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PALEY–WIENER-TYPE THEOREM FOR POLYNOMIAL ULTRADIFFERENTIABLE FUNCTIONS

The image of the space of ultradifferentiable functions with compact supports under Fourier-Laplace transformation is described. An analogue of Paley-Wiener theorem for polynomial ultradifferentiable functions is proved.

Key words and phrases: ultradifferentiable function, ultradistribution, polynomial test function, Paley–Wiener-type theorem.

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INTRODUCTION

In general Paley–Wiener theorem is any theorem that relates decay properties of a function or distribution at infinity with analyticity of its Fourier transform [16]. For example, the Paley–Wiener theorem for the space of smooth functions with compact supports gives a characterization of its image as rapidly decreasing functions having a holomorphic extension to $\mathbb{C}$ of exponential type.

There are plenty of Paley–Wiener-type theorems since there are many kinds of bound for decay rates of functions and many types of characterizations of smoothness. In this regard a wide number of papers have been devoted to the extension of the theory on many other integral transforms and different classes of functions (see [1–3, 6, 9, 15, 17, 18, 20–22] and the references given there).

Let $\mathcal{G}'_\beta := \mathcal{G}'_\beta(\mathbb{R}^d)$ be the space of Roumieu ultradistributions on $\mathbb{R}^d$ and $\mathcal{G}_\beta := \mathcal{G}_\beta(\mathbb{R}^d)$ be its predual. A Fréchet-Schwartz space (briefly, (FS) space) is one that is Fréchet and Schwartz simultaneously (see [23]). It is known (see e.g. [10, 19]) that the spaces $\mathcal{G}'_\beta$ and $\mathcal{G}_\beta$ are nuclear Fréchet-Schwartz and dual Fréchet-Schwartz spaces ((DFS) for short), respectively. These facts are crucial for our investigation.

In this article we consider Fourier-Laplace transformation, defined on the space $\mathcal{G}_\beta$ of ultradifferentiable functions and on the corresponding algebra $\mathcal{P}(\mathcal{G}'_\beta)$ of polynomials over $\mathcal{G}'_\beta$ [12], which have the tensor structure of the form $\bigoplus_{fin} \mathcal{G}_\beta^{\hat{\otimes} n}$ (see Theorem 1).

We completely describe the image of test space $\mathcal{G}_\beta$ under Fourier-Laplace transformation (see Corollary 1 and Theorem 2) and prove Paley–Wiener-type Theorem 3 for polynomial ultradifferentiable functions.

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1 PRELIMINARIES AND NOTATIONS

Let $\mathscr{L}(X)$ denote the space of continuous linear operators over a locally convex space X and let X' be the dual of X. Throughout, we will endow $\mathscr{L}(X)$ and X' with the locally convex topology of uniform convergence on bounded subsets of X.

Let $\otimes_{\mathfrak{p}}$ denote completion of algebraic tensor product with respect to the projective topology $\mathfrak{p}$. Let $X^{\hat{\otimes} n}$, $n \in \mathbb{N}$, be the symmetric nth tensor degree of X, completed in the projective tensor topology. Note, that here and subsequently we omit the index $\mathfrak{p}$ to simplify notations. For any $x \in X$ we denote $x^{\otimes n} := \underbrace{x \otimes \cdots \otimes x}_{n} \in X^{\hat{\otimes} n}$, $n \in \mathbb{N}$. Set $X^{\hat{\otimes} 0} := \mathbb{C}$, $x^{\otimes 0} := 1 \in \mathbb{C}$.

To define the locally convex space $\mathcal{P}_n(\mathcal{X}')$ of n-homogeneous polynomials on $\mathcal{X}'$ we use the canonical topological linear isomorphism

$$\mathcal{P}_n(\mathcal{X}') \simeq (\mathcal{X}'^{\hat{\otimes} n})'$$

described in [4]. Namely, given a functional $p_n \in (\mathcal{X}'^{\hat{\otimes} n})'$, we define an n-homogeneous polynomial $P_n \in \mathcal{P}_n(\mathcal{X}')$ by $P_n(x) := p_n(x^{\otimes n})$, $x \in \mathcal{X}'$. We equip $\mathcal{P}_n(\mathcal{X}')$ with the locally convex topology $\mathfrak{b}$ of uniform convergence on bounded sets in $\mathcal{X}'$. Set $\mathcal{P}_0(\mathcal{X}') := \mathbb{C}$. The space $\mathcal{P}(\mathcal{X}')$ of all continuous polynomials on $\mathcal{X}'$ is defined to be the complex linear span of all $\mathcal{P}_n(\mathcal{X}')$, $n \in \mathbb{Z}_+$, endowed with the topology $\mathfrak{b}$. Denote

$$\Gamma(\mathcal{X}) := \bigoplus_{n \in \mathbb{Z}_+}{}_{fin} \mathcal{X}^{\hat{\otimes} n} \subset \bigoplus_{n \in \mathbb{Z}_+} \mathcal{X}^{\hat{\otimes} n}.$$

Note, that we consider only the case when the elements of direct sum consist of finite but not fixed number of addends. For simplicity of notation we write $\Gamma(\mathcal{X})$ instead of commonly used $\Gamma_{fin}(\mathcal{X})$.

We have the following assertion (see also [12, Proposition 2.1]).

Theorem 1. *There exists the linear topological isomorphism*

$$\Upsilon_{\mathcal{X}} \colon \Gamma(\mathcal{X}) \longrightarrow \mathcal{P}(\mathcal{X}')$$

for any nuclear (F) or (DF) space $\mathcal{X}$.

Let $A : X \longrightarrow Y$ be any linear and continuous operator, where X, Y are locally convex spaces. It is easy to see, that the operator $A \otimes I_Y$, defined on the tensor product $X \otimes Y$ by the formula

$$(A \otimes I_Y)(x \otimes y) := Ax \otimes y, \qquad x \in X, \quad y \in Y,$$

is linear, where I_Y denotes the identity on Y. The operator $A \otimes I_Y$ is continuous in projective topology $\mathfrak{p}$ and it has a unique extension to linear continuous operator onto the space $X \otimes_{\mathfrak{p}} Y$.

The following assertion essentially will be used in the proof of Theorem 3.

Proposition 1 ([13]). *For any nuclear (F) or (DF) spaces X, Y, and any operator $A \in \mathscr{L}(X, Y)$ the following equality holds*

$$\ker(A \otimes I_Y) = \ker(A) \otimes_{\mathfrak{p}} Y.$$

2 SPACES OF FUNCTIONS

Let us consider the definition and some properties of the space of Gevrey ultradifferentiable functions with compact supports. For more details we refer the reader to [10, 11].

We use the following notations: $t^k := t_1^{k_1} \cdot \ldots \cdot t_d^{k_d}$, $k^{k\beta} := k_1^{k_1\beta} \cdot \ldots \cdot k_d^{k_d\beta}$, $|k| := k_1 + \cdots + k_d$ for all $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$(or $\mathbb{C}^d$), $k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$ and $\beta > 1$. Let $\partial^k := \partial_1^{k_1} \ldots \partial_d^{k_d}$, where $\partial_j^{k_j} := \partial^{k_j}/\partial t_j^{k_j}$, $j = 1, \ldots, d$. The notation $\mu \prec \nu$ with $\mu, \nu \in \mathbb{R}^d$ means that $\mu_1 < \nu_1, \ldots, \mu_d < \nu_d$ (similarly, $\mu \succ \nu$). Let $[\mu, \nu] := [\mu_1, \nu_1] \times \ldots \times [\mu_d, \nu_d]$ and $(\mu, \nu) := (\mu_1, \nu_1) \times \ldots \times (\mu_d, \nu_d)$ for any $\mu \prec \nu$. In the following $t \in [\mu, \nu]$ means that $t_j \in [\mu_j, \nu_j]$ and $t \to \infty$ (resp. $t \to 0$) means that $t_j \to \infty$ (resp. $t_j \to 0$) for all $j = 1, \ldots, d$.

A complex infinitely smooth function φ on $\mathbb{R}^d$ is called a Gevrey ultradifferentiable with $\beta > 1$ (see [10, II.2.1]) if for every $[\mu, \nu] \subset \mathbb{R}^d$ there exist constants $h > 0$ and $C > 0$ such that

$$\sup_{t \in [\mu,\nu]} |\partial^k \varphi(t)| \le C h^{|k|} k^{k\beta} \tag{1}$$

holds for all $k \in \mathbb{Z}_+^d$.

For a fixed $h > 0$, consider the subspace $\mathcal{G}_\beta^h[\mu, \nu]$ of all functions supported by $[\mu, \nu] \subset \mathbb{R}^d$ and such that there exists a constant $C = C(\varphi) > 0$, that inequality (1) holds for all $k \in \mathbb{Z}_+^d$. Therefore, the space of ultradifferentiable functions with compact supports is defined as follows

$$\mathcal{G}_\beta^h[\mu, \nu] := \left\{ \varphi \in C^\infty(\mathbb{R}^d) : \operatorname{supp} \varphi \subset [\mu, \nu], \|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]} < \infty \right\},$$

with the norm

$$\|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]} := \sup_{k \in \mathbb{Z}_+^d} \sup_{t \in [\mu,\nu]} \frac{|\partial^k \varphi(t)|}{h^{|k|} k^{k\beta}}.$$

Proposition 2 ([10]). *Each $\mathcal{G}_\beta^h[\mu, \nu]$ is a Banach space, and all inclusions $\mathcal{G}_\beta^h[\mu, \nu] \hookrightarrow \mathcal{G}_\beta^l[\mu, \nu]$ with $h < l$ are compact. Moreover, if $[\mu, \nu] \subset [\mu', \nu']$, then $\mathcal{G}_\beta^h[\mu, \nu]$ is closed subspace in $\mathcal{G}_\beta^h[\mu', \nu']$.*

This proposition implies that the set of Banach spaces

$$\left\{ \mathcal{G}_\beta^h[\mu, \nu] : \; [\mu, \nu] \subset \mathbb{R}^d, \; h > 0 \right\}$$

is partially ordered. Therefore we can consider this set as inductive system with respect to stated above compact inclusions. Hence, we define the space

$$\mathcal{G}_\beta(\mathbb{R}^d) := \bigcup_{\mu \prec \nu, h > 0} \mathcal{G}_\beta^h[\mu, \nu], \qquad \mathcal{G}_\beta(\mathbb{R}^d) \simeq \liminf_{\mu \prec \nu, h > 0} \mathcal{G}_\beta^h[\mu, \nu],$$

and endow it with the topology of inductive limit.

The strong dual space $\mathcal{G}'_\beta(\mathbb{R}^d)$ is called the space of Roumieu ultradistributions on $\mathbb{R}^d$.

Proposition 3 ([10]). *The spaces $\mathcal{G}_\beta(\mathbb{R}^d)$ and $\mathcal{G}'_\beta(\mathbb{R}^d)$ are nonempty locally convex nuclear reflexive spaces. Moreover, $\mathcal{G}_\beta(\mathbb{R}^d)$ is (DFS) space, and $\mathcal{G}'_\beta(\mathbb{R}^d)$ is (FS) space.*

Next define the space of entire functions of exponential type, which will be an image of the space $\mathcal{G}_\beta(\mathbb{R}^d)$ under the Fourier-Laplace transformation (see Section 3).

Let M be a set in $\mathbb{R}^d$. The support function of the set M is defined to be a function

$$H_M(x) = \sup_{t \in M}(t, x), \qquad x \in \mathbb{R}^d,$$

where $(t, x) := t_1x_1 + \cdots + t_dx_d$ denotes the scalar product. It is known [7], that $H_M(\eta)$ is convex, lower semi-continuous function, that may take the value $+\infty$. If M is bounded set, then its support function is continuous.

Let $B_r \subset \mathbb{C}^d$ be a ball of a radius $r > 0$. The space $E(\mathbb{C}^d)$ of entire functions of exponential type we will endow with locally convex topology of uniform convergence on compact sets. This topology can be defined by the system of seminorms

$$p_{r,M}(\psi) := \sup_{z \in B_r} |\psi(z)| e^{-H_M(\eta)},$$

where $\eta = (\eta_1, \dots, \eta_d) \in \mathbb{R}^d$ is imaginary part of $z = (z_1, \dots, z_d) \in \mathbb{C}^d$.

Fix an arbitrary real $\beta > 1$. For a positive number $h > 0$ and vectors $\mu = (\mu_1, \dots, \mu_d)$, $\nu = (\nu_1, \dots, \nu_d) \in \mathbb{R}^d$, such that $\mu \prec \nu$, in the space of entire functions of exponential type we define the subspace $E^h_\beta[\mu, \nu]$ of functions $\mathbb{C}^d \ni z \longmapsto \psi(z) \in \mathbb{C}$ with finite norm

$$\|\psi\|_{E^h_\beta[\mu,\nu]} := \sup_{k \in \mathbb{Z}^d_+} \sup_{z \in \mathbb{C}^d} \frac{|z^k \psi(z) e^{-H_{[\mu,\nu]}(\eta)}|}{h^{|k|} k^{k\beta}}. \tag{2}$$

Since for any $r > 0$ and $\psi \in E^h_\beta[\mu, \nu]$ the next inequality $p_{r,[\mu,\nu]}(\psi) \le \|\psi\|_{E^h_\beta[\mu,\nu]}$ is valid, then all inclusions $E^h_\beta[\mu, \nu] \hookrightarrow E(\mathbb{C}^d)$ are continuous.

Proposition 4. *Each space $E^h_\beta[\mu, \nu]$ is Banach space, and all inclusions*

$$E^h_\beta[\mu, \nu] \hookrightarrow E^{h'}_\beta[\mu', \nu'] \qquad \text{with} \qquad [\mu, \nu] \subset [\mu', \nu'],\ h < h',$$

are compact.

Proof. Let us prove the completeness of the space $E^h_\beta[\mu, \nu]$. Let $\{\psi_m\}_{m \in \mathbb{N}}$ be a Cauchy sequence in $E^h_\beta[\mu, \nu]$. It means that for every $\varepsilon > 0$ there exists an integer $N_\varepsilon \in \mathbb{N}$ such that $\forall\, m, n > N_\varepsilon$ the next inequality $\|\psi_m - \psi_n\|_{E^h_\beta[\mu,\nu]} < \varepsilon$ is valid.

The following inequality

$$\sup_{z \in B_r} \frac{|z^k \psi(z)|}{h^{|k|} k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} \le \|\psi\|_{E^h_\beta[\mu,\nu]}, \qquad \psi \in E^h_\beta[\mu, \nu],$$

is obvious for all $k \in \mathbb{Z}^d_+$ and $r > 0$. It follows that $\{\varphi_m\}_{m \in \mathbb{N}}$, where $\varphi_m(z) := \dfrac{z^k \psi_m(z)}{h^{|k|} k^{k\beta}}$, is fundamental sequence in the space of entire functions of exponential type. Therefore for any $k \in \mathbb{Z}^d_+$ and $r > 0$ we have

$$\sup_{z \in B_r} \frac{|z^k (\psi_m(z) - \psi_n(z))|}{h^{|k|} k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} < \varepsilon, \quad \forall\, m, n > N_\varepsilon. \tag{3}$$

Since $\{\varphi_m\}_{m\in\mathbb{N}}$ is fundamental sequence, it is bounded in $E(\mathbb{C}^d)$. From the Bernstein theorem on compactness [14, theorem 3.3.6] it follows that there exist a subsequence $\{\varphi_{k_m}\}_{k_m\in\mathbb{N}}$ and a function $\varphi \in E(\mathbb{C}^d)$ such that the following equality is satisfied

$$\lim_{k_m\to\infty} \sup_{z\in B_r} \frac{|z^k(\psi_{k_m}(z) - \psi(z))|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} = 0, \qquad k \in \mathbb{Z}^d_+, \quad r > 0.$$

Let us pass to the limit in (3) as $m = k_m \to \infty$. Consequently, for all $k \in \mathbb{Z}^d_+$ and $r > 0$ we obtain the inequality

$$\sup_{z\in B_r} \frac{|z^k(\psi(z) - \psi_n(z))|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} < \varepsilon,$$

which satisfies for all $n > N_\varepsilon$. Hence from the triangle inequality we obtain

$$\sup_{z\in B_r} \frac{|z^k\psi(z)|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} \le \sup_{z\in B_r} \frac{|z^k\psi_{n_0}(z)|}{h^{|k|}k^{k\beta}} e^{-H_{[\mu,\nu]}(\eta)} + \varepsilon,$$

where $n_0 = N_\varepsilon + 1$.

Taking a supremum over k and r in the above inequality, we obtain

$$\|\psi\|_{E^h_\beta[\mu,\nu]} \le \|\psi_{n_0}\|_{E^h_\beta[\mu,\nu]} + \varepsilon,$$

therefore $\psi \in E^h_\beta[\mu,\nu]$. Hence, the space $E^h_\beta[\mu,\nu]$ is complete.

The compactness of inclusions $E^h_\beta[\mu,\nu] \hookrightarrow E^{h'}_\beta[\mu',\nu']$ with $[\mu,\nu] \subset [\mu',\nu']$, $h < h'$ follows from obvious inequality $e^{-H_{[\mu',\nu']}} \le e^{-H_{[\mu,\nu]}}$ and from [10, pp. 38–40]. □

Define the space

$$E_\beta(\mathbb{C}^d) := \bigcup_{\mu\prec\nu,\, h>0} E^h_\beta[\mu,\nu], \qquad E_\beta(\mathbb{C}^d) \simeq \liminf_{\mu\prec\nu,\, h>0} E^h_\beta[\mu,\nu],$$

and endow it with the topology of inductive limit with respect to compact inclusions from the Proposition 4.

In what follows to simplify the notations we will write $\mathcal{G}_\beta := \mathcal{G}_\beta(\mathbb{R}^d)$, $\mathcal{G}'_\beta := \mathcal{G}'_\beta(\mathbb{R}^d)$, $E_\beta := E_\beta(\mathbb{C}^d)$, $E'_\beta := E'_\beta(\mathbb{C}^d)$.

3 FOURIER-LAPLACE TRANSFORM AND PALEY–WIENER-TYPE THEOREM

Consider the inductive limits of Banach spaces

$$E_\beta[\mu,\nu] := \bigcup_{h>0} \mathcal{G}^h_\beta[\mu,\nu], \qquad E_\beta[\mu,\nu] \simeq \liminf_{h\to\infty} \mathcal{G}^h_\beta[\mu,\nu],$$

and

$$\mathcal{G}_\beta[\mu,\nu] := \bigcup_{h>0} \mathcal{G}^h_\beta[\mu,\nu], \qquad \mathcal{G}_\beta[\mu,\nu] \simeq \liminf_{h\to\infty} \mathcal{G}^h_\beta[\mu,\nu].$$

On the space $\mathcal{G}_\beta$ we define the Fourier-Laplace transform

$$\widehat{\varphi}(z) := (F\varphi)(z) = \int_{\mathbb{R}^d} e^{-i(t,z)}\varphi(t)\,dt, \qquad \varphi \in \mathcal{G}_\beta, \quad z \in \mathbb{C}^d. \tag{4}$$

Our main task is to show, that the function $\widehat{\varphi}(z)$ belongs to the space E_β, moreover, we will prove that the mapping $F : \mathcal{G}_\beta \longrightarrow E_\beta$ is surjective. For this end we prove the following auxiliary statement, which can be found in [8, Lemma 1], but our proof is different.

Proposition 5. *The image of the space $\mathcal{G}_\beta[\mu,\nu]$ with respect to mapping F is the space $E_\beta[\mu,\nu]$.*

Proof. Let $\varphi \in \mathcal{G}_\beta[\mu,\nu]$. Properties of the Fourier transform imply $\widehat{\partial^k\varphi}(z) = z^k\widehat{\varphi}(z)$ for all $k \in \mathbb{Z}_+^d$. Therefore for any z and k we have

$$|z^k\widehat{\varphi}(z)| = \Big|\int_{\mathbb{R}^d} e^{-i(t,z)}\partial^k\varphi(t)\,dt\Big| \le \int_{[\mu,\nu]} |e^{-i(t,\xi)}e^{(t,\eta)}\partial^k\varphi(t)|\,dt$$

$$\le h^{|k|}k^{k\beta}e^{H_{[\mu,\nu]}(\eta)}\|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]}\int_{[\mu,\nu]} dt.$$

It follows

$$\|\widehat{\varphi}\|_{E_\beta^h[\mu,\nu]} \le C\|\varphi\|_{\mathcal{G}_\beta^h[\mu,\nu]}, \tag{5}$$

where $C = \prod_{j=1}^d(\nu_j - \mu_j)$. Hence, $F(\mathcal{G}_\beta^h[\mu,\nu]) \subset E_\beta^h[\mu,\nu]$.

Vice versa. Let $\psi \in E_\beta^h[\mu,\nu]$. It is known, that the norm of the space $E_\beta^h[\mu,\nu]$ can be defined by the formula

$$\|\psi\|_{E_\beta^h[\mu,\nu]} := \sup_{k\in\mathbb{Z}_+^d}\sup_{z\in\mathbb{C}^d}\frac{|z^k\psi(z)e^{-H_{[\mu,\nu]}(\eta)}|}{h^{|k|}|k|!^\beta},$$

moreover, the topology, defined by this norm, is equivalent to earlier defined (see (2)). It follows that for each function $\psi \in E_\beta^h[\mu,\nu]$ there exists a constant C such that the inequality

$$|z^k\psi(z)| \le Ch^{|k|}|k|!^\beta e^{H_{[\mu,\nu]}(\eta)} \tag{6}$$

holds for all $z \in \mathbb{C}^d$.

The following inequality

$$e^{\beta t^{1/\beta}} = \left(e^{t^{1/\beta}}\right)^\beta = \Big(\sum_{m=0}^{\infty}\frac{t^{m/\beta}}{m!}\Big)^\beta \ge \frac{|t|^m}{m!^\beta},$$

holds for all $t \in \mathbb{R}$ and $m \in \mathbb{Z}_+$. In particular, for $t = |z|/h$ and $m = |k|$, we obtain

$$e^{\beta\left(\frac{|z|}{h}\right)^{1/\beta}} \ge \frac{|z|^{|k|}}{h^{|k|}|k|!^\beta}.$$

Hence from the inequality $|z^k| \le |z|^{|k|}$ it follows

$$\frac{h^{|k|}|k|!^\beta}{|z^k|}e^{H_{[\mu,\nu]}(\eta)} \ge \frac{e^{H_{[\mu,\nu]}(\eta)}}{e^{(L|z|)^{1/\beta}}},$$

where $L = \frac{\beta^\beta}{h}$. So, if the function ψ satisfies the inequality (6), i.e. belongs to the space $E_\beta^h[\mu,\nu]$, then it satisfies the inequality

$$|\psi(z)| \le Ce^{-(L|z|)^{1/\beta}+H_{[\mu,\nu]}(\eta)}.$$

From the theorem [10, theorem 2.22] it follows that there exists a function $\varphi \in \mathcal{G}_\beta[\mu,\nu]$ such that $\widehat{\varphi} = \psi$, i.e. $E_\beta^h[\mu,\nu] \subset F(\mathcal{G}_\beta^h[\mu,\nu])$.

Hence, we have proved $F(\mathcal{G}_\beta^h[\mu,\nu]) = E_\beta^h[\mu,\nu]$. Since the constant $h > 0$ is arbitrary, properties of inductive limit imply the desired equality

$$F(\mathcal{G}_\beta[\mu,\nu]) = E_\beta[\mu,\nu].$$

□

The immediate consequence of the Proposition 5 and of the properties of inductive limit is the following assertion.

Corollary 1. *The image of the space $\mathcal{G}_\beta$ with respect to mapping F is the space E_β.*

Therefore, we may consider the adjoint mapping $F' : E'_\beta \longrightarrow \mathcal{G}'_\beta$.

Theorem 2. *There exist the following topological isomorphisms*

$$F(\mathcal{G}_\beta) \simeq E_\beta \qquad \textit{and} \qquad F'(E'_\beta) \simeq \mathcal{G}'_\beta.$$

Proof. The inequality (5) implies, that the mapping

$$F : \mathcal{G}_\beta[\mu,\nu] \ni \varphi \longmapsto \widehat{\varphi} \in E_\beta[\mu,\nu]$$

is continuous. From the Proposition 5 we obtain the surjectivity of the map. Therefore, the open map theorem [5, theorem 6.7.2] implies the topological isomorphism $F(\mathcal{G}_\beta[\mu,\nu]) \simeq E_\beta[\mu,\nu]$. Since the segment $[\mu,\nu]$ is arbitrary, the properties of inductive limit imply the desired topological isomorphisms. □

Using the Theorem 1 and a tensor structure of the space

$$\Gamma(\mathcal{G}_\beta) := \bigoplus_{n\in\mathbb{Z}_+}{}_{fin}\, \mathcal{G}_\beta^{\widehat{\otimes}n} \subset \bigoplus_{n\in\mathbb{Z}_+} \mathcal{G}_\beta^{\widehat{\otimes}n},$$

we extend the mapping F to the mapping $F^{\otimes}$, that defined on $\Gamma(\mathcal{G}_\beta)$.

At first, take an element $\varphi^{\otimes n} \in \mathcal{G}_\beta^{\widehat{\otimes}n}$, with $\varphi \in \mathcal{G}_\beta$, from the total subset of $\mathcal{G}_\beta^{\widehat{\otimes}n}$. Define the operator $F^{\otimes n}$ as follows

$$F^{\otimes n} : \varphi^{\otimes n} \longmapsto \widehat{\varphi}^{\otimes n} \quad \text{and} \quad F^{\otimes 0} := I_{\mathbb{C}},$$

where $\widehat{\varphi}^{\otimes n} := (F\varphi)^{\otimes n}$. Next, we extend the map $F^{\otimes n}$ onto whole space $\mathcal{G}_\beta^{\widehat{\otimes}n}$ by linearity and continuity. So, we obtain $F^{\otimes n} \in \mathscr{L}\big(\mathcal{G}_\beta^{\widehat{\otimes}n}, E_\beta^{\widehat{\otimes}n}\big)$. Finally, we define $F^{\otimes}$ as the mapping

$$F^{\otimes} := \big(F^{\otimes n}\big) : \Gamma(\mathcal{G}_\beta) \ni \boldsymbol{p} := \big(p_n\big) \quad \longmapsto \quad \widehat{\boldsymbol{p}} := \big(\widehat{p}_n\big) \in \Gamma(E_\beta), \tag{7}$$

where $p_n \in \mathcal{G}_\beta^{\widehat{\otimes}n}$, $\widehat{p}_n := F^{\otimes n} p_n \in E_\beta^{\widehat{\otimes}n}$.

The following commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\mathcal{G}'_\beta) & \xrightarrow{F^{\otimes}_{\mathcal{P}}} & \mathcal{P}(E'_\beta) \\ {\scriptstyle \Upsilon^{-1}_{\mathcal{G}_\beta}}\big\downarrow & & \big\uparrow{\scriptstyle \Upsilon_{E_\beta}} \\ \Gamma(\mathcal{G}_\beta) & \xrightarrow{F^{\otimes}} & \Gamma(E_\beta) \end{array} \tag{8}$$

uniquely defines the operator $F^{\otimes}_{\mathcal{P}} : \mathcal{P}(\mathcal{G}'_\beta) \longrightarrow \mathcal{P}(E'_\beta)$. The map $F^{\otimes}_{\mathcal{P}}$ we will call the polynomial Fourier-Laplace transformation.

We proved above that the mappings $F : \mathcal{G}_\beta \longrightarrow E_\beta$ and $F' : E'_\beta \longrightarrow \mathcal{G}'_\beta$ are topological isomorphisms. Let us prove the analogue of this result. The next theorem may be considered as Paley–Wiener-type theorem.

Theorem 3. *Polynomial Fourier-Laplace transformation is topological isomorphism from the algebra $\mathcal{P}(\mathcal{G}'_\beta)$ into the algebra $\mathcal{P}(E'_\beta)$.*

Proof. From the Theorem 1 and commutativity of the diagram (8) it follows that it is enough to show that the mapping $F^{\otimes} : \Gamma(\mathcal{G}_\beta) \longrightarrow \Gamma(E_\beta)$ is topological isomorphism.

Theorem 2 and Corollary 1 imply the following equalities

$$\ker F = \{0\}, \qquad \ker F^{-1} = \{0\}.$$

Let us consider the operators

$$\begin{aligned} I_{\mathcal{G}_\beta} \otimes F : \mathcal{G}_\beta \otimes \mathcal{G}_\beta \longrightarrow \mathcal{G}_\beta \otimes E_\beta, \qquad & F \otimes I_{E_\beta} : \mathcal{G}_\beta \otimes E_\beta \longrightarrow E_\beta \otimes E_\beta, \\ I_{E_\beta} \otimes F^{-1} : E_\beta \otimes E_\beta \longrightarrow E_\beta \otimes \mathcal{G}_\beta, \qquad & F^{-1} \otimes I_{\mathcal{G}_\beta} : E_\beta \otimes \mathcal{G}_\beta \longrightarrow \mathcal{G}_\beta \otimes \mathcal{G}_\beta. \end{aligned}$$

Since spaces $\mathcal{G}_\beta$ and E_β are nuclear (DF) spaces, Proposition 1 implies the equalities

$$\begin{aligned} \ker(I_{\mathcal{G}_\beta} \otimes F) = \{0\}, \qquad & \ker(F \otimes I_{E_\beta}) = \{0\}, \\ \ker(I_{E_\beta} \otimes F^{-1}) = \{0\}, \qquad & \ker(F^{-1} \otimes I_{\mathcal{G}_\beta}) = \{0\}. \end{aligned}$$

Therefore, compositions of these operators have the trivial kernels, i.e.

$$\begin{aligned} \ker\big((F \otimes I_{E_\beta}) \circ (I_{\mathcal{G}_\beta} \otimes F)\big) &= \ker(F \otimes F) = \{0\}, \\ \ker\big((F^{-1} \otimes I_{\mathcal{G}_\beta}) \circ (I_{E_\beta} \otimes F^{-1})\big) &= \ker(F^{-1} \otimes F^{-1}) = \{0\}. \end{aligned}$$

Proceeding inductively finite times, we obtain

$$\begin{aligned} \ker F^{\otimes n} &= \ker\big(\underbrace{F \otimes \cdots \otimes F}_{n}\big) = \{0\}, \\ \ker (F^{-1})^{\otimes n} &= \ker\big(\underbrace{F^{-1} \otimes \cdots \otimes F^{-1}}_{n}\big) = \{0\}, \end{aligned}$$

for all natural n. Note, that the mappings $F^{\otimes n}$, $(F^{-1})^{\otimes n}$ are continuous as tensor products of continuous operators. Since $(F^{\otimes n})^{-1} = (F^{-1})^{\otimes n}$, the mapping $F^{\otimes n} : \mathcal{G}_\beta^{\widehat{\otimes} n} \longrightarrow E_\beta^{\widehat{\otimes} n}$ is topological isomorphism. Finally, the map $F^{\otimes} : \Gamma(\mathcal{G}_\beta) \longrightarrow \Gamma(E_\beta)$ is topological isomorphism via the properties of direct sum topology. ☐

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У статті описано образ простору ультрадиференційовних функцій з компактними носіями відносно перетворення Фур'є-Лапласа. Доведено аналог теореми Пелі-Вінера для поліноміальних ультрадиференційовних функцій.

Ключові слова і фрази: ультрадиференційовна функція, ультрарозподіл, поліноміальна основна функція, теорема типу Пелі-Вінера.