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GERASIMENKO V.I.

NEW APPROACH TO DERIVATION OF QUANTUM KINETIC EQUATIONS WITH INITIAL CORRELATIONS

We propose a new approach to the derivation of kinetic equations from dynamics of large particle quantum systems, involving correlations of particle states at initial time. The developed approach is based on the description of the evolution within the framework of marginal observables in scaling limits. As a result the quantum Vlasov-type kinetic equation with initial correlations is constructed and the statement relating to the property of a propagation of initial correlations is proved in a mean field limit.

Key words and phrases: marginal observable, kinetic equation with initial correlations.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01601, Kyiv, Ukraine
E-mail: gerasym@imath.kiev.ua

INTRODUCTION

As it is well known the collective behavior of large particle quantum systems can be effectively described within the framework of a one-particle (marginal) density operator governed by the kinetic equation [1–4]. In this paper we consider the problem of the rigorous description of the kinetic evolution in the presence of initial correlations of quantum particles. Such initial states are typical for the condensed states of quantum gases [5–8] in contrast to the gaseous state. For example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition specified by correlations of the condensed state [5]. One more example is the influence of initial correlations on ultrafast relaxation processes in plasmas [9], [10].

The conventional approach to the rigorous derivation of the quantum kinetic equations is based on the consideration of an asymptotic behavior of a solution of the quantum BBGKY hierarchy for marginal density operators constructed within the framework of the theory of perturbations in case of initial states specified by a one-particle (marginal) density operator without correlations [11–14], i.e. such that satisfy a chaos condition. This method of the derivation of quantum kinetic equations can not be extended on case of initial states specified by initial correlations.

In the paper for the rigorous derivation of the quantum kinetic equations in the presence of initial correlations we develop a new approach based on the description of the evolution of large particle quantum systems within the framework of marginal observables governed by the dual quantum BBGKY hierarchy [15]. In article [16] a rigorous formalism of the description of the kinetic evolution of observables of quantum particles in a mean field scaling limit was developed. In this case the limit dynamics is described by the set of recurrence evolution

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equations, namely by the dual quantum Vlasov hierarchy. In this paper, using established relationships of initial states specified by initial correlations and constructed solution of the dual quantum Vlasov hierarchy for the limit marginal observables, we derive the quantum Vlasov-type kinetic equation with initial correlations. The statement relating to the property of a propagation of initial correlations is also proved.

1 PRELIMINARY FACTS

We consider a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics in the space $\mathbb{R}^3$. We will use units where $h = 2\pi\hbar = 1$ is a Planck constant, and $m = 1$ is the mass of particles.

Let the space $\mathcal{H}$ be a one-particle Hilbert space, then the n-particle space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ is a tensor product of n Hilbert spaces $\mathcal{H}$. We adopt the usual convention that $\mathcal{H}^{\otimes 0} = \mathbb{C}$. The Fock space over the Hilbert space $\mathcal{H}$ we denote by $\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty}\mathcal{H}_n$.

Let $\mathfrak{L}^1(\mathcal{H}_n)$ be the space of trace class operators $f_n \equiv f_n(1,\ldots,n) \in \mathfrak{L}^1(\mathcal{H}_n)$ that satisfy the symmetry condition: $f_n(1,\ldots,n) = f_n(i_1,\ldots,i_n)$ for arbitrary $(i_1,\ldots,i_n) \in (1,\ldots,n)$, and equipped with the norm: $\|f_n\|_{\mathfrak{L}^1(\mathcal{H}_n)} = \mathrm{Tr}_{1,\ldots,n}|f_n(1,\ldots,n)|$, where $\mathrm{Tr}_{1,\ldots,n}$ are partial traces over $1,\ldots,n$ particles. We denote by $\mathfrak{L}^1_0(\mathcal{H}_n)$ the everywhere dense set of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

We shall consider initial states of a quantum many-particle system specified by the one-particle (marginal) density operator $F_1^{0,\varepsilon} \in \mathfrak{L}^1(\mathcal{H})$ in the presence of correlations, i.e. initial states specified by the following sequence of marginal (s-particle) density operators

$$F^{(c)} = \big(I, F_1^{0,\varepsilon}(1), g_2^{\varepsilon}(1,2)\prod_{i=1}^{2}F_1^{0,\varepsilon}(i), \ldots, g_n^{\varepsilon}(1,\ldots,n)\prod_{i=1}^{n}F_1^{0,\varepsilon}(i), \ldots\big), \tag{1}$$

where I is an identity operator, the operators $g_n^{\varepsilon}(1,\ldots,n) \equiv g_n^{\varepsilon} \in \mathfrak{L}^1_0(\mathcal{H}_n)$, $n \geq 2$, are specified the initial correlations and the parameter $\varepsilon > 0$ is a mean field scaling parameter [17].

Traditionally correlations of quantum many-particle systems are described within the framework of marginal (s-particle) correlation operators which are introduced by means of the cluster expansions of the marginal density operators

$$F_s^{0,\varepsilon}(1,\ldots,s) = \sum_{\mathrm{P}\,:\,(1,\ldots,s) = \bigcup_i X_i}\ \prod_{X_i\subset \mathrm{P}} G_{|X_i|}^{0,\varepsilon}(X_i), \quad s \geq 1, \tag{2}$$

where $\sum_{\mathrm{P}:(1,\ldots,s)=\bigcup_i X_i}$ is the sum over all partitions P of the set $(1,\ldots,s)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset (1,\ldots,s)$. Hereupon solutions of cluster expansions (2)

$$G_s^{0,\varepsilon}(1,\ldots,s) = \sum_{\mathrm{P}\,:\,(1,\ldots,s) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i\subset \mathrm{P}} F_{|X_i|}^{0,\varepsilon}(X_i), \quad s \geq 1, \tag{3}$$

are interpreted as the operators that describe correlations. Hence in the case of initial data (1) sequence (3) of marginal correlation operators has the form

$$G^{(c)} = \big(I, F_1^{0,\varepsilon}(1), \tilde{g}_2^{\varepsilon}(1,2)\prod_{i=1}^{2}F_1^{0,\varepsilon}(i), \ldots, \tilde{g}_n^{\varepsilon}(1,\ldots,n)\prod_{i=1}^{n}F_1^{0,\varepsilon}(i), \ldots\big), \tag{4}$$

where the operators $\tilde{g}_n^{\varepsilon}(1,\ldots,n) \equiv \tilde{g}_n^{\varepsilon} \in \mathfrak{L}^1_0(\mathcal{H}_n)$, $n \geq 2$, specified the initial correlations are determined by the expansions

$$\tilde{g}_s^{\varepsilon} = \sum_{\mathrm{P}\,:\,Y = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i\subset \mathrm{P}} g_{|X_i|}^{\varepsilon}, \quad s \geq 2. \tag{5}$$

We remark that in case of initial data satisfying a chaos condition [2] sequence (3) of marginal correlation operators has the form

$$G^0 = \big(I, G_1^{0,\varepsilon}(1), 0, \ldots, 0, \ldots\big), \tag{6}$$

and concequently sequence (2) of marginal density operators takes the form

$$F^0 = \big(I, F_1^{0,\varepsilon}(1), \prod_{i=1}^{2} F_1^{0,\varepsilon}(i), \ldots, \prod_{i=1}^{n} F_1^{0,\varepsilon}(i), \ldots\big). \tag{7}$$

Such assumption about initial states, i.e. (7) (or (6)), is intrinsic for the kinetic description of a gas [1]. On the other hand, initial states (1) (or (4)) are typical for the condensed states of quantum gases, for example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition with the correlations which characterize the condensed state [5].

We note that the evolution of large particle quantum systems can be described not only within the framework of marginal density operators governed by the quantum BBGKY hierarchy [2] but also in terms of marginal observables governed by the dual quantum BBGKY hierarchy [15].

Let a sequence $g = (g_0, g_1, \ldots, g_n, \ldots)$ be an infinite sequence of self-adjoint bounded operators g_n defined on the Fock space $\mathcal{F}_{\mathcal{H}}$. An operator g_n defined on the n-particle Hilbert space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ will be also denoted by the symbol $g_n(1, \ldots, n)$. Let the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}})$ be the space of sequences $g = (g_0, g_1, \ldots, g_n, \ldots)$ of bounded operators g_n defined on the Hilbert space $\mathcal{H}_n$ that satisfy symmetry condition: $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$, for arbitrary $(i_1, \ldots, i_n) \in (1, \ldots, n)$, equipped with the operator norm $\|.\|_{\mathfrak{L}(\mathcal{H}_n)}$. We will also consider a more general space $\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ with the norm $\|g\|_{\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})} \doteq \max_{n\geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathfrak{L}(\mathcal{H}_n)}$, where $0 < \gamma < 1$. We denote by $\mathfrak{L}_{\gamma,0}(\mathcal{F}_{\mathcal{H}}) \subset \mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ the everywhere dense set in the space $\mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

In terms of observables the evolution of quantum many-particle systems is described by the sequence $B(t) = (B_0, B_1(t,1), \ldots, B_s(t,1,\ldots,s), \ldots)$ of marginal observables (or s-particle observables) $B_s(t,1,\ldots,s)$, $s \geq 1$, determined by the following expansions [15]:

$$B_s(t,Y) = \sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1\neq\ldots\neq j_n=1}^{s} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X)\, B_{s-n}^{0,\varepsilon}(Y \setminus X), \quad s \geq 1, \tag{8}$$

where $B(0) = (B_0, B_1^{0,\varepsilon}(1), \ldots, B_s^{0,\varepsilon}(1,\ldots,s), \ldots) \in \mathfrak{L}_\gamma(\mathcal{F}_{\mathcal{H}})$ is a sequence of initial marginal observables, and the generating operator $\mathfrak{A}_{1+n}(t)$ of expansion (8) is the $(1+n)th$-order cumulant of groups of operators (10) defined by the expansion

$$\mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) \doteq \sum_{\mathrm{P}:\, (\{Y\setminus X\}, X) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i \subset \mathrm{P}} \mathcal{G}_{|\theta(X_i)|}(t, \theta(X_i)), \tag{9}$$

where we hold abridged notations: $Y \equiv (1, \ldots, s)$, $X \equiv (j_1, \ldots, j_n) \subset Y$, and $\{Y \setminus X\}$ is the set, consisting of a single element $Y \setminus X = (1, \ldots, s) \setminus (j_1, \ldots, j_n)$, thus, the set $\{Y \setminus X\}$ is a connected subset of the set Y, the symbol $\sum_{\mathrm{P}}$ means the sum over all partitions P of the set $(\{Y \setminus X\}, j_1, \ldots, j_n)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset (\{Y \setminus X\}, X)$, and $\theta(\cdot)$ is the declusterization mapping defined as follows: $\theta(\{Y \setminus X\}, X) = Y$. In expansion (9) for $g_n \in \mathfrak{L}(\mathcal{H}_n)$ the one-parameter mapping $\mathcal{G}_n(t)$ is defined by the formula

$$\mathbb{R}^1 \ni t \mapsto \mathcal{G}_n(t) g_n \doteq e^{itH_n} g_n e^{-itH_n}, \tag{10}$$

where the Hamilton operator H_n of a system of n particles is a self-adjoint operator with the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$ has the structure

$$H_n = \sum_{i=1}^{n} K(i) + \varepsilon \sum_{i_1<i_2=1}^{n} \Phi(i_1, i_2), \tag{11}$$

and $K(i)$ is the operator of a kinetic energy of the i particle, $\Phi(i_1, i_2)$ is the operator of a two-body interaction potential and $\varepsilon > 0$ is a scaling parameter [17]. The operator $K(i)$ acts on functions ψ_n, that belong to the subspace $L^2_0(\mathbb{R}^{3n}) \subset \mathcal{D}(H_n) \subset L^2(\mathbb{R}^{3n})$ of infinitely differentiable functions with compact supports, according to the formula: $K(i)\psi_n = -\frac{1}{2}\Delta_{q_i}\psi_n$. Correspondingly, we have: $\Phi(i_1, i_2)\psi_n = \Phi(q_{i_1}, q_{i_2})\psi_n$, and we assume that the function $\Phi(q_{i_1}, q_{i_2})$ is symmetric with respect to permutations of its arguments, translation-invariant and bounded function.

On the space $\mathfrak{L}(\mathcal{H}_n)$ one-parameter mapping (10) is an isometric $*$-weak continuous group of operators. The infinitesimal generator $\mathcal{N}_n$ of this group of operators is a closed operator for the $*$-weak topology, and on its domain of the definition $\mathcal{D}(\mathcal{N}_n) \subset \mathfrak{L}(\mathcal{H}_n)$ it is defined in the sense of the $*$-weak convergence of the space $\mathfrak{L}(\mathcal{H}_n)$ by the operator

$$\mathrm{w}^*-\lim_{t\to 0} \frac{1}{t}\big(\mathcal{G}_n(t)g_n - g_n\big) = -i\,(g_n H_n - H_n g_n) \doteq \mathcal{N}_n g_n, \tag{12}$$

where H_n is the Hamiltonian (11) and the operator $\mathcal{N}_n g_n$ defined on the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$ has the structure

$$\mathcal{N}_n = \sum_{j=1}^{n} \mathcal{N}(j) + \varepsilon \sum_{j_1<j_2=1}^{n} \mathcal{N}_{\mathrm{int}}(j_1, j_2),$$

where

$$\mathcal{N}(j)g_n \doteq -i\,(g_n K(j) - K(j)g_n), \tag{13}$$

$$\mathcal{N}_{\mathrm{int}}(j_1, j_2)g_n \doteq -i\,(g_n \Phi(j_1, j_2) - \Phi(j_1, j_2)g_n). \tag{14}$$

Therefore on the space $\mathfrak{L}(\mathcal{H}_n)$ a unique solution of the Heisenberg equation for observables of a n-particle system is determined by group (10).

The simplest examples of marginal observables (8) are given by the expansions:

$$B_1(t,1) = \mathfrak{A}_1(t,1)B_1^{0,\varepsilon}(1),$$
$$B_2(t,1,2) = \mathfrak{A}_1(t,\{1,2\})B_2^{0,\varepsilon}(1,2) + \mathfrak{A}_2(t,1,2)(B_1^{0,\varepsilon}(1) + B_1^{0,\varepsilon}(2)),$$

where the corresponding order cumulants (9) of groups of operators (10) are given by the formulas

$$\mathfrak{A}_1(t,\{1,2\}) = \mathcal{G}_s(t,1,2),$$
$$\mathfrak{A}_2(t,1,2) = \mathcal{G}_s(t,1,2) - \mathcal{G}_1(t,1)\mathcal{G}_1(t,2).$$

If $\gamma < e^{-1}$, for the sequence of operators (8) the following estimate is true: $\|B(t)\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})} \leq e^2(1-\gamma e)^{-1}\|B(0)\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})}$.

A sequence of marginal observables (8) is the non-perturbative solution of recurrence evolution equations known as the dual quantum BBGKY hierarchy [15].

We note that in case of initial states specified by sequences (23) the average values (mean values) of marginal observables (8) are determined by the following positive continuous linear functional

$$\big(B(t),F^{(c)}\big) \doteq \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathrm{Tr}_{1,...,n}\, B_n(t,1,...,n)\, g_n^{\varepsilon}(1,\ldots,n) \prod_{i=1}^{n} F_1^{0,\varepsilon}(i). \tag{15}$$

For operators $B(t) \in \mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ and $F_1^{0,\varepsilon} \in \mathfrak{L}^1(\mathcal{H})$, functional (24) exists under the condition that $\|F_1^{0,\varepsilon}\|_{\mathfrak{L}^1(\mathcal{H})} < \gamma$.

2 THE DESCRIPTION OF THE KINETIC EVOLUTION WITHIN THE FRAMEWORK OF MARGINAL OBSERVABLES

In scaling limits the kinetic evolution of many-particle systems can be described within the framework of observables. We consider this problem on an example of the mean field asymptotic behavior of non-perturbative solution (8) of the dual quantum BBGKY hierarchy for marginal observables.

A mean field asymptotic behavior of marginal observables (8) is described by the following proposition [16].

Let for $B_n^{0,\varepsilon} \in \mathfrak{L}(\mathcal{H}_n)$, in the sense of the $*$-weak convergence on the space $\mathfrak{L}(\mathcal{H}_n)$ it holds

$$\mathrm{w}^*\!-\lim_{\varepsilon\to 0}(\varepsilon^{-n}B_n^{0,\varepsilon} - b_n^0) = 0, \quad n \geq 1,$$

then for arbitrary finite time interval there exists mean field scaling limit of marginal observables (8)

$$\mathrm{w}^*\!-\lim_{\varepsilon\to 0}(\varepsilon^{-s}B_s(t) - b_s(t)) = 0, \quad s \geq 1, \tag{16}$$

that are determined by the following expansions:

$$\begin{aligned} b_s(t,Y) = &\sum_{n=0}^{s-1} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{l_1\in Y} \mathcal{G}_1(t-t_1,l_1) \sum_{i_1\neq j_1=1}^{s} \mathcal{N}_{\mathrm{int}}(i_1,j_1) \\ &\times \prod_{l_2\in Y\setminus(j_1)} \mathcal{G}_1(t_1-t_2,l_2)\ldots \prod_{l_n\in Y\setminus(j_1,\ldots,j_{n-1})} \mathcal{G}_1(t_{n-1}-t_n,l_n) \\ &\times \sum_{\substack{i_n\neq j_n=1,\\ i_n,j_n\neq(j_1,\ldots,j_{n-1})}}^{s} \mathcal{N}_{\mathrm{int}}(i_n,j_n) \prod_{l_{n+1}\in Y\setminus(j_1,\ldots,j_n)} \mathcal{G}_1(t_n,l_{n+1}) b_{s-n}^0(Y\setminus(j_1,\ldots,j_n)), \end{aligned} \tag{17}$$

where the operator $\mathcal{N}_{\mathrm{int}}(i_1,j_2)$ is defined on operators $g_n \in \mathfrak{L}(\mathcal{H}_n)$ by formula (14).

The proof of this statement is based on formulas for cumulants of asymptotically perturbed groups of operators (10).

Indeed, for arbitrary finite time interval the asymptotically perturbed group of operators (10) has the following scaling limit in the sense of the $*$-weak convergence on the space $\mathfrak{L}(\mathcal{H}_s)$:

$$\mathrm{w}^*\!-\lim_{\varepsilon\to 0}\big(\mathcal{G}_s(t,Y) - \prod_{j=1}^{s}\mathcal{G}_1(t,j)\big)g_s = 0. \tag{18}$$

Taking into account analogs of the Duhamel equations for cumulants of asymptotically perturbed groups of operators [17], in view of formula (18) we have

$$\begin{aligned}
&\mathrm{w}^*-\lim_{\varepsilon\to 0}\Big(\varepsilon^{-n}\frac{1}{n!}\mathfrak{A}_{1+n}(t,\{Y\setminus X\},j_1,\dots,j_n)\\
&-\int_0^t dt_1\dots\int_0^{t_{n-1}}dt_n\prod_{l_1\in Y}\mathcal{G}_1(t-t_1,l_1)\sum_{i_1\neq j_1=1}^{s}\mathcal{N}_{\mathrm{int}}(i_1,j_1)\prod_{l_2\in Y\setminus(j_1)}\mathcal{G}_1(t_1-t_2,l_2)\dots\\
&\prod_{l_n\in Y\setminus(j_1,\dots,j_{n-1})}\mathcal{G}_1(t_{n-1}-t_n,l_n)\sum_{\substack{i_n\neq j_n=1,\\ i_n,j_n\neq(j_1,\dots,j_{n-1})}}^{s}\mathcal{N}_{\mathrm{int}}(i_n,j_n)\\
&\times\prod_{l_{n+1}\in Y\setminus(j_1,\dots,j_n)}\mathcal{G}_1(t_n,l_{n+1})\Big)g_{s-n}=0,
\end{aligned}$$

where we used notations accepted in (17) and $g_{s-n}\equiv g_{s-n}((1,\dots,s)\setminus(j_1,\dots,j_n))$, $n\geq 1$. As a result of this equality we establish the validity of statement (16) for expansion (8) of marginal observables.

If $b^0\in\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, then the sequence $b(t)=(b_0,b_1(t),\dots,b_s(t),\dots)$ of limit marginal observables (17) is a generalized global solution of the Cauchy problem of the dual quantum Vlasov hierarchy

$$\frac{\partial}{\partial t}b_s(t,Y)=\sum_{j=1}^{s}\mathcal{N}(j)\,b_s(t,Y)+\sum_{j_1\neq j_2=1}^{s}\mathcal{N}_{\mathrm{int}}(j_1,j_2)\,b_{s-1}(t,Y\setminus(j_1)),\tag{19}$$

$$b_s(t)|_{t=0}=b_s^0,\quad s\geq 1,\tag{20}$$

where the infinitesimal generator $\mathcal{N}(j)$ of the group of operators $\mathcal{G}_1(t,j)$ of j particle is defined on $g_1\in\mathfrak{L}_0(\mathcal{H})$ by formula (13). It should be noted that equations set (19) has the structure of recurrence evolution equations. We give several examples of the evolution equations of the dual quantum Vlasov hierarchy (19) in terms of operator kernels of the limit marginal observables

$$\begin{aligned}
i\,\frac{\partial}{\partial t}b_1(t,q_1;q_1')&=-\frac{1}{2}(-\Delta_{q_1}+\Delta_{q_1'})b_1(t,q_1;q_1'),\\
i\,\frac{\partial}{\partial t}b_2(t,q_1,q_2;q_1',q_2')&=-\frac{1}{2}\sum_{i=1}^{2}(-\Delta_{q_i}+\Delta_{q_i'})b_2(t,q_1,q_2;q_1',q_2')\\
&\quad+\big(\Phi(q_1'-q_2')-\Phi(q_1-q_2)\big)\big(b_1(t,q_1;q_1')+b_1(t,q_2;q_2')\big).
\end{aligned}$$

We consider the mean field limit of a particular case of marginal observables, namely the additive-type marginal observables $B^{(1)}(0)=(0,B_1^{0,\varepsilon}(1),0,\dots)$ (the k-ary marginal observable is represented by the sequence $B^{(k)}(0)=(0,\dots,0,B_k^{0,\varepsilon}(1,\dots,k),0,\dots)$). In case of additive-type marginal observables expansions (8) take the following form:

$$B_s^{(1)}(t,Y)=\mathfrak{A}_s(t)\sum_{j=1}^{s}B_1^{0,\varepsilon}(j),\quad s\geq 1,\tag{21}$$

where the operator $\mathfrak{A}_s(t)$ is sth-order cumulant (9) of groups of operators (10).

If for the additive-type marginal observable $B_1^{0,\varepsilon}\in\mathfrak{L}(\mathcal{H})$ it holds

$$\mathrm{w}^*-\lim_{\varepsilon\to 0}(\varepsilon^{-1}B_1^{0,\varepsilon}-b_1^0)=0,$$

then for additive-type marginal observables (21) there exists the following mean field limit

$$\mathrm{w}^*-\lim_{\varepsilon\to 0}(\varepsilon^{-s}B_s^{(1)}(t)-b_s^{(1)}(t))=0,\quad s\geq 1,$$

where the limit additive-type marginal observable $b_s^{(1)}(t)$ is determined by a special case of expansion (17)

$$\begin{aligned}
b_s^{(1)}(t,Y)=&\int\limits_0^t dt_1\ldots\int\limits_0^{t_{s-2}}dt_{s-1}\prod_{l_1\in Y}\mathcal{G}_1(t-t_1,l_1)\sum_{i_1\neq j_1=1}^{s}\mathcal{N}_{\mathrm{int}}(i_1,j_1)\\
&\times\prod_{l_2\in Y\setminus(j_1)}\mathcal{G}_1(t_1-t_2,l_2)\ldots\prod_{l_{s-1}\in Y\setminus(j_1,\ldots,j_{s-2})}\mathcal{G}_1(t_{s-2}-t_{s-1},l_{s-1})\\
&\times\sum_{\substack{i_{s-1}\neq j_{s-1}=1,\\ i_{s-1},j_{s-1}\neq(j_1,\ldots,j_{s-2})}}^{s}\mathcal{N}_{\mathrm{int}}(i_{s-1},j_{s-1})\prod_{l_s\in Y\setminus(j_1,\ldots,j_{s-1})}\mathcal{G}_1(t_{s-1},l_s)b_1^0(Y\setminus(j_1,\ldots,j_{s-1})).
\end{aligned}\tag{22}$$

We make several examples of expansions (22) for the limit additive-type marginal observables

$$\begin{aligned}
b_1^{(1)}(t,1)&=\mathcal{G}_1(t,1)\,b_1^0(1),\\
b_2^{(1)}(t,1,2)&=\int\limits_0^t dt_1\prod_{i=1}^{2}\mathcal{G}_1(t-t_1,i)\mathcal{N}_{\mathrm{int}}(1,2)\sum_{j=1}^{2}\mathcal{G}_1(t_1,j)\,b_1^0(j).
\end{aligned}$$

Thus, for arbitrary initial states in the mean field scaling limit the kinetic evolution of quantum many-particle systems is described in terms of limit marginal observables (17) governed by the dual quantum Vlasov hierarchy (19).

Furthermore, the relation between the evolution of observables (17) and the kinetic evolution of initial states described in terms of a one-particle (marginal) density operator and correlation operators (1) is considered.

3 THE QUANTUM VLASOV-TYPE KINETIC EQUATION WITH INITIAL CORRELATIONS

We assume that for the initial one-particle (marginal) density operator $F_1^{0,\varepsilon}\in\mathfrak{L}^1(\mathcal{H})$ there exists the mean field limit $\lim_{\varepsilon\to 0}\|\varepsilon F_1^{0,\varepsilon}-f_1^0\|_{\mathfrak{L}^1(\mathcal{H})}=0$, and $\lim_{\varepsilon\to 0}\|g_n^\varepsilon-g_n\|_{\mathfrak{L}^1(\mathcal{H}_n)}=0$, $n\geq 2$, then in the mean field limit the initial state is specified by the following sequence of limit operators

$$f^{(c)}=\big(I,f_1^0(1),g_2(1,2)\prod_{i=1}^{2}f_1^0(i),\ldots,g_n(1,\ldots,n)\prod_{i=1}^{n}f_1^0(i),\ldots\big).\tag{23}$$

We note that in case of initial states specified by sequence (23) the average values (mean values) of limit marginal observables (17) are determined by the limit positive continuous linear functional (15)

$$\big(b(t),f^{(c)}\big)\doteq\sum_{n=0}^{\infty}\frac{1}{n!}\,\mathrm{Tr}_{1,\ldots,n}\,b_n(t,1,\ldots,n)\,g_n(1,\ldots,n)\prod_{i=1}^{n}f_1^0(i).\tag{24}$$

For operators $b(t)\in\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $f_1^0\in\mathfrak{L}^1(\mathcal{H})$, functional (24) exists under the condition that $\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})}<\gamma$.

We shall establish the relations of mean value functional (24) represented in terms of constructed mean field asymptotics of marginal observables (17) with its representation in terms of a solution of the quantum Vlasov-type kinetic equation with initial correlations, i.e. in case of initial states (23).

For the limit additive-type marginal observables (22) the following equality is true

$$\big(b^{(1)}(t),f^{(c)}\big)=\sum_{s=0}^{\infty}\frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\,b_{s}^{(1)}(t,1,\ldots,s)g_{s}(1,\ldots,s)\prod_{i=1}^{s}f_{1}^{0}(i)=\mathrm{Tr}_{1}\,b_{1}^{0}(1)f_{1}(t,1),$$

where the operator $b_s^{(1)}(t)$ is determined by expansion (22) and the one-particle (marginal) density operator $f_1(t,1)$ is represented by the series expansion

$$\begin{aligned}f_{1}(t,1)=\sum_{n=0}^{\infty}\int_{0}^{t}dt_{1}\ldots\int_{0}^{t_{n-1}}dt_{n}\,\mathrm{Tr}_{2,\ldots,n+1}\mathcal{G}_{1}^{*}(t-t_{1},1)\mathcal{N}_{\mathrm{int}}^{*}(1,2)\prod_{j_{1}=1}^{2}\mathcal{G}_{1}^{*}(t_{1}-t_{2},j_{1})\ldots\\ \times\prod_{i_{n}=1}^{n}\mathcal{G}_{1}^{*}(t_{n}-t_{n},i_{n})\sum_{k_{n}=1}^{n}\mathcal{N}_{\mathrm{int}}^{*}(k_{n},n+1)\prod_{j_{n}=1}^{n+1}\mathcal{G}_{1}^{*}(t_{n},j_{n})g_{1+n}(1,\ldots,n+1)\prod_{i=1}^{n+1}f_{1}^{0}(i).\end{aligned}\tag{25}$$

In series expansion(25) the operator $\mathcal{N}_{\mathrm{int}}^{*}(j_1,j_2)f_n=-\mathcal{N}_{\mathrm{int}}(j_1,j_2)f_n$ is an adjoint operator to operator (12) and the group $\mathcal{G}_1^*(t,i)=\mathcal{G}_1(-t,i)$ is dual to group (10) in the sense of functional (24). For bounded interaction potentials series (25) is norm convergent on the space $\mathfrak{L}^1(\mathcal{H})$ under the condition that $t<t_0\equiv(2\,\|\Phi\|_{\mathfrak{L}(\mathcal{H}_2)}\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$.

The operator $f_1(t)$ represented by series (25) is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations:

$$\begin{aligned}\frac{\partial}{\partial t}f_{1}(t,1)&=\mathcal{N}^{*}(1)f_{1}(t,1)\\&+\mathrm{Tr}_{2}\,\mathcal{N}_{\mathrm{int}}^{*}(1,2)\prod_{i_{1}=1}^{2}\mathcal{G}_{1}^{*}(t,i_{1})g_{2}(1,2)\prod_{i_{2}=1}^{2}(\mathcal{G}_{1}^{*})^{-1}(t,i_{2})f_{1}(t,1)f_{1}(t,2),\end{aligned}\tag{26}$$

$$f_{1}(t)|_{t=0}=f_{1}^{0},\tag{27}$$

where the operator $\mathcal{N}^*(1)=-\mathcal{N}(1)$ is an adjoint operator to operator (13) in the sense of functional (24) and the group $(\mathcal{G}_1^*)^{-1}(t)=\mathcal{G}_1^*(-t)=\mathcal{G}_1(t)$ is inverse to the group $(\mathcal{G}_1^*)(t)$. This fact is proved similarly as in case of a solution of the quantum BBGKY hierarchy represented by the iteration series [13].

Thus, in case of initial states specified by one-particle (marginal) density operator (23) we establish that the dual quantum Vlasov hierarchy (19) for additive-type marginal observables describes the evolution of a quantum large particle system just as the non-Markovian quantum Vlasov-type kinetic equation with initial correlations (26).

4 THE PROPAGATION OF INITIAL CORRELATIONS IN A MEAN FIELD LIMIT

We consider the evolution of initial correlations in a mean field scaling limit.

The property of the propagation of initial correlations is a consequence of the validity of the following equality for the mean value functional of the limit k-ary marginal observables, i.e. the sequences $b^{(k)}(0)=(0,\ldots,0,b_k^0(1,\ldots,k),0,\ldots)$ [15] at initial instant, in case of $k\geq 2$

$$\begin{aligned}\big(b^{(k)}(t),f^{(c)}\big)&=\sum_{s=0}^{\infty}\frac{1}{s!}\,\mathrm{Tr}_{1,\ldots,s}\,b_{s}^{(k)}(t,1,\ldots,s)g_{s}(1,\ldots,s)\prod_{j=1}^{s}f_{1}^{0}(j)\\&=\frac{1}{k!}\mathrm{Tr}_{1,\ldots,k}\,b_{k}^{0}(1,\ldots,k)\prod_{i_{1}=1}^{k}\mathcal{G}_{1}^{*}(t,i_{1})g_{k}(1,\ldots,k)\prod_{i_{2}=1}^{k}(\mathcal{G}_{1}^{*})^{-1}(t,i_{2})\prod_{j=1}^{k}f_{1}(t,j),\end{aligned}\tag{28}$$

where the limit one-particle (marginal) density operator $f_1(t,j)$ is represented by series expansion (25) and therefore it is governed by the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations (26), (27).

This fact is proved similarly to the proof of a property on the propagation of initial chaos in a mean field scaling limit [18].

Therefore in case of initial states specified by sequence (1) mean field dynamics of all possible states is described in terms of the sequence $f = (I, f_1(t), f_2(t), \ldots, f_n(t), \ldots)$ of the limit marginal density operators $f_n(t,1,\ldots,n)$, $n \geq 1$, which are represented within the framework of the one-particle density operator $f_1(t)$ as follows

$$f_n(t,1,\ldots,n) = \prod_{i_1=1}^{n} \mathcal{G}_1^*(t,i_1) g_n(1,\ldots,n) \prod_{i_2=1}^{n} (\mathcal{G}_1^*)^{-1}(t,i_2) \prod_{j=1}^{n} f_1(t,j), \quad n \geq 2,$$

where the one-particle density operator $f_1(t,j)$ is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with initial correlations (26),(27). In case of initial states specified by sequence (4) of the marginal correlation operators the evolution of all possible correlations is described by the following sequence of the limit marginal correlation operators

$$g_n(t,1,\ldots,n) = \prod_{i_1=1}^{n} \mathcal{G}_1^*(t,i_1) \tilde{g}_n(1,\ldots,n) \prod_{i_2=1}^{n} (\mathcal{G}_1^*)^{-1}(t,i_2) \prod_{j=1}^{n} f_1(t,j), \quad n \geq 2,$$

where the operators $\tilde{g}_n$ related to operators g_n by expansions (5).

We note that the general approach to the description of the evolution of states of quantum many-particle systems within the framework of correlation operators and marginal correlation operators was given in paper [19].

Thus, in case of the limit k-ary marginal observables solution (22) of the dual quantum Vlasov hierarchy (19) is equivalent to a property of the propagation of initial correlations for the k-particle marginal density operator in the sense of equality (28) or in other words the mean field scaling dynamics does not create correlations.

5 Conclusion and outlook

In the paper the concept of quantum kinetic equations in case of the kinetic evolution, involving correlations of particle states at initial time, for instance, correlation operators characterizing the condensed states, was considered.

This paper deals with a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics. The obtained results can be extended to quantum systems of bosons or fermions.

In case of pure states the quantum Vlasov-type kinetic equation with initial correlations (26) can be reduced to the Gross–Pitaevskii-type kinetic equation [14]. Indeed, in this case the one-particle density operator $f_1(t) = |\psi_t\rangle\langle\psi_t|$ is a one-dimensional projector onto a unit vector $|\psi_t\rangle \in \mathcal{H}$ and its kernel has the following form: $f_1(t,q,q') = \psi(t,q)\psi^*(t,q')$. Then, if we consider quantum particles, interacting by the potential which kernel $\Phi(q) = \delta(q)$ is the Dirac measure, from kinetic equation (26) we derive the Gross–Pitaevskii-type kinetic equation [20]

$$i\frac{\partial}{\partial t}\psi(t,q) = -\frac{1}{2}\Delta_q\psi(t,q) + \int dq'dq''\mathfrak{g}(t,q,q;q',q'')\psi(t,q'')\psi^*(t,q)\psi(t,q),$$

where the coupling ratio $\mathfrak{g}(t,q,q;q',q'')$ of the collision integral is the kernel of the scattering length operator $\mathcal{G}_1^*(t,1)\mathcal{G}_1^*(t,2)g_2(1,2)$. If we consider a system of quantum particles without initial correlations (7) (or (6)), then this kinetic equation is the cubic nonlinear Schrödinger equation [13].

We note also that in paper [21] it was developed one more method of the derivation of quantum kinetic equations. By means of a non-perturbative solution of the quantum BBGKY hierarchy it was established that, if initial data is completely specified by a one-particle marginal density operator (in case of initial data with correlations see paper [20]), then all possible states of quantum many-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator governed by the generalized quantum kinetic equation. The actual quantum kinetic equations can be derived from the generalized quantum kinetic equation in the appropriate scaling limit, for example, in a mean field limit [18]. We emphasize that one of the advantages of such an approach to the derivation of the quantum kinetic equations from underlying dynamics governed by the generalized quantum kinetic equation consists in an opportunity to construct the higher-order corrections to the scaling asymptotic behavior of large particle quantum systems.

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Запропоновано новий метод виведення кінетичних рівнянь з динаміки квантових систем багатьох частинок за наявності кореляцій станів частинок в початковий момент. Розвинутий підхід ґрунтується на описі еволюції за допомогою маргінальних спостережуваних в скейлінгових границях. В результаті побудовано власовського типу квантове кінетичне рівняння з початковими кореляціями та доведено твердження стосовно властивості поширення початкових кореляцій в ганиці самоузгодженого поля.

Ключові слова і фрази: маргінальні спостережувані, кінетичне рівняння з початковими кореляціями.