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# GELFAND LOCAL BEZOUT DOMAINS ARE ELEMENTARY DIVISOR RINGS

We introduce the Gelfand local rings. In the case of commutative Gelfand local Bezout domains we show that they are an elementary divisor domains.

*Key words and phrases:* Gelfand ring, Bezout domain, elementary divisor domain.

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## INTRODUCTION

As a common generalization of local and (von Neumann) regular rings, Contessa in [1] called that a ring $R$ is a *VNL (von Neuman local)* ring if for each $a \in R$ either $a$ or $1 - a$ is a (von Neumann) regular element. In this analogy, we consider Gelfand local rings which are generalizations of commutative domains in which each nonzero prime ideal is contained in a unique maximal ideal. In this paper we show that a commutative Gelfand local Bezout domain is an elementary divisor ring. Note that these results are responses to open questions in [6].

We introduce the necessary definitions and facts. All rings considered will be commutative and have identity. A ring is a *Bezout ring*, if every its finitely generated ideal is principal. A ring $R$ is *an elementary divisor ring* if every matrix $A$ (not necessarily square one) over $R$ admits diagonal reduction, that is, there exist invertible matrices $P$ and $Q$ such that $PAQ$ is a diagonal matrix, say $(d_{ii})$, for which $d_{ii}$ is a divisor of $d_{i+1,i+1}$ for each $i$.

Two rectangular matrices $A$ and $B$ are *equivalent* if there exist invertible matrices $P$ and $Q$ of appropriate sizes such that $B = PAQ$ (see [5], [6]). Recall that a ring $R$ is called a *Gelfand ring* if for every $a, b \in R$ such that $a + b = 1$ there exist $r, s \in R$ such that $(1 + ar)(1 + bs) = 0$. A ring $R$ is called a *PM-ring* if each prime ideal is contained in a unique maximal ideal.

## RESULTS

**Definition 1.** *An element $a \in R$ of a commutative ring $R$ is called a Gelfand element if the factor ring $R/aR$ is a PM-ring.*

**Proposition 1.** *An element $a$ of a commutative Bezout domain $R$ is a Gelfand element if and only if for every elements $b, c \in R$ such that $aR + bR + cR = R$ an element $a$ can be represented as $a = rs$, where $rR + bR = R$, $sR + cR = R$.*

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*Proof.* Denote $\overline{R} = R/aR$ and $\overline{b} = b + aR$, $\overline{c} = c + aR$. Since $aR + bR + cR = R$, we have $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Let $\overline{r} = r + aR$, $\overline{s} = s + aR$. Since $a = rs$, then $\overline{0} = \overline{rs}$, where $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Then $\overline{R}$ is a Gelfand ring. By [4], $\overline{R}$ is a PM-ring.

If $\overline{R}$ is a PM-ring, then $\overline{R}$ is a Gelfand ring and $\overline{0} = \overline{rs}$, where $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$, $\overline{s}\overline{R} + \overline{c}\overline{R} = \overline{R}$ for arbitrary $\overline{b}, \overline{c} \in \overline{R}$ such that $\overline{b}\overline{R} + \overline{c}\overline{R} = \overline{R}$. Whence we obtain $aR + bR + cR = R$ and $rs \in aR$, that is, $rs = at$ for some $t \in R$.

Let $rR + aR = r_1R$, $sR + aR = s_1R$, where $r = r_1r_0$, $a = r_1a_0$, $s = s_1s_2$, $a = s_1a_2$, such that $r_0R + a_0R = R$ and $s_2R + a_2R = R$. Since $r_0R + a_0R = R$, we obtain $r_0u + a_0v = 1$ for some elements $u, v \in R$. Since $rs = at$, then $r_1r_0s = r_1a_0t$ and $r_0s = a_0$. By the equality $r_0u + a_0v = 1$ we have $a_0(tu + sv) = s$. Therefore, $a = r_1a_0$ where $r_1R + bR = R$ and $a_0R + cR = R$. □

**Proposition 2.** *The set of all Gelfand elements of a commutative Bezout domain $R$ is a saturated multiplicatively closed set.*

*Proof.* Let $a, b$ be Gelfand elements of $R$. We show that $ab$ is a Gelfand element. Suppose the contrary. Then there exists a prime ideal $P$ and maximal ideals $M_1, M_2$ of $R$ such that $M_1 \neq M_2$ and $ab \in P \subset M_1 \cap M_2$. Since $ab \in P$ and $P$ is a prime ideal of $R$, we obtain that $a \in P$ or $b \in P$. This is impossible, because $a, b$ are Gelfand elements and $P \subset M_1 \cap M_2$. Therefore, the set of Gelfand elements is multiplicatively closed.

Let $ab$ be a Gelfand element of $R$. If $a$ is not a Gelfand element then there exists a prime ideal $P$ such that $a \in P$ and $P \subset M_1 \cap M_2$ for some maximal ideals $M_1, M_2$ for which $M_1 \neq M_2$. Therefore, $ab \in P$ and $P \subset M_1 \cap M_2$, $M_1 \neq M_2$. This is impossible, because $ab$ is a Gelfand element. □

**Definition 2.** *A commutative ring is a Gelfand local ring (GLR) if for each $a \in R$ either $a$ or $1 - a$ is a Gelfand element.*

Since in a commutative domain in which each nonzero prime ideal is contained in a unique maximal ideal every nonzero element is a Gelfand element, we obtain the following result.

**Proposition 3.** *A commutative domain in which each nonzero prime ideal is contained in a unique maximal ideal is a Gelfand local ring.*

The following example of a Gelfand ring is due to Henriksen [2].

Let $R = \{z_0 + a_1x + a_1x^2 + \ldots | z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}, i = 1, 2, \ldots\}$. The Jacobson radical of $R$ is $J(R) = \{a_1x + a_1x^2 + \ldots | a_i \in \mathbb{Q}, i = 1, 2, \ldots\}$. Obviously, if $0 \neq a \notin J(R)$ then $a$ is a Gelfand element. If $a \in J(R)$ then $1 - a$ is a Gelfand element.

**Proposition 4.** *A commutative domain is a GLR ring if and only if for every $a, b \in R$ such that $aR + bR = R$ either $a$ or $b$ is a Gelfand element.*

*Proof.* Let $R$ be a GLR ring and $aR + bR = R$. Then $au + bv = 1$ for some elements $u, v \in R$. By the definition of $R$ we obtain that $au$ or $bv = 1 - au$ is a Gelfand element. If $au$ is a Gelfand element, then by Proposition 2, $a$ is a Gelfand element as well. If $bv$ is a Gelfand element then by Proposition 2, $b$ is a Gelfand element as well. Sufficiency is obvious. □

The main result of this paper is the following theorem.

**Theorem 1.** *Any GLR Bezout domain is an elementary divisor ring.*

*Proof.* Let $R$ be a commutative GLR Bezout domain. Let $a, b, c \in R$ be such that $aR + bR + cR = R$. Let $aR + cR = dR$. Since $aR + bR + cR = R$, then $bR + dR = R$. Since $R$ is GLR, then there two cases are possible:

1) $b$ is a Gelfand element;

2) $d$ is a Gelfand element.

Let us consider the first case. If $b$ is a Gelfand element, we have $b = rs$ where $rR + aR = R$, $sR + cR = R$. Let $p \in R$ be such that $sp + ck = 1$ for some $k \in R$. Hence $rsp + rck = r$ and $bp + crk = r$. Denoting $rk = q$, we obtain $(bp + cq)R + aR = R$. Let $pR + qR = \delta R$ and $\delta = pp_1 + qq_1$ with $p_1R + q_1R = R$. Hence $p_1R + (bp_1 + cq_1)R = R$. Since $pR \subset p_1R$, we obtain $p_1R + cR = R$ and $p_1R + (bp_1 + cq_1)R = R$. Since $bp + cq = \delta(bp_1 + cq_1)$ and $(bp + cq)R + aR = R$, we obtain $(bp_1 + cq_1)R + aR = R$. Finally, we have $ap_1R + (bp_1 + cq_1)R = R$. By [3] a commutative Bezout domain $R$ is an elementary divisor ring if and only if the matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $aR + bR + cR = R$ has a diagonal reduction. Note that a matrix $A$ has a diagonal reduction if and only if there exist $p, q \in R$ such that $apR + (bp + cq)R = R$. That is, if $b$ is a Gelfand element, $R$ is an elementary divisor domain.

Consider the second case. Let $d$ be a Gelfand element. Since $dR = aR + cR$ then $a = da_0$, $c = dc_0$, where $a_0R + c_0R = R$. Since $R$ is a GLR ring, by Proposition 4 we obtain that an element $a_0$ or $c_0$ is a Gelfand element. Note, according to the Proposition 2 then a matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $aR + bR + cR = R$ is equivalent to the matrix $B$ and $B = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$, where $\beta$ is a Gelfand element and $\alpha R + \beta R + \gamma R = R$. By similar considerations as in case 1, we conclude that a matrix $B$ and hence a matrix $A$ has a diagonal reduction. Therefore $R$ is an elementary divisor domain. ☐

## REFERENCES


[1] Contessa M. *On certain classes of PM-rings*. Comm. Algebra 1984, **12** (12), 1447–1469. doi:10.1080/00927878408823063

[2] Henriksen M. *Some remarks about elementary divisor rings. II*. Michigan Math. J. 1955, **56** (3), 159–163. doi:10.1307/mmj/1028990029

[3] Kaplansky I. *Elementary divisors and modules*. Trans. Amer. Math. Soc. 1949, **66**, 464–491. doi:10.1090/S0002-9947-1949-0031470-3

[4] McGovern W. Wm. *Neat rings*. J. Pure Applied Alg. 2006, **205** (2), 243–265. doi:10.1016/j.jpaa.2005.07.012

[5] Zabavsky B.V. Diagonal reduction of matrices over rings. In: Math. Stud., Monograph Series, 16. Lviv, 2012.

[6] Zabavsky B.V. *Questions related to the K-theoretical aspect of Bezout rings with various stable range conditions*. Math. Stud. 2014, **42** (1), 89–109.


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Введено локально гельфандові кільця. У випадку комутативних локально гельфандових областей Безу показано, що вони є областями елементарних дільників.

*Ключові слова і фрази:* гельфандове кільце, область Безу, область елементарних дільників.