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STEFLUK S.D.

PARTITION POLYNOMIALS DEFINED BY PARAFUNCTIONS OF TRIANGULAR MATRICES WITH ARBITRARY FIRST TWO COLUMNS

We research a wide class of partition polynomials that satisfy paradeterminants of sloping triangular matrix with arbitrary first two columns.

Key words and phrases: partition polynomial, parafunction, paradeterminant, parapermanent.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: ljanys_89@mail.ru

INTRODUCTION

Partition polynomials arise in many areas of mathematics: in differentiation of composite functions (Faa di Bruno's formula), in algebra, combinatorics (see [2, p. 1]), number theory [1]. Partition polynomials are studied by many analysts: Beel [3], Riordan [4], Platonov [5], Kuzmyn and Leonova [6,7]. They are usually associated with linear recurrence relations that allow to generate them in an effective way. But for historical reasons the recurrence relations and the corresponding partition polynomials were studied mostly separately. Due to the introduction for triangular matrices in particular their parafunctions it became possible to construct binary relations between parafunctions of triangular matrices, polynomial partitions and linear recurrence relations. Moreover it became possible to apply a unified approach to the study of all partition polynomials, to introduce the concept of inverse partition of polynomials, etc. In [8] a class of partition polynomials that are defined by parafunctions of triangular matrices with arbitrary first column was studied. This paper describes the partition polynomials, that are defined by parafunctions of triangular matrices with any first two columns.

1 PRELIMINARIES AND DENOTATIONS

Let K be a fixed number field.

Definition 1.1. *A triangular table of numbers from some field K*

$$A=\begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{1}$$

is called a triangular matrix, and number n — its order.

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Note that a triangular matrix in the definition is not a matrix in the usual sense, because it is triangular rather than rectangular table of numbers.

Every element a_{ij} of the matrix (1) corresponds with the $(i-j+1)$ elements $a_{ik},\ k=j,\dots,i,$ which are called *the derived elements* of the matrix generated by the *key element* a_{ij}.

The product of all derived elements generated by the element a_{ij} can be denoted $\{a_{ij}\}$ and called the *factorial product of the key element* a_{ij}, i.e.

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

Definition 1.2. *If A — is a triangular matrix (1), then the paradeterminant and the parapermanent of the triangular matrix are, respectively, the following numbers:*

$$ddet(A) = \sum_{r=1}^{n} \sum_{\alpha_1+\ldots+\alpha_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s,\alpha_1+\ldots+\alpha_{s-1}+1}\},$$

$$pper(A) = \sum_{r=1}^{n} \sum_{\alpha_1+\ldots+\alpha_r=n} \prod_{s=1}^{r} \{a_{\alpha_1+\ldots+\alpha_s,\alpha_1+\ldots+\alpha_{s-1}+1}\},$$

where the summation is made by a set of natural solutions of the equation $\alpha_1+\ldots+\alpha_r=n$.

Theorem 1 ([9]). *For a triangular matrix the following equalities hold:*

$$\left\langle \begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \ldots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right\rangle_n = \left\langle \begin{matrix} (a_{11}-a_{21})\cdot a_{22} & & & \\ (a_{11}-a_{31})\cdot a_{32} & a_{33} & & \\ \vdots & \ldots & \ddots & \\ (a_{11}-a_{n1})\cdot a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix} \right\rangle_{n-1}, \tag{2}$$

$$\left[\begin{matrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \ldots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix} \right]_n = \left[\begin{matrix} (a_{11}+a_{21})\cdot a_{22} & & & \\ (a_{11}+a_{31})\cdot a_{32} & a_{33} & & \\ \vdots & \ldots & \ddots & \\ (a_{11}+a_{n1})\cdot a_{n2} & a_{n3} & \cdots & a_{nn} \end{matrix} \right]_{n-1}. \tag{3}$$

Theorem 2. *Let the polynomials* $y_n(x_1,x_2,\ldots,x_n), n=0,1,\ldots,$ *be given by the recurrence equation*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + \ldots + (-1)^{n-2} x_{n-1} y_1 + (-1)^{n-1} \tau_{n1} x_n y_0, \tag{4}$$

where $y_0=1$, *then the following equalities hold:*

$$y_n = \mathrm{ddet} \left(\begin{matrix} \tau_{11} x_1 & & & \\ \tau_{21}\frac{x_2}{x_1} & x_1 & & \\ \vdots & \ldots & \ddots & \\ \tau_{n1}\frac{x_n}{x_{n-1}} & \ldots & \frac{x_2}{x_1} & x_1 \end{matrix} \right), \tag{5}$$

$$y_n = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{n-k} \left(\sum_{i=1}^{n} \lambda_i \tau_{i1} \right) \frac{(k-1)!}{\lambda_1!\lambda_2!\cdot\ldots\cdot\lambda_n!} x_1^{\lambda_1} x_2^{\lambda_2} \cdot\ldots\cdot x_n^{\lambda_n}, \tag{6}$$

where $k=\lambda_1+\lambda_2+\ldots+\lambda_n$.

2 The main results

Theorem 3. *Let the polynomials be given by the recurrence equation*

$$y_n = x_1 y_{n-1} - x_2 y_{n-2} + x_3 y_{n-3} - \ldots + (-1)^{n-2}\tau_{n2}x_{n-1}y_1 + (-1)^{n-1}\tau_{n1}\tau_{n2}x_n y_0, \tag{7}$$

where $y_0 = 1$, then the following equalities hold:

$$y_n = \left\langle \begin{array}{cccccc} \tau_{11}x_1 & & & & & \\ \tau_{21}\frac{x_2}{x_1} & \tau_{22}x_1 & & & & \\ \tau_{31}\frac{x_3}{x_2} & \tau_{32}\frac{x_2}{x_1} & x_1 & & & \\ \vdots & \cdots & \cdots & \ddots & & \\ \tau_{n-1,1}\frac{x_{n-1}}{x_{n-2}} & \tau_{n-1,2}\frac{x_{n-2}}{x_{n-3}} & \frac{x_{n-3}}{x_{n-4}} & \cdots & x_1 & \\ \tau_{n1}\frac{x_n}{x_{n-1}} & \tau_{n2}\frac{x_{n-1}}{x_{n-2}} & \frac{x_{n-2}}{x_{n-3}} & \cdots & \frac{x_2}{x_1} & x_1 \end{array} \right\rangle_n \tag{8}$$

$$y_n = \sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n} (-1)^{n-k}\frac{A(\lambda,\tau)}{\lambda_1!\cdot\ldots\cdot\lambda_n!}x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}, \tag{9}$$

where

$$A(\lambda,\tau) = \left(\left(\lambda_1(\lambda_1-1)\tau_{11}\tau_{22} + \sum_{i=2}^{n-1}\lambda_1\lambda_i\tau_{11}\tau_{i+1,2}\right)\cdot(k-2)! + \sum_{i=1}^{n-1}\lambda_{i+1}\tau_{i+1,1}\tau_{i+1,2}\cdot(k-1)!\right)$$

Proof.

$$y_n = \left\langle \begin{array}{cccccc} \tau_{11}x_1 & & & & & \\ \tau_{21}\frac{x_2}{x_1} & \tau_{22}x_1 & & & & \\ \tau_{31}\frac{x_3}{x_2} & \tau_{32}\frac{x_2}{x_1} & x_1 & & & \\ \vdots & \cdots & \cdots & \ddots & & \\ \tau_{n-1,1}\frac{x_{n-1}}{x_{n-2}} & \tau_{n-1,2}\frac{x_{n-2}}{x_{n-3}} & \frac{x_{n-3}}{x_{n-4}} & \cdots & x_1 & \\ \tau_{n1}\frac{x_n}{x_{n-1}} & \tau_{n2}\frac{x_{n-1}}{x_{n-2}} & \frac{x_{n-2}}{x_{n-3}} & \cdots & \frac{x_2}{x_1} & x_1 \end{array} \right\rangle_n$$

$$= \left\langle \begin{array}{cccccc} \left(\tau_{11}x_1 - \tau_{21}\frac{x_2}{x_1}\right) & \tau_{22}x_1 & & & & \\ \left(\tau_{11}x_1 - \tau_{31}\frac{x_3}{x_2}\right) & \tau_{32}\frac{x_2}{x_1} & x_1 & & & \\ \vdots & & \cdots & \ddots & & \\ \left(\tau_{11}x_1 - \tau_{n-1,1}\frac{x_{n-1}}{x_{n-2}}\right) & \tau_{n-1,2}\frac{x_{n-2}}{x_{n-3}} & \frac{x_{n-3}}{n-4} & \cdots & x_1 & \\ \left(\tau_{11}x_1 - \tau_{n1}\frac{x_n}{x_{n-1}}\right) & \tau_{n2}\frac{x_{n-1}}{x_{n-2}} & \frac{x_{n-2}}{n-3} & \cdots & \frac{x_2}{x_1} & x_1 \end{array} \right\rangle_{n-1}$$

$$= \sum_{\lambda_1+\ldots+(n-1)\lambda_{n-1}=n-1} (-1)^{n-1-k}\sum_{i=1}^{n-1}\lambda_i\left(\tau_{11}x_1 - \tau_{i+1,1}\frac{x_{i+1}}{x_i}\right)\tau_{i+1,2}\frac{(k-1)!}{\lambda_1!\lambda_2!\ldots\lambda_{n-1}!}x_1^{\lambda_1}\cdot\ldots\cdot x_{n-1}^{\lambda_{n-1}}$$

$$= \sum_{\lambda_1+\ldots+(n-1)\lambda_{n-1}=n-1} (-1)^{n-1-k}\tau_{11}\sum_{i=1}^{n-1}\lambda_i\tau_{i+1,2}\frac{(k-1)!}{\lambda_1!\lambda_2!\ldots\lambda_{n-1}!}x_1^{\lambda_1+1}\cdot\ldots\cdot x_{n-1}^{\lambda_{n-1}}$$

$$+ \sum_{\lambda_1+\ldots+(n-1)\lambda_{n-1}=n-1} (-1)^{n-k}\sum_{i=1}^{n-1}\lambda_i\tau_{i+1,1}\tau_{i+1,2}\frac{(k-1)!}{\lambda_1!\ldots\lambda_{n-1}!}x_1^{\lambda_1}\cdot\ldots\cdot x_i^{\lambda_i-1}x_{i+1}^{\lambda_{i+1}+1}\cdot\ldots\cdot x_{n-1}^{\lambda_{n-1}}$$

where $\lambda_1+\lambda_2+\ldots+\lambda_{n-1} = k$.

The first sum after substituting $\lambda_1+1=\lambda_1^{'}, \lambda_2=\lambda_2^{'}, \ldots, \lambda_n=\lambda_n^{'}=0$ will have the form

$$\sum_{\lambda_1^{'}+2\lambda_2^{'}+\ldots+n\lambda_n^{'}=n}(-1)^{n-k^{'}}\left((\lambda_1^{'}-1)\tau_{11}\tau_{22}+\tau_{11}\sum_{i=2}^{n-1}\lambda_i^{'}\tau_{i+1,2}\right)\frac{(k^{'}-2)!}{(\lambda_1^{'}-1)!\lambda_2^{'}!\ldots\lambda_n^{'}!}x_1^{\lambda_1^{'}}x_2^{\lambda_2^{'}}\cdot\ldots\cdot x_n^{\lambda_n^{'}}$$

and the second one — after substituting $\lambda_1=\lambda_1^{'}, \ldots, \lambda_{i-1}=\lambda_{i-1}^{'}, \lambda_i-1=\lambda_i^{'}, \lambda_{i+1}+1=\lambda_{i+1}^{'}, \lambda_{i+2}=\lambda_{i+2}^{'}, \ldots, \lambda_n=\lambda_n^{'}=0$ — will be in the form

$$\sum_{\lambda_1^{'}+\ldots+n\lambda_n^{'}=n}(-1)^{n-k^{'}}\sum_{i=1}^{n-1}\tau_{i+1,1}\tau_{i+1,2}\frac{(k^{'}-1)!}{\lambda_1^{'}!\cdot\ldots\lambda_i^{'}!(\lambda_{i+1}^{'}-1)!\lambda_{i+2}^{'}!\cdot\ldots\lambda_n^{'}!}\cdot x_1^{\lambda_1^{'}}\cdot\ldots x_i^{\lambda_i^{'}}x_{i+1}^{\lambda_{i+1}^{'}}\cdot\ldots x_n^{\lambda_n^{'}}.$$

Finally, we note that the expansion of the paradeterminant (8) according to the elements of the last range leads to the recurrence relations (7). □

This theorem can be proved in a similar way.

Theorem 4. *Let the polynomials be given by the recurrence equation*

$$y_n=x_1y_{n-1}+x_2y_{n-2}+x_3y_{n-3}+\ldots+x_{n-2}y_2+\tau_{n2}x_{n-1}y_1+\tau_{n1}\tau_{n2}x_ny_0, \tag{10}$$

where $y_0=1$, and τ_{ij} are some parameters, then the following equalities hold:

$$y_n=\begin{bmatrix}\tau_{11}x_1 & & & & & \\ \tau_{21}\frac{x_2}{x_1} & \tau_{22}x_1 & & & & \\ \tau_{31}\frac{x_3}{x_2} & \tau_{32}\frac{x_2}{x_1} & x_1 & & & \\ \vdots & \ldots & \ldots & \ddots & & \\ \tau_{n-1,1}\frac{x_{n-1}}{x_{n-2}} & \tau_{n-1,2}\frac{x_{n-2}}{x_{n-3}} & \frac{x_{n-3}}{x_{n-4}} & \cdots & x_1 & \\ \tau_{n1}\frac{x_n}{x_{n-1}} & \tau_{n2}\frac{x_{n-1}}{x_{n-2}} & \frac{x_{n-2}}{x_{n-3}} & \cdots & \frac{x_2}{x_1} & x_1\end{bmatrix}_n \tag{11}$$

$$y_n=\sum_{\lambda_1+2\lambda_2+\ldots+n\lambda_n=n}A(\lambda,\tau)x_1^{\lambda_1}\cdot\ldots\cdot x_n^{\lambda_n}, \tag{12}$$

where

$$A(\lambda,\tau)=\left(\left(\lambda_1(\lambda_1-1)\tau_{11}\tau_{22}+\sum_{i=2}^{n-1}\lambda_1\lambda_i\tau_{11}\tau_{i+1,2}\right)\frac{(k-2)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!}\right.$$
$$\left.+\sum_{i=1}^{n-1}\lambda_{i+1}\tau_{i+1,1}\tau_{i+1,2}\cdot\frac{(k-1)!}{\lambda_1!\cdot\ldots\cdot\lambda_n!}\right).$$

3 EXAMPLE

Let's find $A(\lambda,\tau)$ in the partition polynomials (9) for case when $n=15$, $\lambda_1=3$, $\lambda_2=1$, $\lambda_5=2$. In that case $k=6$ and

$$A(\lambda,\tau)=4!\left(3\cdot 2\tau_{11}\tau_{22}+\sum_{i=2}^{14}3\lambda_i\tau_{11}\tau_{i+1,2}\right)+5!\left(\sum_{i=1}^{14}\lambda_{i+1}\tau_{i+1,1}\tau_{i+1,2}\right).$$

Thus, the coefficient of $x_1^3x_2x_5^2$ is equal to

$$(-1)^{15-6}A(\lambda,\tau)=\frac{3\cdot 2\cdot 4!}{3!\cdot 1!\cdot 2!}a_{11}a_{22}-\frac{3\cdot 4!}{3!\cdot 1!\cdot 2!}a_{11}a_{32}$$
$$-\frac{3\cdot 2\cdot 4!}{3!\cdot 1!\cdot 2!}a_{11}a_{62}-\frac{1\cdot 5!}{3!\cdot 1!\cdot 2!}a_{21}a_{22}-\frac{2\cdot 5!}{3!\cdot 1!\cdot 2!}a_{51}a_{52}.$$

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Стефлюк С.Д. *Многочлени розбиттів, що задаються парафункціями трикутних матриць з двома першими довільними стовпцями.* // Карпатські матем. публ. — 2014. — Т.6, №2. — С. 367–371.

Досліджується широкий клас многочленів розбиттів, які задовольняють парадетермінанти трикутної матриці з двома довільними першими стовпцями.

Ключові слова і фрази: многочлени розбиттів, парадетермінант, параперманент, парафункція.

Стефлюк С.Д. *Многочлены разбиений, задаваемые парафункциями треугольных матриц с двумя первыми произвольными столбцами* // Карпатские матем. публ. — 2014. — Т.6, №2. — С. 367–371.

Исследуется широкий класс многочленов разбиений, порождающихся парадетерминантами треугольных матриц с двумя первыми произвольными столбцами.

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