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ON A NECESSARY CONDITION FOR $L^p(0<p<1)-$CONVERGENCE (UPPER BOUNDEDNESS) OF TRIGONOMETRIC SERIES

In this paper we prove that the condition $\sum_{k=\left[\frac{n}{2}\right]}^{2n}\frac{\lambda_k(p)}{(|n-k|+1)^{2-p}}=o(1)\ (=O(1))$, is a necessary condition for the $L^p(0<p<1)-$convergence (upper boundedness) of a trigonometric series. Precisely, the results extend some results of A. S. Belov [1].

Key words and phrases: trigonometric series, L^p-convergence, Hardy-Littlewood's inequality, Bernstein-Zygmund inequalities.

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1 INTRODUCTION AND PRELIMINARIES

Let

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \left(\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+b_n\sin nx\right) \tag{1}$$

be a trigonometric series in the complex or real form respectively, and we use the following standard notations for all $n\geq 0$

$$\begin{aligned}
a_n &= c_n+c_{-n},\\
b_n &= (c_n-c_{-n})i,\\
\lambda_n(p) &= \sqrt{2\left(|c_n|^{2p}+|c_{-n}|^{2p}\right)},\\
r_n &= \sqrt{|a_n|^2+|b_n|^2}=\sqrt{2(|c_n|^2+|c_{-n}|^2)},\\
A_n(x) &= c_ne^{inx}+c_{-n}e^{-inx}=a_n\cos nx+b_n\sin nx,\\
S_n(x) &= c_0+\sum_{k=1}^{n}A_k(x),\\
\sigma_n(x) &= \frac{1}{n+1}\sum_{k=1}^{n}S_k(x),\\
\widetilde{S}_n(x) &= \sum_{k=1}^{n}(a_k\sin kx-b_k\cos kx)=-i\sum_{k=0}^{n}\left(c_ke^{ikx}-c_{-k}e^{-ikx}\right).
\end{aligned}$$

The square brackets denote the integer part of a number.

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For $f \in L_{2\pi}$ the L^1-metric is defined by the equality

$$\|f\|_{L^1} = \|f\| = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|dx.$$

Regarding to the series (1) Belov [1] has proved a necessary condition, expressed in terms of its coefficients, for the L^1-convergence or L^1-boundedness of its partial sums proving the following statement.

Theorem 1. *If $n \geq 2$ is an integer, then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{r_k}{|n-k|+1} \leq 100 \max_{m=[n/2]-1,\dots,2n} \|\sigma_m - S_m\|.$$

In particular:

1. *If*
$$\|\sigma_m - S_m\| = o(1)\ \ (= O(1))\,, \tag{2}$$
then
$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{r_k}{|n-k|+1} = o(1)\ \ (= O(1) \textit{ respectively})\,. \tag{3}$$

2. *Assume that series (1) converges (possesses bounded partial sums) in the L^1-metric, then condition (3) holds.*

Also the author has considered the cosine and sine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{4}$$

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{5}$$

where for the series (4) or (5) the coefficients a_n are the same as in the trigonometric series (1) except for coefficients of series (5) which are denoted a_n instead of b_n, and the following corollary has been proved by him.

Corollary 1. *1. Assume that series* (4) *or* (5) *satisfies condition* (2)*, then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{|a_k|}{|n-k|+1} = o(1)\ \ (O(1) \textit{ respectively})\,. \tag{6}$$

2. Assume that series (4) *or* (5) *converges (possesses bounded partial sums) in the L^1-metric, then condition* (6) *holds.*

For $f \in L^p_{2\pi}$, $0 < p < 1$, the L^p-metric is defined by the equality

$$\|f\|_{L^p} = \|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}.$$

Of course $\|\cdot\|_p$ for $0 < p < 1$ is not a norm, it does not satisfy the triangle property, and it is known as quasi-norm.

The above statements, for $r-$th derivative of the series (1.1), has been generalized by present author in [2]. But nothing seems to be done so far concerning L^p-convergence $(0 < p < 1)$ of the series (1) in the direction as Belov did in [1]. Therefore, our main goal in this paper is studying of L^p-convergence of the series (1) for $0 < p < 1$.

Our main tools in proving the main results are Bernstein-Zygmund's inequality and Hardy-Littelwood's theorem in the spaces L^p $(0 < p < 1)$, and H^p $(0 < p < 1)$, respectively.

Lemma 1 ([3] or [5]). *Let $T_n(x)$ be a trigonometrical polynomial of order n and $0 < p < 1$. Then the inequality $\|T_n'\|_p \le C_p n \|T_n\|_p$ holds true.*

Lemma 2 ([4]). *If $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$, $|z| < 1$ and $\varphi \in H^p$, $0 < p < 1$, then*

$$\sum_{k=0}^{\infty} (k+1)^{p-2} |c_k|^p \le C_p \|\varphi\|_p^p.$$

Throughout in this paper C_p denotes a positive constant that depends only on p, not necessarily the same at each occurrences.

2 Main results

We begin with the following helpful statements.

Lemma 3. *For every $m \in \mathbb{N}$ and $0 < p < 1$, we have*

$$\min\left\{\left\|\sum_{k=0}^{m} c_k e^{ikx}\right\|_p^p, \left\|\sum_{k=0}^{m} c_k e^{-ikx}\right\|_p^p\right\} \ge \frac{1}{C_p} \max\left\{\sum_{k=0}^{m} (k+1)^{p-2}|c_k|^p, \sum_{k=0}^{m} (m-k+1)^{p-2}|c_k|^p\right\}.$$

Proof. The proof of this lemma is an immediate result of the Lemma 2. Indeed, we have that

$$\begin{aligned}\left\|\sum_{k=0}^{m} c_k e^{ikx}\right\|_p^p &= \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m} c_k e^{ikx}\right|^p dx = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m} \bar{c}_k e^{-ikx}\right|^p dx \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|e^{imx}\sum_{k=0}^{m} \bar{c}_k e^{-ikx}\right|^p dx = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m} \bar{c}_k e^{i(m-k)x}\right|^p dx \\ &= \left\|\sum_{k=0}^{m} \bar{c}_k e^{i(m-k)x}\right\|_p^p \ge \frac{1}{C_p}\sum_{k=0}^{m}(m-k+1)^{p-2}|c_k|^p.\end{aligned}$$

The inequalities for $\sum_{k=0}^{m} c_k e^{-ikx}$ one can prove in the same way in view of equality

$$\left\|\sum_{k=0}^{m} c_k e^{-ikx}\right\|_p = \left\|\sum_{k=0}^{m} \bar{c}_k e^{ikx}\right\|_p.$$

□

Lemma 4. *Given an arbitrary trigonometric series (1) and arbitrary natural numbers n and N such that $n \le N \le 2n+1$. Then for $0 < p < 1$ the following estimates hold:*

$$\max_{k=n,\dots,N} \|\widetilde{S}_{n-1} - \widetilde{S}_k\|_p \le C_p \max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p, \tag{7}$$

$$\max_{m=n,\dots,N} \left\| \left(\sum_{j=n}^{m} c_j e^{ijx} \right) \right\|_p \le C_p \max_{m=n,\dots,N} \|S_m - S_{n-1}\|_p, \tag{8}$$

$$\max_{m=n,\dots,N} \left\| \left(\sum_{j=n}^{m} c_{-j} e^{-ijx} \right) \right\|_p \le C_p \max_{m=n,\dots,N} \|S_m - S_{n-1}\|_p, \tag{9}$$

$$\max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p \le C_p \max_{k=n-1,\dots,N} \|S_k - \sigma_k\|_p, \tag{10}$$

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(k+1-n)^{2-p}} \right)^{1/p} \le C_p \max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p, \tag{11}$$

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(N+1-k)^{2-p}} \right)^{1/p} \le C_p \|S_N - S_{n-1}\|_p, \tag{12}$$

where C_p is a positive constant depending only on p.

Proof. (7). Let m, n be two natural numbers such that $m \ge n$. Using the equality

$$\widetilde{S}_{n-1}(x) - \widetilde{S}_m(x) = \frac{1}{m}\left(S'_m(x) - S'_{n-1}(x)\right) + \sum_{k=n}^{m-1} \frac{1}{k(k+1)}\left(S'_k(x) - S'_{n-1}(x)\right),$$

Lemma 1 and well-known inequalities

$$|a+b|^\beta \le \begin{cases} |a|^\beta + |b|^\beta, & \text{if } 0 < \beta < 1 \\ 2^\beta\left(|a|^\beta + |b|^\beta\right), & \text{if } \beta \ge 1 \end{cases}$$

we have that $\|S'_k - S'_{n-1}\|_p \le C_p k \|S_k - S_{n-1}\|_p$, and

$$\begin{aligned} \|\widetilde{S}_{n-1} - \widetilde{S}_m\|_p &\le 2^{\frac{1}{p}} C_p \left\{ \|S_m - S_{n-1}\|_p + \sum_{k=n}^{m-1} \frac{1}{k+1} \|S_k - S_{n-1}\|_p \right\} \\ &\le C_p \left(1 + \sum_{k=n}^{m-1} \frac{1}{k+1}\right) \max_{k=n,\dots,m} \|S_k - S_{n-1}\|_p. \end{aligned}$$

Thus for $n \le N \le 2n+1$

$$1 + \sum_{k=n}^{N-1} \frac{1}{k+1} \le 1 + \frac{1}{n+1}(N-n) \le 2,$$

we obtain

$$\max_{k=n,\dots,N} \|\widetilde{S}_{n-1} - \widetilde{S}_k\|_p \le C_p \max_{k=n,\dots,N} \|S_k - S_{n-1}\|_p.$$

(8). From the equality

$$2\sum_{j=n}^{m} c_j e^{ijx} = (S_m(x)-S_{n-1}(x)) + i\left(\widetilde{S}_m(x)-\widetilde{S}_{n-1}(x)\right)$$

and (7) we get

$$\begin{aligned} 2\max_{m=n,\dots,N}\left\|\left(\sum_{j=n}^{m} c_j e^{ijx}\right)\right\|_p &\le 2^{\frac{1}{p}}\left\{\max_{m=n,\dots,N}\|S_m-S_{n-1}\|_p+\max_{m=n,\dots,N}\|\widetilde{S}_m-\widetilde{S}_{n-1}\|_p\right\} \\ &\le 2^{\frac{1}{p}}\left(1+C_p\right)\max_{m=n,\dots,N}\|S_m-S_{n-1}\|_p, \end{aligned}$$

which is the required estimate. The estimate (9) we can prove in the same manner as the estimate (8). It is sufficient to use the equality

$$2\sum_{j=n}^{m} c_{-j} e^{-ijx} = (S_m(x)-S_{n-1}(x)) - i\left(\widetilde{S}_m(x)-\widetilde{S}_{n-1}(x)\right),$$

therefore by the reason of similarity we omit the details.

(10). By the equality

$$\begin{aligned} S_m(x)-S_{n-1}(x) = {} & \frac{m+1}{m}\left(S_m(x)-\sigma_m(x)\right) \\ & + \sum_{k=n}^{m-1}\frac{1}{k}\left(S_k(x)-\sigma_k(x)\right) - \left(S_{n-1}(x)-\sigma_{n-1}(x)\right) \end{aligned}$$

we have that

$$\begin{aligned} \|S_m-S_{n-1}\|_p &\le 2^{\frac{1}{p}}\left\{\frac{m+1}{m}\|S_m-\sigma_m\|_p+\sum_{k=n}^{m-1}\frac{1}{k}\|S_k-\sigma_k\|_p+\|S_{n-1}-\sigma_{n-1}\|_p\right\} \\ &= 2^{\frac{1}{p}}\left\{\|S_m-\sigma_m\|_p+\sum_{k=n}^{m}\frac{1}{k}\|S_k-\sigma_k\|_p+\|S_{n-1}-\sigma_{n-1}\|_p\right\} \\ &\le 2^{\frac{1}{p}}\left(2+\sum_{k=n}^{m}\frac{1}{k}\right)\max_{k=n-1,\dots,m}\|S_k-\sigma_k\|_p. \end{aligned}$$

Thus

$$\max_{k=n,\dots,N}\|S_k-S_{n-1}\|_p \le 5\cdot 2^{\frac{1}{p}}\max_{k=n-1,\dots,N}\|S_k-\sigma_k\|_p \quad \text{for} \quad n=1.$$

The case when $n\ge 2$ can be treated in a similar manner. Indeed, since for $n\le N\le 2n+1$ we have

$$2+\sum_{k=n}^{m}\frac{1}{k}\le 2+\frac{N-n+1}{n}\le 3+\frac{2}{n}\le 4,$$

then the estimate (10) holds for all $n\ge 1$.

(11). From the estimate (8) we have

$$HL := \left\|\sum_{j=n}^{N} c_j e^{ijx}\right\|_p \le C_p\max_{k=n,\dots,N}\|S_k-S_{n-1}\|_p. \tag{13}$$

On the other hand, by the Lemma 3 we obtain

$$HL := \left\| \sum_{j=n}^{N} c_j e^{ijx} \right\|_p \geq \frac{1}{C_p} \left(\sum_{k=n}^{N} \frac{|c_k|^p}{(k+1-n)^{2-p}} \right)^{1/p} . \tag{14}$$

Hence, from (13) and (14) we get

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p}{(k+1-n)^{2-p}} \right)^{1/p} \leq C_p \max_{k=n,\ldots,N} \|S_k - S_{n-1}\|_p. \tag{15}$$

In a similiar manner one can find the following estimate

$$\left(\sum_{k=n}^{N} \frac{|c_{-k}|^p}{(k+1-n)^{2-p}} \right)^{1/p} \leq C_p \max_{k=n,\ldots,N} \|S_k - S_{n-1}\|_p. \tag{16}$$

It is obvious that from (15) and (16) follows

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(k+1-n)^{2-p}} \right)^{1/p} \leq C_p \max_{k=n,\ldots,N} \|S_k - S_{n-1}\|_p,$$

which proves the estimate (11).

(12). The equality $S_N(x) - S_{n-1}(x) = \sum_{j=n}^{N} c_j e^{ijx} + \sum_{j=n}^{N} c_{-j} e^{-ijx}$ and Lemma 3 give

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p}{(N+1-k)^{2-p}} \right)^{1/p} \leq C_p \|S_N - S_{n-1}\|_p,$$

and

$$\left(\sum_{k=n}^{N} \frac{|c_{-k}|^p}{(N+1-k)^{2-p}} \right)^{1/p} \leq C_p \|S_N - S_{n-1}\|_p.$$

Using the last two estimates we obtain

$$\left(\sum_{k=n}^{N} \frac{|c_k|^p + |c_{-k}|^p}{(N+1-k)^{2-p}} \right)^{1/p} \leq C_p \|S_N - S_{n-1}\|_p.$$

This completes the proof of the Lemma 4. □

We shall prove now an another lemma which in this paper do not need us. The only its importance is that it extends the Lemma 2 in [1] from the case $p = 1$ to the case $0 < p < 1$. It may be useful for the other aspects.

Lemma 5. *For any trigonometric series (1) and arbitrary natural number n, the following estimate holds ($0 < p < 1$):*

$$\|\sigma_n - S_n\|_p \leq C_p \left\{ \frac{1}{n+1} \sum_{j=1}^{n-1} \|S_j - S_{[j/2]}\|_p + 2 \max_{k=[n/2],\ldots,n} \|S_k - S_{[n/2]}\|_p \right\}. \tag{17}$$

If

$$\max_{k=[n/2],\ldots,n} \|S_k - S_{[n/2]}\|_p = o(1) \ \left(= O(1)\right), \tag{18}$$

then condition (20) (see below in this paper) is satisfied.

Proof. Applying Lemma 1 to the equality

$$(n+1)\left(S_n(x)-\sigma_n(x)\right)=\sum_{j=1}^{n-1}\left(S_j(x)-S_{[j/2]}(x)\right)$$
$$+n\left(S_n(x)-S_{[n/2]}(x)\right)-2\sum_{j=[n/2]+1}^{n-1}\left(S_j(x)-S_{[n/2]}(x)\right),$$

we obtain

$$\begin{aligned}(n+1)\|S_n-\sigma_n\|_p &\le 4^{\frac{1}{p}}\left\{\sum_{j=1}^{n-1}\|S_j-S_{[j/2]}\|_p+n\|S_n-S_{[n/2]}\|_p\right\}\\ &+2^{1+\frac{1}{p}}\sum_{j=[n/2]+1}^{n-1}\|S_j-S_{[n/2]}\|_p\le 4^{\frac{1}{p}}\left\{\sum_{j=1}^{n-1}\|S_j-S_{[j/2]}\|_p+n\|S_n-S_{[n/2]}\|_p\right\}\\ &+2^{1+\frac{1}{p}}\left(n-[n/2]-1\right)\max_{k=[n/2],\ldots,n}\|S_k-S_{[n/2]}\|_p\\ &\le 4^{\frac{1}{p}}\left\{\sum_{j=1}^{n-1}\|S_j-S_{[j/2]}\|_p+(2n-1)\max_{k=[n/2],\ldots,n}\|S_k-S_{[n/2]}\|_p\right\}.\end{aligned}$$

Supposing that (18) holds, then from (17) obviously the estimate (20) holds. □

The main results of this paper are the following statements which extends Theorem 1 and Corollary 1 from the case $p=1$ to the case $0<p<1$.

Theorem 2. *If $n\ge 2$ is an integer and $0<p<1$, then*

$$\left(\sum_{k=\left[\frac{n}{2}\right]}^{2n}\frac{\lambda_k(p)}{(|n-k|+1)^{2-p}}\right)^{1/p}\le C_p\max_{k=\left[\frac{n}{2}\right]-1,\ldots,2n}\|\sigma_k-S_k\|_p. \tag{19}$$

In particular:

1. *If*
$$\|\sigma_n-S_n\|_p=o(1)\ \ (=O(1)), \tag{20}$$
then
$$\sum_{k=\left[\frac{n}{2}\right]}^{2n}\frac{\lambda_k(p)}{(|n-k|+1)^{2-p}}=o(1)\ \ (=O(1)\ \textit{respectively}). \tag{21}$$

2. *Assume that series* (1) *converges (possesses bounded partial sums) in the $L^p(0<p<1)-$metric; then condition* (20) *holds.*

Proof. From Lemma 4, according to the estimates (11) and (10)

$$\begin{aligned}\left(\sum_{k=n}^{2n}\frac{\lambda_k(p)}{(k+1-n)^{2-p}}\right)^{1/p}&\le C_p\left(\sum_{k=n}^{2n}\frac{|c_k|^p+|c_{-k}|^p}{(k+1-n)^{2-p}}\right)^{1/p}\\ &\le C_p\max_{k=n,\ldots,2n}\|S_k-S_{n-1}\|_p\le C_p\max_{k=n-1,\ldots,2n}\|S_k-\sigma_k\|_p.\end{aligned} \tag{22}$$

On the other hand according to the estimates (12) and (10), for $2[n/2]+1 \geq n$ we have

$$\begin{aligned}\left(\sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{\lambda_k(p)}{(n+1-k)^{2-p}}\right)^{1/p} &\leq C_p \left(\sum_{k=\left[\frac{n}{2}\right]}^{n} \frac{|c_k|^p+|c_{-k}|^p}{(n+1-k)^{2-p}}\right)^{1/p} \\ &\leq C_p \left\|S_n - S_{\left[\frac{n}{2}\right]-1}\right\|_p \leq C_p \max_{k=\left[\frac{n}{2}\right]-1,\ldots,n} \|S_k-\sigma_k\|_p.\end{aligned} \tag{23}$$

Adding (22) and (23) we obtain (19). In addition, from (20) and (19) imply (21).

Let the series (1) converges (possesses bounded partial sums) in the $L^p(0<p<1)$-metric, then

$$\|\sigma_n - S_n\|_p \leq 2^{\frac{1}{p}}\left\{\|f-S_n\|_p + \|\sigma_n - f\|_p\right\} = o(1)\ \ (=O(1)).$$

Therefore (20) implies (21). This completes the proof of the Theorem 3.1. ☐

The following corollary is a direct consequence of the Theorem 2.

Corollary 2. *1. Assume that series* (4) *or* (5) *satisfies condition* (20)*, then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{|a_k|^p}{(|n-k|+1)^{2-p}} = o(1)\ \ (=O(1)\ \text{respectively}). \tag{24}$$

2. Assume that series (4) *or* (5) *converges (possesses bounded partial sums) in the* $L^p(0<p<1)$*-metric, then condition* (24) *holds.*

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В цій статті доведено, що умова $\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{\lambda_k(p)}{(|n-k|+1)^{2-p}} = o(1)\ (=O(1))$, є необхідною умовою для $L^p(0<p<1)-$збіжності (обмеженості зверху) тригонометричного ряду. Результати статті узагальнюють деякі результати Бєлова А.С. [1].

Ключові слова і фрази: тригонометричний ряд, L^p-збіжність, нерівність Харді-Літтлвуда, нерівності Бернштейна-Зігмунда.