ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2015, **7** (2), 172–187
doi:10.15330/cmp.7.2.172-187

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2015, Т.7, №2, С.172–187

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GENERALIZED TYPES OF THE GROWTH OF DIRICHLET SERIES

Let Φ be a continuous function on $[\sigma_0, A)$ such that $\Phi(\sigma) \to +\infty$ as $\sigma \to A-0$, where $A \in (-\infty, +\infty]$. We establish a necessary and sufficient condition on a nonnegative sequence $\lambda = (\lambda_n)$, increasing to $+\infty$, under which the equality

$$\varlimsup_{\sigma\uparrow A} \frac{\ln M(\sigma, F)}{\Phi(\sigma)} = \varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)}$$

holds for every Dirichlet series of the form $F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s = \sigma + it$, which is absolutely convergent in the half-plane $\operatorname{Re} s < A$. Here $M(\sigma, F) = \sup\{|F(s)| : \operatorname{Re} s = \sigma\}$ and $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$ are the maximum modulus and maximal term of this series respectively.

Key words and phrases: Dirichlet series, maximum modulus, maximal term, generalized type.

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INTRODUCTION

Let $\mathbb{N}_0$ be the set of all nonnegative integer numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, Λ be the class of all nonnegative sequences $\lambda = (\lambda_n)$, increasing to $+\infty$, $A \in (-\infty, +\infty]$, and Ω_A be the class of all continuous functions Φ on $[\sigma_0, A)$, such that

$$\forall x \in \mathbb{R}: \quad \lim_{\sigma\uparrow A}(x\sigma - \Phi(\sigma)) = -\infty. \tag{1}$$

Note that in the case $A < +\infty$ the condition (1) is equivalent to the condition $\Phi(\sigma) \to +\infty$, $\sigma \to A-0$, and in the case $A = +\infty$ this condition is equivalent to the condition $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$.

For a sequence $\lambda \in \Lambda$ let

$$\tau(\lambda) = \varlimsup_{n\to\infty} \frac{\ln n}{\lambda_n}.$$

Consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \tag{2}$$

and put

$$E_1(F) = \left\{\sigma \in \mathbb{R} : \sum_{n=0}^{\infty} |a_n| e^{\sigma\lambda_n} < +\infty\right\}, \quad E_2(F) = \left\{\sigma \in \mathbb{R} : \lim_{n\to\infty} |a_n| e^{\sigma\lambda_n} = 0\right\},$$

$$\sigma_a(F) = \begin{cases} -\infty, & \text{if } E_1(F) = \varnothing, \\ \sup E_1(F), & \text{if } E_1(F) \neq \varnothing, \end{cases} \qquad \beta(F) = \begin{cases} -\infty, & \text{if } E_2(F) = \varnothing, \\ \sup E_2(F), & \text{if } E_2(F) \neq \varnothing \end{cases}$$

УДК 517.53
2010 *Mathematics Subject Classification:* 30B50, 30D10, 30D15, 30D20.

($\sigma_a(F)$ is the abscissa of absolute convergence for the Dirichlet series (2)).

It is easy to show that

$$\beta(F) = \varliminf_{n\to\infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}.$$

Also, it is well known (see, for example, [7, p. 114–115]), that

$$\sigma_a(F) \le \beta(F) \le \sigma_a(F) + \tau(\lambda)$$

and these inequalities are sharp (more precisely, for every $A, B \in \overline{\mathbb{R}}$ such that $A \le B \le A + \tau(\lambda)$ there exists [3] a Dirichlet series F of the form (2) such that $\sigma_a(F) = A$ and $\beta(F) = B$).

If $\sigma_a(F) > -\infty$, then for each $\sigma < \sigma_a(F)$ let $M(\sigma, F) = \sup\{|F(s)| : \operatorname{Re} s = \sigma\}$ be the maximum modulus of the series (2). If $\beta(F) > -\infty$, then for each $\sigma < \beta(F)$ let $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \in \mathbb{N}_0\}$ be the maximal term of this series. As is well known, in the case $\sigma_a(F) > -\infty$ we have $\mu(\sigma, F) \le M(\sigma, F)$ for all $\sigma < \sigma_a(F)$.

By $\mathcal{D}_A(\lambda)$ we denote the class of all Dirichlet series of the form (2) such that $\sigma_a(F) \ge A$. Put $\mathcal{D}_A = \cup_{\lambda\in\Lambda}\mathcal{D}_A(\lambda)$. For $\Phi \in \Omega_A$ and $F \in \mathcal{D}_A$, the value

$$T_\Phi(F) = T_{\Phi,A}(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln M(\sigma, F)}{\Phi(\sigma)}$$

will be called Φ-*type* of the series F in the half-plane $\{s : \operatorname{Re} s < A\}$.

By $\mathcal{D}^*_A(\lambda)$ we denote the class of all Dirichlet series of the form (2) such that $\beta(F) \ge A$. Set $\mathcal{D}^*_A = \cup_{\lambda\in\Lambda}\mathcal{D}^*_A(\lambda)$. For $\Phi \in \Omega_A$ and $F \in \mathcal{D}^*_A$ we put

$$t_\Phi(F) = t_{\Phi,A}(F) = \varlimsup_{\sigma\uparrow A} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)}.$$

If $F \in \mathcal{D}_A$, then $\mu(\sigma, F) \le M(\sigma, F)$ for each $\sigma < A$, so $t_\Phi(F) \le T_\Phi(F)$.

Note that $\mathcal{D}_A(\lambda) \subset \mathcal{D}^*_A(\lambda)$ for every sequence $\lambda \in \Lambda$. From what has been said above it follows that in the case $A < +\infty$ we have $\mathcal{D}_A(\lambda) = \mathcal{D}^*_A(\lambda)$ if and only if $\tau(\lambda) = 0$. Furthermore, $\mathcal{D}_{+\infty}(\lambda) = \mathcal{D}^*_{+\infty}(\lambda)$ if and only if $\tau(\lambda) < +\infty$. It is clear that $\mathcal{D}_A \subset \mathcal{D}^*_A$ and $\mathcal{D}_A \ne \mathcal{D}^*_A$.

The notion of Φ-type generalizes the classical notion of the type for entire Dirichlet series.

Let F be an entire Dirichlet series, i. e. $F \in \mathcal{D}_{+\infty}$, and ρ be a fixed positive number. Recall that

$$T(F) = \varlimsup_{\sigma\uparrow+\infty} \frac{\ln M(\sigma, F)}{e^{\rho\sigma}}$$

is called the *type* of the series F. If $\lambda \in \Lambda$ and $\tau(\lambda) = 0$, then the type of every entire Dirichlet series of the form (2) can be calculated (see, for example, [7, p. 178]) by the formula

$$T(F) = \varlimsup_{n\to\infty} \frac{\lambda_n}{e\rho} |a_n|^{\frac{\rho}{\lambda_n}}. \tag{3}$$

Let $\Phi \in \Omega_A$. The function

$$\widetilde{\Phi}(x) = \sup\{x\sigma - \Phi(\sigma) : \sigma \in [\sigma_0, A)\}, \quad x \in \mathbb{R},$$

is said to be *Young conjugate to* Φ (see, for example, [1, pp. 86–88]). The following properties of the function $\widetilde{\Phi}$ are well known (see also Lemmas 2 and 3 below): $\widetilde{\Phi}$ is convex on $\mathbb{R}$; if φ is the right-hand derivative of $\widetilde{\Phi}$, then $\widetilde{\Phi}(x) = x\varphi(x) - \Phi(\varphi(x))$, $x \in \mathbb{R}$, $\varphi(x) < A$ on $\mathbb{R}$ and

$\varphi(x) \nearrow A$ as $x \uparrow +\infty$; if $x_0 = \inf\{x > 0 : \Phi(\varphi(x)) > 0\}$, then the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ increase to A on $(x_0, +\infty)$. Since $\widetilde{\Phi}$ is convex on $\mathbb{R}$, $\widetilde{\Phi}$ is continuous on $\mathbb{R}$. Thus, the function $\overline{\Phi}$ is continuous on $(x_0, +\infty)$. Let $A_0 = \overline{\Phi}(x_0 + 0)$ and $\psi : (A_0, A) \to (x_0, +\infty)$ be the inverse function of $\overline{\Phi}$. Set $\psi(\sigma) = +\infty$ for $\sigma \in [A, +\infty]$. Let $F \in \mathcal{D}^*_A$ be a Dirichlet series of the form (2). Then $\beta(F) \geq A$, so that

$$\frac{1}{\lambda_n} \ln \frac{1}{|a_n|} \geq A_0, \quad n \geq n_0.$$

Let $t > 0$ be a fixed number and $h(\sigma) = t\Phi(\sigma)$, $\sigma \in [\sigma_0, A)$. Then $\widetilde{h}(x) = t\widetilde{\Phi}(x/t)$, $x \in \mathbb{R}$, and hence $\widetilde{h}(x) = x\overline{\Phi}(x/t)$, $x \geq tx_0$. Using Lemma 5, given below, we obtain

$$t_\Phi(F) = \varlimsup_{n\to\infty} \frac{\lambda_n}{\psi\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}. \tag{4}$$

Therefore, for every Dirichlet series $F \in \mathcal{D}^*_A$ of the form (2) we have (4). Consequently, if $F \in \mathcal{D}_A$ is a Dirichlet series of the form (2) such that $T_\Phi(F) = t_\Phi(F)$, then Φ-type of this series can be calculated by the formula

$$T_\Phi(F) = \varlimsup_{n\to\infty} \frac{\lambda_n}{\psi\left(\frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}. \tag{5}$$

Note, that in the classical case, considered above ($A = +\infty$, $\Phi(\sigma) = e^{\rho\sigma}$), the formula (5) coincides with the formula (3). In this connection the following problem arises.

Problem 1. *Let $\lambda \in \Lambda$, $\Phi \in \Omega_A$. Find a necessary and sufficient condition on the sequence λ and the function Φ under which $T_\Phi(F) = t_\Phi(F)$ for every Dirichlet series $F \in \mathcal{D}_A$.*

In particular cases Problem 1 is solved in [2, 4, 5, 8, 6]. Denote by Ω^*_A the class of all function $\Phi \in \Omega_A$, convex on $[\sigma_0, A)$. If $\Phi \in \Omega^*_A$, then the one-sided derivatives Φ'_- and Φ'_+ are nondecreasing functions on $[\sigma_0, A)$ and $\Phi'_-(\sigma) \to +\infty$, $x \uparrow A$. Besides, using the definition of the function $\widetilde{\Phi}$ and Lemma 3, given below, it is easy to prove that

$$\Phi'_-(\varphi(x)) \leq x \leq \Phi'_+(\varphi(x)), \quad x > x_0 := \Phi'_+(\sigma_0). \tag{6}$$

The solution of Problem 1, in the case of the sequence $\lambda = (n)$ and an arbitrary function $\Phi \in \Omega^*_A$, was obtained practically in [2, 4] for $A = +\infty$ and in [5] for every $A \in (-\infty, +\infty]$ (actually, the growth of power series was investigated in [2, 4, 5]). We state a result from [5] in the following equivalent formulation.

Theorem A. *Let $\lambda = (n)$, $A \in (-\infty, +\infty]$, and $\Phi \in \Omega^*_A$. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ the equality $T_\Phi(F) = t_\Phi(F)$ holds if and only if*

$$\ln \Phi'_+(\sigma) = o(\Phi(\sigma)), \quad \sigma \uparrow A.$$

Let $\Phi : [\sigma_0, A) \to \mathbb{R}$ be a continuously differentiable function from the class Ω^*_A such that Φ' is a positive function, increasing on $[\sigma_0, A)$. From (6) it follows that the restriction of the right-hand derivative φ of the function $\widetilde{\Phi}$ to $(x_0, +\infty)$ is the inverse function of Φ'. Put

$$\Psi(\sigma) = \sigma - \frac{\Phi(\sigma)}{\Phi'(\sigma)}, \quad \sigma \in [\sigma_0, A).$$

(As is well known, the function Ψ is called the *Newton transform of* Φ.) It is easy to see that $\Psi(\varphi(x)) = \overline{\Phi}(x)$, $x \in [x_0, +\infty)$. For a sequence $\lambda \in \Lambda$, let $n_\lambda(x) = \sum_{\lambda_n \le x} 1$ be its counting function. The next theorem was proved by M. M. Sheremeta [8].

Theorem B. *Let* $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A^*$ *be a twice continuously differentiable function on* $[\sigma_0, A)$ *such that* $\Phi'(\sigma)/\Phi(\sigma) \nearrow +\infty$ *and* $\ln \Phi'(\sigma) = o(\Phi(\sigma))$ *as* $\sigma \uparrow A$. *Then for every Dirichlet series* $F \in \mathcal{D}_A(\lambda)$ *the inequality* $t_\Phi(F) \le 1$ *implies the inequality* $T_\Phi(F) \le 1$ *if and only if*

$$\ln n_\lambda(x) = o(\Phi(\Psi(\varphi(x)))), \quad x \to +\infty. \tag{7}$$

Remark 1. *We can rewrite* (7) *in the form*

$$\ln n_\lambda(x) = o(\Phi(\overline{\Phi}(x))), \quad x \to +\infty.$$

Furthermore, as is easily seen, the condition (7) *is equivalent to each of the conditions*

$$\ln n_\lambda(\Phi'(\sigma)) = o(\Phi(\Psi(\sigma))), \quad \sigma \uparrow A;$$
$$\ln n = o(\Phi(\overline{\Phi}(\lambda_n))), \quad n \to \infty.$$

Remark 2. *The sufficiency of the condition* (7) *in Theorem B was proved in [8] only by the condition that* $\Phi \in \Omega_A^*$ *is a twice continuously differentiable function such that the function* Φ'/Φ *is nondecreasing on* $[\sigma_0, A)$.

Let $t \in (0, +\infty)$ be a fixed number. If Φ satisfy the conditions of Theorem B, then the function $t\Phi$ also satisfy these conditions. Applying Theorem B with $t\Phi$ instead of Φ and taking into account Remark 1, we see that $T_\Phi(F) = t_\Phi(F)$ for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ if an only if

$$\forall t > 0: \quad \ln n = o(\Phi(\overline{\Phi}(\lambda_n/t))), \quad n \to \infty. \tag{8}$$

Note also that Theorem B does not imply Theorem A. In addition, Theorem B does not give the answer to the next question: whether the condition $\tau(\lambda) = 0$ is necessary in order that (3) holds for every entire Dirichlet series of the form (2)? Note, that the positive answer to this question was obtained in [6].

In connection with Theorem B the next problem arises.

Problem 2. *Let* $T_0 \ge t_0 \ge 0$ *be arbitrary constants,* $\lambda \in \Lambda$, *and* $\Phi \in \Omega_A$. *Find a necessary and sufficient condition on the sequence* λ *and the function* Φ *under which for every Dirichlet series* $F \in \mathcal{D}_A$ *such that* $t_\Phi(F) = t_0$ *the inequality* $T_\Phi(F) \le T_0$ *holds.*

In this article we obtain the complete solutions of Problems 1 and 2.

1 THE STATEMENT OF MAIN RESULTS

For a sequence $\lambda \in \Lambda$, a function $\Phi \in \Omega_A$ and every $t_2 > t_1 > 0$ we put

$$\Delta(t_1, t_2) = \Delta_{\Phi,\lambda}(t_1, t_2) = \varlimsup_{n\to\infty} \frac{\ln n}{t_1\widetilde{\Phi}(\lambda_n/t_1) - t_2\widetilde{\Phi}(\lambda_n/t_2)}.$$

First we mention some properties of $\Delta(t_1, t_2)$.

If d is a fixed number, then for the function $\gamma(t) = t\widetilde{\Phi}(d/t)$, $t \in \mathbb{R}\setminus\{0\}$, we have

$$\gamma'_+(t) = \widetilde{\Phi}\left(\frac{d}{t}\right) - \frac{d}{t}\varphi\left(\frac{d}{t}\right) = -\Phi\left(\varphi\left(\frac{d}{t}\right)\right).$$

Hence,

$$t_1\widetilde{\Phi}\left(\frac{d}{t_1}\right) - t_2\widetilde{\Phi}\left(\frac{d}{t_2}\right) = \int_{t_1}^{t_2} \Phi\left(\varphi\left(\frac{d}{t}\right)\right) dt. \tag{9}$$

Let $a > 0$ be a fixed number. Consider the function $y = \Delta(a,t)$, $t \in (a,+\infty)$. Using (9), Lemmas 2 and 6, given below, and taking into account that the function $\alpha(x) = \Phi(\varphi(x))$ is positive on $(x_0,+\infty)$, for every $t_2 > t_1 > a$ we obtain

$$0 \le y(t_2) \le y(t_1) \le \frac{t_2 - a}{t_1 - a} y(t_2).$$

It follows from this that the next three cases are possible: the function y is identically equal to 0; the function y is identically equal to $+\infty$; the function y is positive continuous nonincreasing on $(a,+\infty)$.

Let $b > 0$ be a fixed number. Consider the function $y = \Delta(t,b)$, $t \in (0,b)$. Using again Lemma 6, for every $0 < t_1 < t_2 < b$ we obtain

$$0 \le y(t_1) \le \frac{b - t_2}{b - t_1} y(t_2).$$

This implies that if $y(t_2) = 0$ for some $t_2 \in (0,b)$, then $y(t) = 0$ on $(0,t_2]$; if $y(t_1) = +\infty$ for some $t_1 \in (0,b)$, then $y(t) = +\infty$ on $[t_1,b)$; if the function y does not take the value 0 and $+\infty$ at some point $t \in (0,b)$, then this function increase at the point t.

Note also that the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0,+\infty)$, by Lemma 3, given below. Consequently, from (9), for every $d \ge 0$ and $t_2 > t_1 > 0$, we have

$$(t_2 - t_1)\Phi\left(\varphi\left(\frac{d}{t_2}\right)\right) \le t_1\widetilde{\Phi}\left(\frac{d}{t_1}\right) - t_2\widetilde{\Phi}\left(\frac{d}{t_2}\right) \le (t_2 - t_1)\Phi\left(\varphi\left(\frac{d}{t_1}\right)\right). \tag{10}$$

The solution of Problem 1 is contained in the following theorem.

Theorem 1. *Let $\lambda \in \Lambda$, $A \in (-\infty,+\infty]$, and $\Phi \in \Omega_A$. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ the equality $T_\Phi(F) = t_\Phi(F)$ holds if and only if*

$$\forall t > 0: \quad \ln n = o(\Phi(\varphi(\lambda_n/t))). \tag{11}$$

Remark 3. *The conditions* (8) *and* (11) *are equivalent for every function $\Phi \in \Omega_A^*$. This fact follows from the inequalities*

$$(1 - q)\Phi(\varphi(qx)) \le \Phi(\overline{\Phi}(x)) < \Phi(\varphi(x)), \tag{12}$$

which hold for every fixed $q \in (0,1)$ and all large enough x (see Lemma 8 below).

Note also that if $F \in \mathcal{D}_A(\lambda)$ and $t_\Phi(F) = +\infty$, then $T_\Phi(F) = +\infty$, by the inequality $\mu(\sigma,F) \le M(\sigma,F)$, $\sigma < A$, so that $T_\Phi(F) = t_\Phi(F)$. In this connection, the next theorem makes more precise Theorem 1 in the part of the sufficiency of (11).

Theorem 2. *Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, and $\Phi \in \Omega_A$. If the condition* (11) *holds, then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) < +\infty$ belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $T_\Phi(F) = t_\Phi(F)$.*

The solution of Problem 2 is contained in the following theorem.

Theorem 3. *Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $T_0 \geq t_0 \geq 0$ be arbitrary constants. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $t_\Phi(F) = t_0$ the inequality $T_\Phi(F) \leq T_0$ holds if and only if*

$$\forall T > T_0 \ \exists c \in (t_0, T) : \quad \Delta(c, T) < 1. \tag{13}$$

By Theorem 3, for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ the inequality $t_\Phi(F) \leq 1$ implies the inequality $T_\Phi(F) \leq 1$ if and only if

$$\forall T > 1 \ \exists c \in (1, T) : \quad \Delta(c, T) < 1. \tag{14}$$

If $A = +\infty$ and $\Phi(\sigma) = \sigma \ln \sigma$, $\sigma \geq e$, then, as is easy to show, the condition (14) becomes

$$\overline{\lim_{n\to\infty}} \frac{\ln \ln n}{\lambda_n} < 1,$$

but the condition (7) from Theorem B takes the form

$$\ln n = o(e^{\lambda_n}), \quad n \to \infty.$$

Hence, generally, the condition (14) does not coincide with the condition (7).

In the part of the sufficiency of (13) the Theorem 3 can be made more precise.

Theorem 4. *Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $T_0 \geq t_0 \geq 0$ be arbitrary constants. If the condition* (13) *holds, then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) = t_0$ belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $T_\Phi(F) \leq T_0$.*

Theorems 3 and 4 follow immediately from Theorems 5 and 6, given below, respectively.

Theorem 5. *Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $T_0 > t_0 \geq 0$ be arbitrary constants. Then for every Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $t_\Phi(F) = t_0$ the inequality $T_\Phi(F) < T_0$ holds if and only if*

$$\exists c \in (t_0, T_0) : \quad \Delta(c, T_0) < 1. \tag{15}$$

Theorem 6. *Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $T_0 > t_0 \geq 0$ be arbitrary constants. If the condition* (15) *holds, then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) = t_0$ belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $T_\Phi(F) < T_0$.*

Theorem 6 follows from the next more general result.

Theorem 7. *Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, and $\Phi, \Gamma \in \Omega_A$. If*

$$\sum_{n=0}^{\infty} \frac{1}{e^{\widetilde{\Phi}(\lambda_n) - \widetilde{\Gamma}(\lambda_n)}} < +\infty, \tag{16}$$

then every Dirichlet series F from the class $\mathcal{D}_A^*(\lambda)$ such that $\ln \mu(\sigma, F) \leq \Phi(\sigma)$, $\sigma \in [\sigma_1, A)$, belong to the class $\mathcal{D}_A(\lambda)$ and for this series we have $\ln M(\sigma, F) \leq \Gamma(\sigma)$, $\sigma \in [\sigma_2, A)$.

2 AUXILIARY RESULTS

Denote by X the class of all functions $h : \mathbb{R} \to \overline{\mathbb{R}}$. Suppose $h \in X$ and let $\widetilde{h} \in X$ be the Young conjugate function to h, i. e.

$$\widetilde{h}(\sigma) = \sup\{\sigma x - h(x) : x \in \mathbb{R}\}, \quad \sigma \in \mathbb{R}.$$

It is clear that if $h, g \in X$ and $h(x) \geq g(x)$ for all $x \in \mathbb{R}$, then $\widetilde{h}(\sigma) \leq \widetilde{g}(\sigma)$ for all $\sigma \in \mathbb{R}$.

Let $h \in X$. Then $\widetilde{\widetilde{h}}(x) \leq h(x)$ for each $x \in \mathbb{R}$, where $\widetilde{\widetilde{h}}$ is the Young conjugate function to $\widetilde{h}$. Indeed, the definition of $\widetilde{h}$ implies that for every $\sigma, x \in \mathbb{R}$ the inequality $\sigma x - h(x) \leq \widetilde{h}(\sigma)$ holds. Then $x\sigma - \widetilde{h}(\sigma) \leq h(x)$ for every $\sigma, x \in \mathbb{R}$. From this it follows that $\widetilde{\widetilde{h}}(x) \leq h(x)$ for each $x \in \mathbb{R}$.

Lemma 1. *Let $h, g \in X$. Then the following conditions are equivalent:*
(i) $\widetilde{h}(\sigma) \leq g(\sigma)$ for all $\sigma \in \mathbb{R}$;
(ii) $h(x) \geq \widetilde{g}(x)$ for all $x \in \mathbb{R}$.

Proof. If the condition (i) holds, then $\widetilde{\widetilde{h}}(x) \geq \widetilde{g}(x)$ for each $x \in \mathbb{R}$. Since $\widetilde{\widetilde{h}}(x) \leq h(x)$ for all $x \in \mathbb{R}$, from this it follows (ii).

If the condition (ii) holds, then $\widetilde{h}(\sigma) \leq \widetilde{\widetilde{g}}(\sigma)$ for each $\sigma \in \mathbb{R}$. Since $\widetilde{\widetilde{g}}(\sigma) \leq g(\sigma)$ for all $\sigma \in \mathbb{R}$, from this it follows (i). □

Lemma 2. *Let $h \in X$. Then $\widetilde{h}$ is a convex function on $\mathbb{R}$, i. e. for every $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1 \leq x_2 \leq x_3$ we have*

$$\widetilde{h}(x_2)(x_3 - x_1) \leq \widetilde{h}(x_1)(x_3 - x_2) + \widetilde{h}(x_3)(x_2 - x_1). \tag{17}$$

Proof. For each $t \in \mathbb{R}$ we have

$$(tx_2 - h(t))(x_3 - x_1) = (tx_1 - h(t))(x_3 - x_2) + (tx_3 - h(t))(x_2 - x_1).$$

From this equality and the definition of $\widetilde{h}$ we have (17). □

For a function $h \in X$ we put $D_h = \{x \in \mathbb{R} : h(x) < +\infty\}$. It is clear that in the definition of $\widetilde{h}(\sigma)$ we can take the supremum by all $x \in D_h$ instead the supremum by all $x \in \mathbb{R}$.

Let $A \in (-\infty, +\infty]$ and $\Phi : [\sigma_0, A) \to \mathbb{R}$ be a function from the class Ω_A. We assume that the function Φ belong to the class X, setting $\Phi(\sigma) = +\infty$ for every $\sigma \notin [\sigma_0, +\infty)$ (then $D_\Phi = [\sigma_0, +\infty)$). Fix some $x \in \mathbb{R}$ and set

$$y(\sigma) = x\sigma - \Phi(\sigma), \quad \sigma \in [\sigma_0, A).$$

The function y is continuous on $[\sigma_0, A)$. In addition, by (1), $y(\sigma) \to -\infty$ as $\sigma \uparrow A$. Hence, this function assumes its supremum on $[\sigma_0, A)$, i. e.

$$\widetilde{\Phi}(x) = \max_{\sigma \geq \sigma_0} y(\sigma).$$

Consider the set

$$S(x) = \{\sigma \geq \sigma_0 : y(\sigma) = \widetilde{\Phi}(x)\}.$$

From what has been said it follows that the set $S(x)$ is nonempty and bounded. Let $\varphi(x) = \sup S(x)$. Then $\varphi(x) \in S(x)$, i.e. $\max S(x)$ exists and $\varphi(x) = \max S(x)$. Indeed, if we assume that $\varphi(x) \notin S(x)$, then the set $S(x)$ is infinite and $\sigma < \varphi(x)$ for every $\sigma \in S(x)$. Let (σ_n) be a sequence of points in $S(x)$, increasing to $\varphi(x)$. For every $n \in \mathbb{N}_0$ we have $y(\sigma_n) = \widetilde{\Phi}(x)$. Letting n to ∞ and using the continuity of the function Φ, we obtain $y(\varphi(x)) = \widetilde{\Phi}(x)$, i.e. $\varphi(x) \in S(x)$, but this contradicts the assumption that $\varphi(x) \notin S(x)$. Hence, $\max S(x)$ exists and $\varphi(x) = \max S(x)$.

Lemma 3. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $\varphi(x) = \max\{\sigma \in [\sigma_0, A) : x\sigma - \Phi(\sigma) = \widetilde{\Phi}(x)\}$, $x \in \mathbb{R}$. Then:*

(i) the function φ is nondecreasing on $\mathbb{R}$;

(ii) the function φ is continuous from the right on $\mathbb{R}$;

(iii) $\varphi(x) \to A$, $x \to +\infty$;

(iv) the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at every point $x \in \mathbb{R}$;

(v) if $x_0 = \inf\{x > 0 : \Phi(\varphi(x)) > 0\}$, then the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ increase to A on $(x_0, +\infty)$;

(vi) the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$.

Proof. (i) Let $x_1 < x_2$. Since $x_j\varphi(x_j) - \Phi(\varphi(x_j)) = \widetilde{\Phi}(x_j)$, $j \in \{1, 2\}$, the definition of $\widetilde{\Phi}$ implies the following inequalities

$$x_1\varphi(x_1) - \Phi(\varphi(x_1)) \geq x_1\varphi(x_2) - \Phi(\varphi(x_2)), \quad x_2\varphi(x_2) - \Phi(\varphi(x_2)) \geq x_2\varphi(x_1) - \Phi(\varphi(x_1)).$$

Adding these inequalities, we obtain $(\varphi(x_2) - \varphi(x_1))(x_2 - x_1) \geq 0$. From this it follows that $\varphi(x_1) \leq \varphi(x_2)$.

(ii) Let $x_0 \in \mathbb{R}$ be a fixed point. By (i) it follows that the right-hand limit $\varphi(x_0 + 0)$ exists and $\varphi(x_0 + 0) \geq \varphi(x_0)$. Let us prove that $\varphi(x_0 + 0) = \varphi(x_0)$, i.e. that φ is continuous from the right at the point x_0. Indeed, the definition of $\widetilde{\Phi}$ implies the inequality

$$x\varphi(x_0) - \Phi(\varphi(x_0)) \leq x\varphi(x) - \Phi(\varphi(x)).$$

Letting x to x_0 from the right, we obtain $\widetilde{\Phi}(x_0) \leq x_0\varphi(x_0 + 0) - \Phi(\varphi(x_0 + 0))$. On the other hand, $\widetilde{\Phi}(x_0) \geq x_0\varphi(x_0 + 0) - \Phi(\varphi(x_0 + 0))$. Hence, $\widetilde{\Phi}(x_0) = x_0\varphi(x_0 + 0) - \Phi(\varphi(x_0 + 0))$. Then from the definition of φ we obtain $\varphi(x_0 + 0) \leq \varphi(x_0)$ and thus $\varphi(x_0 + 0) = \varphi(x_0)$.

(iii) Suppose the contrary, that is $\varphi(+\infty) = B < A$. Let $C \in (B, A)$. Using the definition of the function $\widetilde{\Phi}$, we have

$$xC - \Phi(C) \leq x\varphi(x) - \Phi(\varphi(x))$$

for every $x \in \mathbb{R}$. This implies that

$$x(C - \varphi(x)) \leq \Phi(C) - \Phi(\varphi(x)).$$

Letting x to $+\infty$, we obtain $+\infty \leq \Phi(C) - \Phi(B)$, but this is impossible.

(iv) Let $x \in \mathbb{R}$ be a fixed point and $h > 0$. From the definition of the function $\widetilde{\Phi}$ we have

$$\frac{\widetilde{\Phi}(x+h) - \widetilde{\Phi}(x)}{h} \geq \frac{(x+h)\varphi(x) - \Phi(\varphi(x)) - \widetilde{\Phi}(x)}{h} = \varphi(x),$$
$$\frac{\widetilde{\Phi}(x+h) - \widetilde{\Phi}(x)}{h} \leq \frac{\widetilde{\Phi}(x+h) - (x\varphi(x+h) - \Phi(\varphi(x+h)))}{h} = \varphi(x+h).$$

Hence,

$$\varphi(x) \le \frac{\widetilde{\Phi}(x+h) - \widetilde{\Phi}(x)}{h} \le \varphi(x+h).$$

Letting h to 0 and using (ii), we see that the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$.

(v) Since $x\varphi(x) - \widetilde{\Phi}(x) = \Phi(\varphi(x)) > 0$ for $x > x_0$,

$$(\overline{\Phi}(x))'_+ = \frac{x\varphi(x) - \widetilde{\Phi}(x)}{x^2} > 0, \quad x > x_0.$$

Hence, the function $\overline{\Phi}(x)$ increase on $(x_0, +\infty)$. Furthermore, the inequality $x\varphi(x) - \widetilde{\Phi}(x) > 0$, $x > x_0$, implies that $\overline{\Phi}(x) < \varphi(x) < A$, $x > x_0$. On the other hand, for every fixed x_1 and each $x \ge x_1$ we have

$$\widetilde{\Phi}(x) = \widetilde{\Phi}(x_1) + \int_{x_1}^{x} \varphi(t)dt \ge \widetilde{\Phi}(x_1) + (x - x_1)\varphi(x_1).$$

From this it follows that

$$\varliminf_{x\to+\infty} \overline{\Phi}(x) \ge \varphi(x_1).$$

Letting x_1 to $+\infty$, we see that $\overline{\Phi}(x) \to A$, $x \to +\infty$.

(vi) Let $x_2 > x_1 \ge 0$. Then

$$\begin{aligned}\alpha(x_2) - \alpha(x_1) &= x_2\varphi(x_2) - x_1\varphi(x_1) + \widetilde{\Phi}(x_1) - \widetilde{\Phi}(x_2) \ge x_2\varphi(x_2) - x_1\varphi(x_1) + (x_1 - x_2)\varphi(x_2)\\ &= x_1(\varphi(x_2) - \varphi(x_1)) \ge 0.\end{aligned}$$

Therefore, the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$. □

Lemma 4. *Let $A \in (-\infty, +\infty]$, $\Phi_1, \Phi_2 \in \Omega_A$, and $\Phi_1(\sigma) = \Phi_2(\sigma)$ for all $\sigma \in [\sigma_0, A)$. Then $\widetilde{\Phi}_1(x) = \widetilde{\Phi}_2(x)$ for each $x \ge x_0$.*

Proof. For $j \in \{1, 2\}$ let $D_{\Phi_j} = [\sigma_j, A)$ and

$$\varphi_j(x) = \max\{\sigma \in [\sigma_j, A) : x\sigma - \Phi_j(\sigma) = \widetilde{\Phi}_j(x)\}, \quad x \in \mathbb{R}.$$

Lemma 3 implies that $\min\{\varphi_1(x), \varphi_2(x)\} \ge \max\{\sigma_0, \sigma_1, \sigma_2\}$ for all $x \ge x_0$. Then for every $x \ge x_0$ we get

$$\begin{aligned}\widetilde{\Phi}_1(x) &= x\varphi_1(x) - \Phi_1(\varphi_1(x)) = x\varphi_1(x) - \Phi_2(\varphi_1(x)) \le \max_{\sigma\ge\sigma_2}(x\sigma - \Phi_2(\sigma)) = \widetilde{\Phi}_2(x),\\ \widetilde{\Phi}_2(x) &= x\varphi_2(x) - \Phi_2(\varphi_2(x)) = x\varphi_2(x) - \Phi_1(\varphi_2(x)) \le \max_{\sigma\ge\sigma_1}(x\sigma - \Phi_1(\sigma)) = \widetilde{\Phi}_1(x),\end{aligned}$$

and, hence, $\widetilde{\Phi}_1(x) = \widetilde{\Phi}_2(x)$. □

Lemma 5. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $F \in \mathcal{D}^*_A$ be a Dirichlet series of the form* (2). *Then $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$ if and only if $\ln |a_n| \le -\widetilde{\Phi}(\lambda_n)$ for all $n \ge n_0$.*

Proof. Suppose that $\ln \mu(\sigma, F) \le \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$. We set $\Psi(\sigma) = \Phi(\sigma)$ for every $\sigma \in [\sigma_0, A)$ and $\Psi(\sigma) = +\infty$ for every $\sigma \notin [\sigma_0, A)$. Let $h \in X$ be the function such that $h(\lambda_n) = -\ln |a_n|$ for all $n \in \mathbb{N}_0$ and $h(x) = +\infty$ for all $x \in \mathbb{R}\backslash\{\lambda_0, \lambda_1, \dots\}$. Then $\ln \mu(\sigma, F) = \widetilde{h}(\sigma)$ for $\sigma < \beta(F)$. Consequently, $\widetilde{h}(\sigma) \le \Psi(\sigma)$ for each $\sigma \in \mathbb{R}$. By Lemma 1, $h(x) \ge \widetilde{\Psi}(x)$, $x \in \mathbb{R}$. Therefore, using Lemma 4, we have $\ln |a_n| = -h(\lambda_n) \le -\widetilde{\Psi}(\lambda_n) = -\widetilde{\Phi}(\lambda_n)$ for all $n \ge n_0$.

Now suppose that $\ln|a_n| \leq -\widetilde{\Phi}(\lambda_n)$ for all $n \geq n_0$. If the function $\mu(\sigma, F)$ is bounded on $(-\infty, A)$, then, obviously, $\ln\mu(\sigma, F) \leq \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$. If the function $\mu(\sigma, F)$ is unbounded on $(-\infty, A)$, then we consider, along with F, the Dirichlet series

$$G(s) = \sum_{n=0}^{\infty} b_n e^{s\lambda_n}, \quad s = \sigma + it, \tag{18}$$

such that $b_n = 0$ for $n < n_0$ and $b_n = a_n$ for $n \geq n_0$. It is easy to show that $\mu(\sigma, F) = \mu(\sigma, G)$ for each $\sigma \in [\sigma_0, A)$. Besides, $\ln|b_n| \leq -\widetilde{\Phi}(\lambda_n)$ for all $n \in \mathbb{N}_0$. Hence, by Lemma 1, we have $\ln\mu(\sigma, G) \leq \Phi(\sigma)$, $\sigma < A$. This implies that $\ln\mu(\sigma, F) \leq \Phi(\sigma)$ for each $\sigma \in [\sigma_0, A)$. □

Lemma 6. *Let Ψ be a function, convex on $\mathbb{R}$, and $x_0 \geq 0$. Then for all $t_1, t_2, t_3 \in \mathbb{R}$ such that $t_3 > t_2 > t_1 > 0$ we have*

$$t_1\Psi\left(\frac{x_0}{t_1}\right) - t_2\Psi\left(\frac{x_0}{t_2}\right) \geq \frac{t_2 - t_1}{t_3 - t_1}\left(t_1\Psi\left(\frac{x_0}{t_1}\right) - t_3\Psi\left(\frac{x_0}{t_3}\right)\right),$$
$$t_2\Psi\left(\frac{x_0}{t_2}\right) - t_3\Psi\left(\frac{x_0}{t_3}\right) \leq \frac{t_3 - t_2}{t_3 - t_1}\left(t_1\Psi\left(\frac{x_0}{t_1}\right) - t_3\Psi\left(\frac{x_0}{t_3}\right)\right).$$

Proof. Since Ψ is convex on $\mathbb{R}$, for every $t_1, t_2, t_3 \in \mathbb{R}$ such that $t_3 > t_2 > t_1 > 0$ we have the following inequality

$$\Psi\left(\frac{x_0}{t_2}\right)\left(\frac{x_0}{t_1} - \frac{x_0}{t_3}\right) \leq \Psi\left(\frac{x_0}{t_1}\right)\left(\frac{x_0}{t_2} - \frac{x_0}{t_3}\right) + \Psi\left(\frac{x_0}{t_3}\right)\left(\frac{x_0}{t_1} - \frac{x_0}{t_2}\right).$$

Multiplying this inequality by $t_1 t_2 t_3$, we obtain

$$\Psi\left(\frac{x_0}{t_2}\right) t_2(t_3 - t_1) \leq \Psi\left(\frac{x_0}{t_1}\right) t_1(t_3 - t_2) + \Psi\left(\frac{x_0}{t_3}\right) t_3(t_2 - t_1).$$

From this it follows that

$$\begin{aligned}
\Psi\left(\frac{x_0}{t_1}\right) t_1(t_3 - t_1) - \Psi\left(\frac{x_0}{t_2}\right) t_2(t_3 - t_1) &\geq \Psi\left(\frac{x_0}{t_1}\right) t_1(t_3 - t_1) - \Psi\left(\frac{x_0}{t_1}\right) t_1(t_3 - t_2) \\
- \Psi\left(\frac{x_0}{t_3}\right) t_3(t_2 - t_1) &= \Psi\left(\frac{x_0}{t_1}\right) t_1(t_2 - t_1) - \Psi\left(\frac{x_0}{t_3}\right) t_3(t_2 - t_1), \\
\Psi\left(\frac{x_0}{t_2}\right) t_2(t_3 - t_1) - \Psi\left(\frac{x_0}{t_3}\right) t_3(t_3 - t_1) &\leq \Psi\left(\frac{x_0}{t_1}\right) t_1(t_3 - t_2) + \Psi\left(\frac{x_0}{t_3}\right) t_3(t_2 - t_1) \\
- \Psi\left(\frac{x_0}{t_3}\right) t_3(t_3 - t_1) &= \Psi\left(\frac{x_0}{t_1}\right) t_1(t_3 - t_2) - \Psi\left(\frac{x_0}{t_3}\right) t_3(t_3 - t_2).
\end{aligned}$$

Lemma 6 is proved. □

We note, that some of the above properties of the Young conjugate functions are well known (see, for examle, [1, § 3.2]).

Lemma 7. *Let (x_n) be a positive sequence such that*

$$\varlimsup_{n\to\infty} \frac{\ln n}{x_n} = \delta \geq 1.$$

Then, for every $q \in (0, 1)$, the set $E(q) = \{n \in \mathbb{N}_0 : \ln n \geq q x_n \wedge x_{[n/2]} \geq q x_n\}$ is unbounded.

Proof. If $\delta=+\infty$, then, setting $m_k=\min\{n\in\mathbb{N}_0:\ln n\ge (k+1)x_n\}$, we see that $m_k\in E(q)$ for every $k\in\mathbb{N}_0$. If $\delta<+\infty$, then, for some increasing sequence (p_k) of nonnegative integers, we have $\ln p_k\sim\delta x_{p_k}$, $k\to\infty$. Therefore,

$$\varlimsup_{k\to\infty}\frac{x_{p_k}}{x_{[p_k/2]}}=\frac{1}{\delta}\varlimsup_{k\to\infty}\frac{\ln p_k}{x_{[p_k/2]}}=\frac{1}{\delta}\varlimsup_{k\to\infty}\frac{\ln[p_k/2]}{x_{[p_k/2]}}\le\frac{1}{\delta}\varlimsup_{n\to\infty}\frac{\ln n}{x_n}=1.$$

It is clear that $p_k\in E_q$ for all $k\ge k_0(q)$. □

Theorem 8. *Let $A\in(-\infty,+\infty]$, $\lambda\in\Lambda$ be a sequence such that $\tau(\lambda)>0$ in the case $A<+\infty$ and $\tau(\lambda)=+\infty$ in the case $A=+\infty$, and $G\in\mathcal{D}_A^*(\lambda)\backslash\mathcal{D}_A(\lambda)$ be a Dirichlet series of the form* (18) *such that $b_n\ge 0$, $n\in\mathbb{N}_0$. Then for every positive on $(-\infty,A)$ function l there exists a Dirichlet series $F\in\mathcal{D}(\lambda)$ of the form* (2) *such that either $a_n=b_n$ or $a_n=0$ for every $n\in\mathbb{N}_0$ and $M(\sigma,F)=F(\sigma)\ge l(\sigma)$ for all $\sigma\in[\sigma_0,A)$.*

Proof. We may assume without loss of generality that the function l is nondecreasing on $(-\infty,A)$.

Since $G\in\mathcal{D}_A^*(\lambda)\backslash\mathcal{D}_A(\lambda)$, we have $\beta(G)\ge A$ and $\sigma_a(G)<A$. The inequality $\beta(G)\ge A$ implies that there exists a sequence (η_n), increasing to A, such that

$$\frac{1}{\lambda_n}\ln\frac{1}{b_n}\ge\eta_n,\quad n\in\mathbb{N}_0.$$

Then $b_n\le e^{-\eta_n\lambda_n}$, $n\in\mathbb{N}_0$. Since $\sigma_a(G)<A$, for all $\sigma\in(\sigma_a(G),A)$ and every $m\in\mathbb{N}_0$ we have

$$\sum_{n\ge m}b_ne^{\sigma\lambda_n}=+\infty.$$

Fix some sequence (σ_n), increasing to A. We choose a sequence (m_k) of nonnegative integers to be so rapidly increasing that the inequalities

$$\eta_{m_k}\ge\sigma_k,\quad e^{(\sigma_k-\sigma_{k+1})\lambda_{m_{k+1}}}(l(\sigma_{k+2})+1)<\frac{1}{(k+1)^2},\quad \sum_{n=m_k}^{m_{k+1}-1}b_ne^{\sigma_k\lambda_n}\ge l(\sigma_{k+1})$$

hold for every $k\in\mathbb{N}_0$. Put

$$p_k=\min\left\{p\ge m_k:\sum_{n=m_k}^{p}b_ne^{\sigma_k\lambda_n}\ge l(\sigma_{k+1})\right\},\quad k\in\mathbb{N}_0.$$

Note that $m_k\le p_k\le m_{k+1}-1$ and

$$l(\sigma_{k+1})\le\sum_{n=m_k}^{p_k}b_ne^{\sigma_k\lambda_n}<l(\sigma_{k+1})+b_{p_k}e^{\sigma_k\lambda_{p_k}}\le l(\sigma_{k+1})+e^{(\sigma_k-\eta_{p_k})\lambda_{p_k}}\le l(\sigma_{k+1})+1.$$

Let $n\in\mathbb{N}_0$. If $n\in[m_k,p_k]$ for some $k\in\mathbb{N}_0$, then we put $a_n=b_n$. If $n\notin[m_k,p_k]$ for every $k\in\mathbb{N}_0$, then let $a_n=0$. Consider the Dirichlet series F of the form (2) and let us prove that $\sigma_a(G)\ge A$. Indeed, for every fixed $j\in\mathbb{N}_0$ we have

$$\begin{aligned}\sum_{n\ge m_{j+1}}a_ne^{\sigma_j\lambda_n}&=\sum_{k\ge j+1}\sum_{n=m_k}^{p_k}b_ne^{\sigma_j\lambda_n}=\sum_{k\ge j+1}\sum_{n=m_k}^{p_k}b_ne^{\sigma_k\lambda_n}e^{(\sigma_j-\sigma_k)\lambda_n}\\&\le\sum_{k\ge j+1}e^{(\sigma_j-\sigma_k)\lambda_{m_k}}\sum_{n=m_k}^{p_k}b_ne^{\sigma_k\lambda_n}\\&\le\sum_{k\ge j+1}e^{(\sigma_{k-1}-\sigma_k)\lambda_{m_k}}(l(\sigma_{k+1})+1)<\sum_{k\ge j+1}\frac{1}{k^2}<+\infty,\end{aligned}$$

so that $\sigma_a(F) \geq A$. Moreover, if $\sigma \in [\sigma_0, A)$, then $\sigma \in [\sigma_k, \sigma_{k+1})$ for some $k \in \mathbb{N}_0$ and therefore

$$F(\sigma) \geq \sum_{n=m_k}^{p_k} a_n e^{\sigma\lambda_n} = \sum_{n=m_k}^{p_k} b_n e^{\sigma\lambda_n} \geq \sum_{n=m_k}^{p_k} b_n e^{\sigma_k\lambda_n} \geq l(\sigma_{k+1}) \geq l(\sigma).$$

Theorem 8 is proved. □

Lemma 8. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A^*$, and $q \in (0,1)$. Then the inequalities* (12) *hold for all* $x \geq x_0$.

Proof. If $\Phi \in \Omega_A^*$, then the function Φ is increasing on $[\sigma_1, A)$. Since

$$\overline{\Phi}(x) = \varphi(x) - \frac{\Phi(\varphi(x))}{x} < \varphi(x), \quad x > x_1,$$

we have $\Phi(\overline{\Phi}(x)) < \Phi(\varphi(x))$, $x > x_2$, i.e. the right of the inequalities (12) holds.

Further, using the convexity of the function Φ and the inequalities (6), we have

$$\Phi(\varphi(x)) - \Phi(\varphi(qx)) \leq (\varphi(x) - \varphi(qx))\Phi'_-(\varphi(x)) \leq (\varphi(x) - \varphi(qx))x, \quad x > x_3,$$

and, hence, for all $x > x_4$ we obtain

$$\begin{aligned}\Phi(\varphi(qx)) - \Phi(\overline{\Phi}(x)) &\leq (\varphi(qx) - \overline{\Phi}(x))\Phi'_-(\varphi(qx)) \leq \left(\varphi(qx) - \varphi(x) + \frac{\Phi(\varphi(x))}{x}\right) qx \\ &\leq \left(\frac{\Phi(\varphi(qx)) - \Phi(\varphi(x))}{x} + \frac{\Phi(\varphi(x))}{x}\right) qx = q\Phi(\varphi(qx)).\end{aligned}$$

This implies the left of the inequalities (12). □

3 The proofs of main results

Proof of Theorem 7. Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, and $\Phi, \Gamma \in \Omega_A$ be functions that satisfy (16).

Consider a Dirichlet series $F \in \mathcal{D}_A^*(\lambda)$ of the form (2) such that $\ln \mu(\sigma, F) \leq \Phi(\sigma)$, $\sigma \in [\sigma_1, A)$. By Lemma 5 we have $\ln |a_n| \leq -\widetilde{\Phi}(\lambda_n)$, $n \geq n_1$.

Fix $n_2 \geq n_1$ such that

$$\sum_{n \geq n_2} \frac{1}{e^{\widetilde{\Phi}(\lambda_n) - \widetilde{\Gamma}(\lambda_n)}} \leq \frac{1}{2}.$$

Then for all $\sigma \in [\sigma_2, A)$ we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| e^{\sigma\lambda_n} &= \sum_{n<n_2} |a_n| e^{\sigma\lambda_n} + \sum_{n \geq n_2} |a_n| e^{\sigma\lambda_n} \leq \frac{1}{2} e^{\Gamma(\sigma)} + \sum_{n \geq n_2} \frac{e^{\sigma\lambda_n}}{e^{\widetilde{\Phi}(\lambda_n)}} \\ &= \frac{1}{2} e^{\Gamma(\sigma)} + e^{\Gamma(\sigma)} \sum_{n \geq n_2} \frac{e^{\sigma\lambda_n - \Gamma(\sigma)}}{e^{\widetilde{\Phi}(\lambda_n)}} \leq e^{\Gamma(\sigma)} \left(\frac{1}{2} + \sum_{n \geq n_2} \frac{e^{\widetilde{\Gamma}(\lambda_n)}}{e^{\widetilde{\Phi}(\lambda_n)}}\right) \leq e^{\Gamma(\sigma)}.\end{aligned}$$

Hence, $\sigma_a(F) \geq A$, so that $F \in \mathcal{D}_A(\lambda)$. Furthermore, $\ln M(\sigma, F) \leq \Gamma(\sigma)$, $\sigma \in [\sigma_2, A)$. □

Proof of Theorem 6. Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and $T_0 > t_0 \geq 0$ be some constants. Assume that the condition (15) holds, i.e. for some $c \in (t_0, T_0)$ we have $\Delta(c, T_0) < 1$. Consider the function $y = \Delta(c, t)$, $t \in (c, +\infty)$. It follows from the properties of this function, described above, that there exists a point $T \in (c, T_0)$ such that $\Delta(c, T) < 1$. Let $q \in (\Delta(c, T), 1)$. Then there exists $n_0 \in \mathbb{N}_0$ such that

$$\ln n \le q\left(c\widetilde{\Phi}\left(\frac{\lambda_n}{c}\right) - T\widetilde{\Phi}\left(\frac{\lambda_n}{T}\right)\right), \quad n \ge n_0,$$

and thus

$$\sum_{n=0}^{\infty} \frac{1}{e^{c\widetilde{\Phi}(\lambda_n/c) - T\widetilde{\Phi}(\lambda_n/T)}} < +\infty. \tag{19}$$

Consider some Dirichlet series $F \in \mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) = t_0$. Then $t_\Phi(F) < c$, and hence $\ln \mu(\sigma, F) \le c\Phi(\sigma)$, $\sigma \in [\sigma_1, A)$. By Theorem 7, in view of (19), the series F belong to the class $\mathcal{D}_A(\lambda)$ and for this series the inequality $\ln M(\sigma, F) \le T\Phi(\sigma)$ holds for all $\sigma \in [\sigma_2, A)$, so that $T_\Phi(F) \le T < T_0$. □

Proof of Theorem 5. In view of Theorem 6, it remains only to prove the necessity of the condition (15).

We suppose that this condition is false, i. e. $\Delta(c, T_0) \ge 1$ for all $c \in (t_0, T_0)$, and prove that there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ of the form (2) such that $t_\Phi(F) = t_0$, but $T_\Phi(F) \ge T_0$.

For every $t_2 > t_1 > 0$ we set

$$\delta(t_1, t_2) = \varlimsup_{n\to\infty} \frac{\ln n}{(t_2 - t_1)\Phi(\varphi(\lambda_n/t_1))}.$$

Note that $\Delta(t_1, t_2) \ge \delta(t_1, t_2)$, by the right of the inequalities (10).

First we consider the case when for every $c \in (t_0, T_0)$ the inequality $\delta(c, T_0) \ge 1$, stronger than the inequality $\Delta(c, T_0) \ge 1$, holds. By Lemma 7, for every fixed $c \in (t_0, T_0)$ and $q \in (0, 1)$, the set $E(c, q)$ of all $n \in \mathbb{N}_0$ such that simultaneously

$$\ln n \ge q(T_0 - c)\Phi\left(\varphi\left(\frac{\lambda_n}{c}\right)\right), \quad \Phi\left(\varphi\left(\frac{\lambda_{[n/2]}}{c}\right)\right) \ge q\Phi\left(\varphi\left(\frac{\lambda_n}{c}\right)\right),$$

is infinite. Let (c_k) be a decreasing to t_0 sequence of points in (t_0, T_0) and (q_k) be a increasing to 1 sequence of points in $(0, 1)$. Choose a sequence (n_k) of nonnegative integers such that for every $k \in \mathbb{N}_0$ the conditions $n_k \in E(c_k, q_k)$ and $[n_{k+1}/2] > n_k$ hold.

Let $n \in \mathbb{N}_0$. Put $b_n = e^{-c_k\widetilde{\Phi}(\lambda_n/c_k)}$, if $n \in [[n_k/2], n_k]$ for some $k \in \mathbb{N}_0$, and let $b_n = 0$, if $n \notin [[n_k/2], n_k]$ for all $k \in \mathbb{N}_0$. Consider the Dirichlet series (18) with the coefficients b_n. This series we can write as

$$G(s) = \sum_{k=0}^{\infty} \sum_{n=[n_k/2]}^{n_k} \frac{e^{s\lambda_n}}{e^{c_k\widetilde{\Phi}(\lambda_n/c_k)}}. \tag{20}$$

For all $n \in \mathbb{N}_0$ such that $n \in [[n_k/2], n_k]$ for some $k \in \mathbb{N}_0$ we obtain

$$\frac{1}{\lambda_n} \ln \frac{1}{b_n} = \frac{c_k}{\lambda_n}\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) = \overline{\Phi}\left(\frac{\lambda_n}{c_k}\right).$$

Since, by Lemma 3, the function $\overline{\Phi}$ is increasing to A on $(x_0, +\infty)$, we have $\beta(G) = A$. Thus, $G \in \mathcal{D}_A^*(\lambda)$. Furthermore, if $\psi : (A_0, A) \to (x_0, +\infty)$ be the inverse function of $\overline{\Phi}$ (here $A_0 = \overline{\Phi}(x_0 + 0)$), then for all $n \in [[n_k/2], n_k]$ and for every $k \ge k_0$ we have

$$\frac{\lambda_n}{\psi\left(\frac{1}{\lambda_n} \ln \frac{1}{b_n}\right)} = c_k.$$

This implies that $t_\Phi(G) = t_0$.

If $G \in \mathcal{D}_A(\lambda)$, then it is enough to set $a_n = b_n$ for all $n \in \mathbb{N}_0$, i. e. it is enough to set $F = G$. Indeed, if $\sigma_k = \varphi(\lambda_{n_k}/c_k)$, then for each $k \in \mathbb{N}_0$ and for all $n \in [[n_k/2], n_k]$ we have

$$\begin{aligned}\sigma_k\lambda_n - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) &= \lambda_n\varphi\left(\frac{\lambda_{n_k}}{c_k}\right) - \lambda_n\varphi\left(\frac{\lambda_n}{c_k}\right) + c_k\Phi\left(\varphi\left(\frac{\lambda_n}{c_k}\right)\right)\\ &\geq c_k\Phi\left(\varphi\left(\frac{\lambda_n}{c_k}\right)\right) \geq c_k\Phi\left(\varphi\left(\frac{\lambda_{[n_k/2]}}{c_k}\right)\right) \geq c_kq_k\Phi\left(\varphi\left(\frac{\lambda_{n_k}}{c_k}\right)\right),\end{aligned}$$

and hence

$$\begin{aligned}M(\sigma_k, G) = G(\sigma_k) &\geq \sum_{n=[n_k/2]}^{n_k} \frac{e^{\sigma_k\lambda_n}}{e^{c_k\widetilde{\Phi}(\lambda_n/c_k)}}\\ &\geq \frac{n_k}{2}e^{c_kq_k\Phi(\varphi(\lambda_{n_k}/c_k))} \geq e^{q_k(T_0-c_k)\Phi(\varphi(\lambda_{n_k}/c_k))-\ln 2}e^{c_kq_k\Phi(\varphi(\lambda_{n_k}/c_k))} = e^{q_kT_0\Phi(\sigma_k)-\ln 2}.\end{aligned}$$

Therefore, $\ln M(\sigma_k, G) \geq q_kT_0\Phi(\sigma_k) - \ln 2$ for each $k \in \mathbb{N}_0$. Since $\sigma_k \to A$, $k \to \infty$, we obtain

$$T_\Phi(F) = T_\Phi(G) \geq \varlimsup_{k\to\infty} \frac{\ln M(\sigma_k, G)}{\Phi(\sigma_k)} \geq T_0 \varlimsup_{k\to\infty} q_k = T_0.$$

If $G \notin \mathcal{D}_A(\lambda)$, then, by Theorem 8, there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ of the form (2) such that either $a_n = b_n$ or $a_n = 0$ for every $n \in \mathbb{N}_0$ and $F(\sigma) \geq e^{T_0\Phi(\sigma)}$ for all $\sigma \in [\sigma_0, A)$. It is clear that $t_\Phi(F) = t_0$ and $T_\Phi(F) \geq T_0$.

Hence, in the case when for every $c \in (t_0, T_0)$ the inequality $\delta(c, T_0) \geq 1$ holds the existence of a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ with $t_\Phi(F) = t_0$ and $T_\Phi(F) \geq T_0$ is proved. Now let us consider the opposite case, i. e. suppose that for some $d_0 \in (t_0, T_0)$ we have $\delta(d_0, T_0) < 1$. Then

$$\ln p < (T_0 - d_0)\Phi\left(\varphi\left(\frac{\lambda_p}{d_0}\right)\right) - \ln 3, \quad p \geq p_0.$$

Since, by Lemma 3, the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0, +\infty)$, for every $c \in (t_0, d_0]$ we obtain

$$\ln p < (T_0 - c)\Phi\left(\varphi\left(\frac{\lambda_p}{c}\right)\right) - \ln 3, \quad p \geq p_0. \tag{21}$$

By the above assumption, $\Delta(c, T_0) \geq 1$ for all $c \in (t_0, T_0)$. Then from the properties of the function $y = \Delta(t, T_0)$, $t \in (0, T_0)$, described above, it follows that for every $c \in (t_0, T_0)$ the stronger inequality $\Delta(c, T_0) > 1$ holds.

Let (c_k) be a decreasing to t_0 sequence of points in $(t_0, c_0]$. Since $\Delta(c_k, T_0) > 1$ for every $k \in \mathbb{N}_0$, there exists a sequence (n_k) of nonnegative integers such that $n_0 \geq 2p_0$ and for all $k \in \mathbb{N}_0$ we have $[n_{k+1}/2] > n_k$ and

$$\ln n_k > c_k\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{c_k}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{T_0}\right). \tag{22}$$

Let $n \in \mathbb{N}_0$. Put $b_n = e^{-c_k\widetilde{\Phi}(\lambda_n/c_k)}$, if $n \in [[n_k/2], n_k]$ for $k \in \mathbb{N}_0$, and let $b_n = 0$, if $n \notin [[n_k/2], n_k]$ for every $k \in \mathbb{N}_0$. Consider the Dirichlet series (18) with the coefficients b_n. This series we can write in the form (20). Arguing as above, we see that $\beta(G) = A$ and $t_\Phi(G) = t_0$.

Using (21) with $c = c_k$ and $p = [n_k/2]$ and also (22), for each $k \in \mathbb{N}_0$ we obtain

$$(T_0 - c_k)\Phi\left(\varphi\left(\frac{\lambda_{[n_k/2]}}{c_k}\right)\right) > \ln\left[\frac{n_k}{2}\right] + \ln 3 > \ln n_k > c_k\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{c_k}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{T_0}\right)$$
$$= \int_{c_k}^{T_0} \Phi\left(\varphi\left(\frac{\lambda_{n_k}}{t}\right)\right) dt \geq (T_0 - c_k)\Phi\left(\varphi\left(\frac{\lambda_{n_k}}{T_0}\right)\right)$$

and thus

$$\frac{\lambda_{[n_k/2]}}{c_k} > \frac{\lambda_{n_k}}{T_0}. \tag{23}$$

Put $\sigma_k = \varphi(\lambda_{n_k}/T_0)$. Then for every $k \in \mathbb{N}_0$ and for all $n \in [[n_k/2], n_k]$, using (22), the monotonicity of the function φ, and (23), we have

$$\begin{aligned}
\sigma_k\lambda_n - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) &= \lambda_n\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) - T_0\Phi\left(\varphi\left(\frac{\lambda_{n_k}}{T_0}\right)\right) + T_0\Phi(\sigma_k) \\
&= (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) + T_0\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{T_0}\right) + T_0\Phi(\sigma_k) \\
&> (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) - c_k\widetilde{\Phi}\left(\frac{\lambda_n}{c_k}\right) + c_k\widetilde{\Phi}\left(\frac{\lambda_{n_k}}{c_k}\right) - \ln n_k + T_0\Phi(\sigma_k) \\
&= (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) + c_k\int_{\lambda_n/c_k}^{\lambda_{n_k}/c_k} \varphi(x)dx - \ln n_k + T_0\Phi(\sigma_k) \\
&\geq (\lambda_n - \lambda_{n_k})\varphi\left(\frac{\lambda_{n_k}}{T_0}\right) + c_k\left(\frac{\lambda_{n_k}}{c_k} - \frac{\lambda_n}{c_k}\right)\varphi\left(\frac{\lambda_n}{c_k}\right) - \ln n_k + T_0\Phi(\sigma_k) \\
&= (\lambda_{n_k} - \lambda_n)\left(\varphi\left(\frac{\lambda_n}{c_k}\right) - \varphi\left(\frac{\lambda_{n_k}}{T_0}\right)\right) - \ln n_k + T_0\Phi(\sigma_k) \\
&\geq -\ln n_k + T_0\Phi(\sigma_k).
\end{aligned}$$

If $G \in \mathcal{D}_A(\lambda)$, then it is enough to set $a_n = b_n$ for all $n \in \mathbb{N}_0$, i.e. it is enough to set $F = G$. Indeed, in this case for every $k \in \mathbb{N}_0$ we obtain

$$M(\sigma_k, G) = G(\sigma_k) \geq \sum_{n=[n_k/2]}^{n_k} \frac{e^{\sigma_k\lambda_n}}{e^{c_k\widetilde{\Phi}(\lambda_n/c_k)}} \geq \frac{n_k}{2}e^{-\ln n_k + T_0\Phi(\sigma_k)} = e^{T_0\Phi(\sigma_k) - \ln 2}.$$

Hence, $\ln M(\sigma_k, G) \geq T_0\Phi(\sigma_k) - \ln 2$ for all $k \in \mathbb{N}_0$. Since $\sigma_k \to A$, $k \to \infty$, we have $T_\Phi(F) = T_\Phi(G) \geq T_0$.

If $G \notin \mathcal{D}_A(\lambda)$, then, by Theorem 8, there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ of the form (2) such that either $a_n = b_n$ or $a_n = 0$ for every $n \in \mathbb{N}_0$ and $F(\sigma) \geq e^{T_0\Phi(\sigma)}$ for all $\sigma \in [\sigma_0, A)$. It is clear that $t_\Phi(F) = t_0$ and $T_\Phi(F) \geq T_0$. □

Proof of Theorem 2. Let $\lambda \in \Lambda$, $A \in (-\infty, +\infty]$, and $\Phi \in \Omega_A$. Suppose that the condition (11) holds and consider a Dirichlet series $F \in \mathcal{D}_A^*(\lambda)$ such that $t_\Phi(F) < +\infty$. Set $t_0 = t_\Phi(F)$. Let $T_0 > t_0$ and $c \in (t_0, T_0)$ be fixed numbers. Using the condition (11) with $t = T_0$ and left of the inequalities (10), for all $n \geq n_0$ we obtain

$$\ln n \leq \frac{T_0 - c}{2}\Phi\left(\varphi\left(\frac{\lambda_n}{T_0}\right)\right) \leq \frac{1}{2}\left(c\widetilde{\Phi}\left(\frac{\lambda_n}{c}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_n}{T_0}\right)\right)$$

and thus $\Delta(c, T_0) \leq 1/2 < 1$. By Theorem 6, the series F belong to the class $\mathcal{D}_A(\lambda)$ and for this series the inequality $T_\Phi(F) < T_0$ holds. Since $T_0 > t_0$ is arbitrary, this inequality implies that $T_\Phi(F) = t_\Phi(F)$. □

Proof of Theorem 1. In view of Theorem 2, it remains only to prove the necessity of the condition (11). Suppose that this condition is false, i. e. there exist positive constants t_0 and δ such that

$$\overline{\lim_{n\to\infty}} \frac{\ln n}{\Phi(\varphi(\lambda_n/t_0)))} \geq \delta. \tag{24}$$

Set $T_0 = t_0 + \delta$. Then, using the right of the inequalities (10), for every $c \in (t_0, T_0)$ we obtain

$$c\widetilde{\Phi}\left(\frac{\lambda_n}{c}\right) - T_0\widetilde{\Phi}\left(\frac{\lambda_n}{T_0}\right) \leq (T_0 - c)\Phi\left(\varphi\left(\frac{\lambda_n}{c}\right)\right) \leq \delta\Phi\left(\varphi\left(\frac{\lambda_n}{t_0}\right)\right), \quad n \geq n_0.$$

Together with (24) this implies that $\Delta(c, T_0) \geq 1$ for every $c \in (t_0, T_0)$. Then, by Theorem 5, there exists a Dirichlet series $F \in \mathcal{D}_A(\lambda)$ such that $t_\Phi(F) = t_0$ and $T_\Phi(F) \geq T_0 > t_0$. This completes the proof of Theorem 1. □

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Received 03.08.2015

Глова Т.Я., Філевич П.В. *Узагальнені типи зростання рядів Діріхле* // Карпатські матем. публ. — 2015. — Т.7, №2. — С. 172–187.

Нехай Φ — така неперервна на $[\sigma_0, A)$ функція, що $\Phi(\sigma) \to +\infty$, якщо $\sigma \to A - 0$, де $A \in (-\infty, +\infty]$. Знайдено необхідну і достатню умову на невід'ємну зростаючу до $+\infty$ послідовність $(\lambda_n)_{n=0}^{\infty}$, за якої для кожного абсолютно збіжного в півплощині $\operatorname{Re} s < A$ ряду Діріхле вигляду $F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}$, $s = \sigma + it$, виконується співвідношення

$$\overline{\lim_{\sigma\uparrow A}} \frac{\ln M(\sigma, F)}{\Phi(\sigma)} = \overline{\lim_{\sigma\uparrow A}} \frac{\ln \mu(\sigma, F)}{\Phi(\sigma)},$$

де $M(\sigma, F) = \sup\{|F(s)| : \operatorname{Re} s = \sigma\}$ і $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n} : n \geq 0\}$ — максимум модуля і максимальний член цього ряду відповідно.

Ключові слова і фрази: ряд Діріхле, максимум модуля, максимальний член, узагальнений тип.