xi, U) = 0. \tag{32}$$

Let $\{e_i : i = 1, 2, \dots, 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X = U = e_i$, in (32) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$-g(((\nabla_W \tau)(e_i,Y)Z), e_i) + \eta((\nabla_W \tau)(e_i,Y)Z)g(\xi, e_i) = 0. \tag{33}$$

By using (29) in (33) and simplifying, we get

$$\begin{aligned}
&-a_0(\nabla_W S)(Y,Z) - (2na_1 + a_2 + a_5)[2(n-1)\{(1-k)g(W,\varphi Y)\eta(Z) + g(W,h\varphi Y)\eta(Z)\\
&\quad + \eta(Y)g(h(\varphi W + \varphi hW), Z)\} + [2(1-n) + 2nk]\{-g(\varphi W, Y)\eta(Z) - g(Y, \varphi hW)\eta(Z)\\
&\quad - \eta(Y)g(\varphi W, Z) - \eta(Y)g(\varphi hW, Z)\}] - (a_3 + a_6)[2(n-1)\{-(1-k)g(\varphi W, Z)\eta(Y)\\
&\quad - g(\varphi hW, Z)\eta(Y) + \eta(Z)g(h(\varphi W + \varphi hW), Y)\} + [2(1-n) + 2nk]\{-g(\varphi W, Z)\eta(Y)\\
&\quad - g(Z, \varphi hW)\eta(Y) - \eta(Z)g(\varphi W, Y) - \eta(Z)g(\varphi hW, Y)\}] - 2na_7(\nabla_W r)g(Y,Z)\\
&\quad + a_0\eta((\nabla_W R)(\xi, Y)Z) + a_2[[2(1-n) + 2nk]\{-g(\varphi W, Z)\eta(Y) - g(\varphi hW, Z)\eta(Y)\}\\
&\quad + 2(n-1)g(h(\varphi W + \varphi hW), Z)\eta(Y)] + a_3[2(n-1)g(h(\varphi W + \varphi hW), Y)\eta(Z)\\
&\quad + [2(1-n) + 2nk]\{-g(\varphi W, Y)\eta(Z) - g(\varphi hW, Y)\eta(Z)\}]\\
&\quad + a_5\eta(Z)[2(n-1)\{(1-k)g(W, \varphi Y) + g(W, h\varphi Y)\} + [2(1-n) + 2nk]\{-g(\varphi W, Y)\\
&\quad - g(Y, \varphi hW)\}] + a_6\eta(Y)[[2(1-n) + 2nk]\{-g(\varphi W, Z) - g(Z, \varphi hW)\}\\
&\quad + 2(n-1)\{(1-k)g(W, \varphi Z) + g(W, h\varphi Z)\}] + (\nabla_W r)a_7[g(Y,Z) - \eta(Y)\eta(Z)] = 0.
\end{aligned} \tag{34}$$

Putting $Z = \xi$ and using the condition

$$(\nabla_W S)(Y, \xi) = S(Y, \varphi W) + S(Y, \varphi hW) - 2nkg(Y, \varphi W) - 2nkg(Y, \varphi hW),$$

in (34) we obtain

$$\begin{aligned}
&-a_0 S(Y,W) + [[2nk - 2(n-1)(k-1)]a_0 + [2na_1 + a_2 + a_6][2(1-n) + 2nk]\\
&\quad + 2(n-1)(1-k)[2na_1 + a_2]]g(Y,W) + [2(n-1)(k-1)[a_0 + 2na_1 + a_2]\\
&\quad - [2na_1 + a_2 + a_6][2(1-n) + 2nk]]\eta(Y)\eta(W) + [[2nk + 2(n-1)]a_0\\
&\quad + [2na_1 + a_2 + a_6][2(1-n) + 2nk] + 2(n-1)[2na_1 + a_2]]g(Y, hW)\\
&\quad - 2(n-1)a_6[(k-1)[g(Y,W) - \eta(Y)\eta(W)] - g(hW, Y)] = 0.
\end{aligned} \tag{35}$$

Replacing W by hW in (35), we obtain

$$g(Y, hW) = \frac{E}{F}[g(Y,W) - \eta(Y)\eta(W)], \tag{36}$$

where, $E = [[2nk - 2(n-1)]a_0 + [2na_1 + a_2 + a_6][2(1-n) + 2nk] + 2(n-1)(a_0 + 2na_1 + a_2 + a_6)](k-1)$ and $F = [[2nk - 2(n-1)]a_0 - 2(n-1)(k-1)(a_0 + 2na_1 + a_2 + a_6) + [2na_1 + a_2 + a_6][2(1-n) + 2nk]]$. By substituting (36) in (35), we obtain

$$S(Y,W) = \alpha g(Y,W) + \beta\eta(Y)\eta(W),$$

where $\alpha = [\frac{N}{a_0} + \frac{LE}{a_0 F}]$ and $\beta = [\frac{P}{a_0} - \frac{LE}{a_0 F}]$,
$N = [[2nk - 2(n-1)(k-1)]a_0 + [2na_1 + a_2 + a_6][2(1-n) + 2nk] + 2(n-1)(1-k)[2na_1 + a_2]]$,
$L = [[2nk + 2(n-1)]a_0 + [2na_1 + a_2 + a_6][2(1-n) + 2nk] + 2(n-1)[2na_1 + a_2 + a_6]]$,
$P = [2(n-1)(k-1)(a_0 + 2na_1 + a_2) - [2na_1 + a_2 + a_6][2(1-n) + 2nk]]$.

Theorem 4. *A globally τ-φ-symmetric $N(k)$-contact metric manifold is an η-Einstein manifold with $a_0 \neq 0$.*

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В цій статті вивчається тензор τ-кривини в $N(k)$-контактному метричному многовиді. Досліджується τ-φ-рекурентний, τ-φ-симетричний і глобально τ-φ-симетричний $N(k)$-контактний метричний многовид.

Ключові слова і фрази: контактний метричний многовид, симетрія.

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В этой статье изучается тензор τ-кривизны в $N(k)$-контактном метрическом многообразии. Исследуется τ-φ-рекуррентное, τ-φ-симметрическое и глобально τ-φ-симметрическое $N(k)$-контактное метрическое многообразие.

Ключевые слова и фразы: контактное метрическое многообразие, симметрия.