ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2016, **8** (1), 83–106
doi:10.15330/cmp.8.1.83-106

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2016, Т.8, №1, С.83–106

KACHANOVSKY N.A.

OPERATORS OF STOCHASTIC DIFFERENTIATION ON SPACES OF NONREGULAR GENERALIZED FUNCTIONS OF LÉVY WHITE NOISE ANALYSIS

The operators of stochastic differentiation, which are closely related with the extended Skorohod stochastic integral and with the Hida stochastic derivative, play an important role in the classical (Gaussian) white noise analysis. In particular, these operators can be used in order to study some properties of the extended stochastic integral and of solutions of stochastic equations with Wick-type nonlinearities.

During recent years the operators of stochastic differentiation were introduced and studied, in particular, in the framework of the Meixner white noise analysis, in the same way as on spaces of regular test and generalized functions and on spaces of nonregular test functions of the Lévy white noise analysis. In the present paper we make the next natural step: introduce and study operators of stochastic differentiation on spaces of nonregular generalized functions of the Lévy white noise analysis (i.e., on spaces of generalized functions that belong to the so-called nonregular rigging of the space of square integrable with respect to the measure of a Lévy white noise functions). In so doing, we use Lytvynov's generalization of the chaotic representation property. The researches of the present paper can be considered as a contribution in a further development of the Lévy white noise analysis.

Key words and phrases: operator of stochastic differentiation, stochastic derivative, extended stochastic integral, Lévy process.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01601, Kyiv, Ukraine
E-mail: nkachano@gmail.com

INTRODUCTION

Let $L = (L_t)_{t\in[0,+\infty)}$ be a Lévy process (i.e., a random process on $[0, +\infty)$ with stationary independent increments and such that $L_0 = 0$, see, e.g., [5, 30, 31] for details) without Gaussian part and drift. In [23] the extended Skorohod stochastic integral with respect to L and the corresponding Hida stochastic derivative on the space of square integrable random variables (L^2) were constructed in terms of Lytvynov's generalization of the *chaotic representation property* (CRP) (see [27] and Subsection 1.2), some properties of these operators were established; and it was shown that the above-mentioned integral coincides with the well-known (constructed in terms of Itô's generalization of the CRP [14]) extended stochastic integral with respect to a Lévy process (e.g., [6, 7]). In [10, 21] the notion of stochastic integral and derivative was widened to spaces of regular and nonregular test and generalized functions that belong to so-called regular parametrized and nonregular riggings of (L^2) respectively, this gives a possibility to extend an area of possible applications of the above-mentioned operators (in particular, now it is possible to define the stochastic integral and derivative as linear *continuous* operators). Together with the stochastic integral and derivative, it is natural to introduce

УДК 517.98
2010 *Mathematics Subject Classification:* 60G51, 60H40, 60H05, 46F05, 46F25.

and to study so-called *operators of stochastic differentiation* in the Lévy white noise analysis, by analogy with the Gaussian analysis [1, 37], the Gamma-analysis [17, 18], and the Meixner analysis [19, 20]. These operators are closely related with the extended Skorohod stochastic integral with respect to a Lévy process and with the corresponding Hida stochastic derivative and, by analogy with the classical Gaussian case, can be used, in particular, in order to study some properties of the extended stochastic integral and of solutions of normally ordered stochastic equations (stochastic equations with Wick-type nonlinearities in another terminology). In [9, 8] the operators of stochastic differentiation on spaces that belong to a *regular parametrized* rigging of (L^2) ([21]) were introduced and studied. This rigging plays a very important role in the Lévy analysis; but, in order to solve some problems that arise in this analysis (in particular, in the theory of normally ordered stochastic equations), it is necessary to introduce into consideration another, *nonregular* rigging of (L^2) (see [21] and Subsection 1.3), and operators (e.g., the extended stochastic integral, the Hida stochastic derivative) on spaces (of nonregular test and generalized functions) that belong to this rigging. Therefore it is natural to introduce and to study operators of stochastic differentiation on the just now mentioned spaces.

In the paper [24] the operators of stochastic differentiation were introduced and studied on the spaces of nonregular test functions of the Lévy white noise analysis. In particular, it was shown that, roughly speaking, these operators are the restrictions to the above-mentioned spaces of the corresponding operators on (L^2). The next natural step is, of course, to consider operators of stochastic differentiation on the spaces of nonregular generalized functions. But here there is a problem: in contrast to the classical Gaussian case and to the "regular case", the operators of stochastic differentiation on (L^2) cannot be naturally continued to the just now mentioned spaces (to the point, actually for the same reason the Hida stochastic derivative also cannot be naturally continued from (L^2) to the spaces of nonregular generalized functions). Nevertheless, it is possible to introduce on these spaces natural analogs of the above-mentioned operators. These analogs have properties quite analogous to the properties of operators of stochastic differentiation, and can be accepted as operators of stochastic differentiation on the spaces of nonregular generalized functions. In the present paper we introduce and study in detail the just now mentioned operators. In forthcoming papers we'll consider elements of the so-called Wick calculus in the Lévy white noise analysis, this will give us the possibility to continue the study of properties and to consider some applications of the operators of stochastic differentiation.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a convenient for our considerations probability triplet connected with L; then, following [21, 23, 27], we describe in detail Lytvynov's generalization of the CRP, the nonregular rigging of (L^2), and stochastic derivatives and integrals on the spaces that belong to this rigging. In the second section we deal with the operators of stochastic differentiation on the spaces of nonregular generalized functions, considering separately the cases of bounded and unbounded operators. Note that some results of this paper were announced without proofs in [25].

1 PRELIMINARIES

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H; by $(\cdot,\cdot)_H$ the scalar product in a space H; and by $\langle\cdot,\cdot\rangle_H$ or $\langle\langle\cdot,\cdot\rangle\rangle_H$ the dual pairing generated by the scalar product in a space H. Another notation for norms, scalar products and dual pairings will be introduced when it will be necessary.

1.1 Lévy processes

Denote $\mathbb{R}_+ := [0, +\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_t)_{t \in \mathbb{R}_+}$ (a random process on $\mathbb{R}_+$ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift (it is comparatively simple to consider such processes from technical point of view). As is well known (e.g., [7]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_t}] = \exp\Big[t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x)\nu(dx)\Big], \tag{1}$$

where ν is the Lévy measure of L, which is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, here and below $\mathcal{B}$ denotes the Borel σ-algebra; $\mathbb{E}$ denotes the expectation. We assume that ν *is a Radon measure whose support contains an infinite number of points,* $\nu(\{0\}) = 0$, *there exists* $\varepsilon > 0$ *such that*

$$\int_{\mathbb{R}} x^2 e^{\varepsilon |x|} \nu(dx) < \infty,$$

and

$$\int_{\mathbb{R}} x^2 \nu(dx) = 1. \tag{2}$$

Let us define a measure of the white noise of L. Let $\mathcal{D}$ denote the set of all real-valued infinite-differentiable functions on $\mathbb{R}_+$ with compact supports. As is well known, $\mathcal{D}$ can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [4]). Let $\mathcal{D}'$ be the set of linear continuous functionals on $\mathcal{D}$. For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle \omega, \varphi \rangle$; note that one can understand $\langle \cdot, \cdot \rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on $\mathbb{R}_+$, see Subsection 1.3 for details. The notation $\langle \cdot, \cdot \rangle$ will be preserved for dual pairings in tensor powers of spaces.

Definition. *A probability measure* μ *on* $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, *where* $\mathcal{C}$ *denotes the cylindrical* σ*-algebra, with the Fourier transform*

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \mu(d\omega) = \exp\Big[\int_{\mathbb{R}_+ \times \mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x) du \nu(dx)\Big], \quad \varphi \in \mathcal{D}, \tag{3}$$

is called the measure of a Lévy white noise.

The existence of μ follows from the Bochner–Minlos theorem (e.g., [13]), see [27]. Below we assume that the σ*-algebra* $\mathcal{C}(\mathcal{D}')$ *is complete with respect to* μ, i.e., $\mathcal{C}(\mathcal{D}')$ contains all subsets of all measurable sets O such that $\mu(O) = 0$.

Denote $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) real-valued square integrable with respect to μ functions on $\mathcal{D}'$; let also $\mathcal{H} := L^2(\mathbb{R}_+)$. Substituting in (3) $\varphi = t\psi$, $t \in \mathbb{R}$, $\psi \in \mathcal{D}$, and using the Taylor decomposition by t and (2), one can show that

$$\int_{\mathcal{D}'} \langle \omega, \psi \rangle^2 \mu(d\omega) = \int_{\mathbb{R}_+} (\psi(u))^2 du \tag{4}$$

(this statement follows also from results of [27] and [7]). Let $f \in \mathcal{H}$ and $\mathcal{D} \ni \varphi_k \to f$ in $\mathcal{H}$ as $k \to \infty$ (it is well known (e.g., [4]) that $\mathcal{D}$ is a dense set in $\mathcal{H}$). It follows from (4) that

$\{\langle \circ, \varphi_k \rangle\}_{k\geq 1}$ is a Cauchy sequence in (L^2), therefore one can define $\langle \circ, f \rangle := (L^2) - \lim_{k\to\infty} \langle \circ, \varphi_k \rangle$. It is easy to show (by the method of "mixed sequences") that $\langle \circ, f \rangle$ does not depend on the choice of an approximating sequence for f and therefore is well defined in (L^2).

Let us consider $\langle \circ, 1_{[0,t)} \rangle \in (L^2)$, $t \in \mathbb{R}_+$ (here and below 1_A denotes the indicator of a set A). It follows from (1) and (3) that $\big(\langle \circ, 1_{[0,t)} \rangle\big)_{t\in\mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$, i.e., one can write $L_t = \langle \circ, 1_{[0,t)} \rangle \in (L^2)$.

Remark. *Note that one can understand the Lévy white noise as a generalized random process (in the sense of [11]) with trajectories from $\mathcal{D}'$: formally $L'_\cdot(\omega) = \langle \omega, 1_{[0,\cdot)} \rangle' = \langle \omega, \delta_\cdot \rangle = \omega(\cdot)$, where $\delta_\cdot$ is the Dirac delta-function concentrated at $\cdot$. Therefore μ is the measure of L' in the classical sense of this notion [12].*

Remark. *A Lévy process L without Gaussian part and drift is a Poisson process if its Lévy measure $\nu(\Delta) = \delta_1(\Delta)$, $\Delta \in \mathcal{B}(\mathbb{R})$, i.e., if ν is a point mass at 1. This measure does not satisfy the conditions accepted above (the support of δ_1 does not contain an infinite number of points); nevertheless, all results of the present paper have natural (and often strong) analogs in the Poissonian analysis. The reader can find more information about peculiarities of the Poissonian case in [23], Subsection 1.2.*

1.2 Lytvynov's generalization of the CRP

As is known, some random processes L have a so-called *chaotic representation property* (CRP) that consists, roughly speaking, in the following: any square integrable random variable can be decomposed in a series of repeated stochastic integrals from nonrandom functions with respect to L (see, e.g., [28] for a detailed presentation). The CRP plays a very important role in the stochastic analysis (in particular, for processes with the CRP this property can be used in order to construct extended stochastic integrals [16, 34, 15], stochastic derivatives and operators of stochastic differentiation, e.g., [37, 1]), but, unfortunately, the only Lévy processes with this property are Wiener and Poisson processes (e.g., [36]).

There are different approaches to a generalization of the CRP for Lévy processes: Itô's approach [14], Nualart-Schoutens' approach [29, 32], Lytvynov's approach [27], Oksendal's approach [7, 6] etc. The interconnections between these generalizations of the CRP are described in, e.g., [27, 2, 7, 35, 6, 23]. In the present paper we deal with Lytvynov's generalization of the CRP that will be described now in detail.

Denote by $\widehat{\otimes}$ a symmetric tensor product and set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Let $\mathcal{P} \equiv \mathcal{P}(\mathcal{D}')$ be the set of polynomials on $\mathcal{D}'$, i.e., $\mathcal{P}$ consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle \omega^{\otimes n}, f^{(n)} \rangle, \quad \omega \in \mathcal{D}', \ N_f \in \mathbb{Z}_+, \ f^{(n)} \in \mathcal{D}^{\widehat{\otimes} n}, \ f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f; $\langle \omega^{\otimes 0}, f^{(0)} \rangle := f^{(0)} \in \mathcal{D}^{\widehat{\otimes} 0} := \mathbb{R}$. Since the measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (3) and properties of the measure ν, see also [27]), $\mathcal{P}$ is a dense set in (L^2) [33]. Denote by $\mathcal{P}_n$ the set of polynomials of power not greater than n, by $\overline{\mathcal{P}}_n$ the closure of $\mathcal{P}_n$ in (L^2). Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear now that

$$(L^2) = \mathop{\oplus}_{n=0}^{\infty} \mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}^{\widehat{\otimes}n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle \circ^{\otimes n}, f^{(n)}\rangle:$ the orthogonal projection in (L^2) of a monomial $\langle \circ^{\otimes n}, f^{(n)}\rangle$ onto $\mathbf{P}_n$. Let us define scalar products $(\cdot,\cdot)_{ext}$ on $\mathcal{D}^{\widehat{\otimes}n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}^{\widehat{\otimes}n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!}\int_{\mathcal{D}'} :\langle \omega^{\otimes n}, f^{(n)}\rangle::\langle \omega^{\otimes n}, g^{(n)}\rangle:\mu(d\omega),$$

and let $|\cdot|_{ext}$ be the corresponding norms, i.e., $|f^{(n)}|_{ext} = \sqrt{(f^{(n)}, f^{(n)})_{ext}}$. Denote by $\mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}^{\widehat{\otimes}n}$ with respect to the norms $|\cdot|_{ext}$. For $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$ define a Wick monomial $:\langle \circ^{\otimes n}, F^{(n)}\rangle: \stackrel{\text{def}}{=} (L^2) - \lim_{k\to\infty} :\langle \circ^{\otimes n}, f^{(n)}_k\rangle:$, where $\mathcal{D}^{\widehat{\otimes}n} \ni f^{(n)}_k \to F^{(n)}$ as $k \to \infty$ in $\mathcal{H}^{(n)}_{ext}$ (well-posedness of this definition can be proved by the method of "mixed sequences"). Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)}\rangle: | f^{(n)} \in \mathcal{D}^{\widehat{\otimes}n}\}$ is a dense one in $\mathbf{P}_n$, we have the next statement (which describes Lytvynov's generalization of the CRP).

Theorem. ([27]) *A random variable $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$, $n \in \mathbb{Z}_+$, such that*

$$F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)}\rangle: \tag{5}$$

(the series converges in (L^2)) and

$$\|F\|^2_{(L^2)} = \int_{\mathcal{D}'} |F(\omega)|^2\mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n!|F^{(n)}|^2_{ext} < \infty.$$

So, for $F, G \in (L^2)$ the scalar product has the form

$$(F,G)_{(L^2)} = \int_{\mathcal{D}'} F(\omega)G(\omega)\mu(d\omega) = \mathbb{E}[FG] = \sum_{n=0}^{\infty} n!(F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}^{(n)}_{ext}$ are the kernels from decompositions (5) for F and G respectively. In particular, for $F^{(n)} \in \mathcal{H}^{(n)}_{ext}$ and $G^{(m)} \in \mathcal{H}^{(m)}_{ext}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned}\big(:\langle \circ^{\otimes n}, F^{(n)}\rangle:, :\langle \circ^{\otimes m}, G^{(m)}\rangle:\big)_{(L^2)} &= \int_{\mathcal{D}'} :\langle \omega^{\otimes n}, F^{(n)}\rangle::\langle \omega^{\otimes m}, G^{(m)}\rangle:\mu(d\omega) \\ &= \mathbb{E}\big[:\langle \circ^{\otimes n}, F^{(n)}\rangle::\langle \circ^{\otimes m}, G^{(m)}\rangle:\big] = \delta_{n,m} n!(F^{(n)}, G^{(n)})_{ext}.\end{aligned}$$

Note that in the space (L^2) we have $:\langle \circ^{\otimes 0}, F^{(0)}\rangle: = \langle \circ^{\otimes 0}, F^{(0)}\rangle = F^{(0)}$ and $:\langle \circ, F^{(1)}\rangle: = \langle \circ, F^{(1)}\rangle$ [27].

Remark. *In order to make calculations connected with the spaces $\mathcal{H}^{(n)}_{ext}$, it is necessary to know explicit formulas for the scalar products $(\cdot,\cdot)_{ext}$. Such formulas were obtained by E.W. Lytvynov in [27]. Here, following [23], we write out it for convenience of a reader. Denote by $\|\cdot\|_\nu$ the norm in the space $L^2(\mathbb{R},\nu)$ of (classes of) square integrable with respect to ν real-valued functions on $\mathbb{R}$. Let*

$$p_n(x) := x^n + a_{n,n-1}x^{n-1} + \cdots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R},\ j \in \{1,\dots,n-1\},\ n \in \mathbb{N}, \tag{6}$$

be orthogonal in $L^2(\mathbb{R},\nu)$ polynomials, i.e., for natural numbers n,m such that $n \neq m$, $\int_{\mathbb{R}} p_n(x)p_m(x)\nu(dx) = 0$. Then for $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, we have

$$
\begin{aligned}
(F^{(n)},G^{(n)})_{ext} = &\sum_{\substack{k,l_j,s_j\in\mathbb{N}:\ j=1,\dots,k,\ l_1>l_2>\dots>l_k,\\ l_1s_1+\dots+l_ks_k=n}} \frac{n!}{s_1!\cdots s_k!}\Big(\frac{\|p_{l_1}\|_\nu}{l_1!}\Big)^{2s_1}\cdots\Big(\frac{\|p_{l_k}\|_\nu}{l_k!}\Big)^{2s_k}\\
&\times \int_{\mathbb{R}_+^{s_1+\dots+s_k}} F^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})\\
&\times G^{(n)}(\underbrace{u_1,\dots,u_1}_{l_1},\dots,\underbrace{u_{s_1},\dots,u_{s_1}}_{l_1},\dots,\underbrace{u_{s_1+\dots+s_k},\dots,u_{s_1+\dots+s_k}}_{l_k})du_1\cdots du_{s_1+\dots+s_k}.
\end{aligned} \tag{7}
$$

In particular, for $n = 1$ $(F^{(1)},G^{(1)})_{ext} = \|p_1\|_\nu^2\int_{\mathbb{R}_+} F^{(1)}(u)G^{(1)}(u)du$; if $n = 2$ then we have $(F^{(2)},G^{(2)})_{ext} = \|p_1\|_\nu^4\int_{\mathbb{R}_+^2} F^{(2)}(u,v)G^{(2)}(u,v)dudv + \frac{\|p_2\|_\nu^2}{2}\int_{\mathbb{R}_+} F^{(2)}(u,u)G^{(2)}(u,u)du$, etc.

It follows from (7) *that $\mathcal{H}_{ext}^{(1)} = \mathcal{H} \equiv L^2(\mathbb{R}_+)$: by* (6) *$p_1(x) = x$ and therefore by* (2) *$\|p_1\|_\nu = 1$; and for $n \in \mathbb{N}\setminus\{1\}$ one can identify $\mathcal{H}^{\widehat{\otimes}n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (i.e., $F^{(n)}(u_1,\dots,u_n) = 0$ if there exist $k,j \in \{1,\dots,n\}$ such that $k \neq j$ but $u_k = u_j$). In this sense the space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}^{\widehat{\otimes}n}$ (this explains why we use the subscript ext in the notations $\mathcal{H}_{ext}^{(n)}$, $(\cdot,\cdot)_{ext}$ and $|\cdot|_{ext}$).*

1.3 A nonregular rigging of (L^2)

Denote by T the set of indexes $\tau = (\tau_1,\tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on $\mathbb{R}_+$ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Let $\mathcal{H}_\tau$ be the Sobolev space on $\mathbb{R}_+$ of order τ_1 weighted by the function τ_2, i.e., $\mathcal{H}_\tau$ is a completion of the set of infinite differentiable functions on $\mathbb{R}_+$ with compact supports with respect to the norm generated by the scalar product

$$(\varphi,\psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+}\Big(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1}\varphi^{[k]}(u)\psi^{[k]}(u)\Big)\tau_2(u)du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ respectively. It is well known (e.g., [4]) that $\mathcal{D} = \mathrm{pr}\lim_{\tau\in T}\mathcal{H}_\tau$ (moreover, $\mathcal{D}^{\widehat{\otimes}n} = \mathrm{pr}\lim_{\tau\in T}\mathcal{H}_\tau^{\widehat{\otimes}n}$, see, e.g., [3] for details) and for each $\tau \in T$ $\mathcal{H}_\tau$ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$, therefore one can consider the chain

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of $\mathcal{H}_\tau$ with respect to $\mathcal{H}$. Note that by the Schwartz theorem [4] $\mathcal{D}' = \mathrm{ind}\lim_{\tau\in T}\mathcal{H}_{-\tau}$ (it is convenient for us to consider $\mathcal{D}'$ as a topological space with the inductive limit topology). By analogy with [22] one can easily show that the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\widetilde{\tau}}$ with some $\widetilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\widetilde{\tau}}) = 1$. Excepting from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that *for each $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.*

Denote the norms in $\mathcal{H}_\tau$ and its tensor powers by $|\cdot|_\tau$, i.e., for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$, $n \in \mathbb{N}$, $|f^{(n)}|_\tau = \sqrt{(f^{(n)},f^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes}n}}}$.

Lemma. ([21]) *There exists $\tau' \in T$ such that for each $n \in \mathbb{N}$ the space $\mathcal{H}_{\tau'}^{\widehat{\otimes}n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$. Moreover, for all $f^{(n)} \in \mathcal{H}_{\tau'}^{\widehat{\otimes}n}$*

$$|f^{(n)}|_{ext}^2 \leq n!c^n|f^{(n)}|_{\tau'}^2,$$

where $c > 0$ is some constant.

It follows from this lemma that if for some $\tau \in T$ the space $\mathcal{H}_\tau$ is continuously embedded into the space $\mathcal{H}_{\tau'}$ then for each $n \in \mathbb{N}$ the space $\mathcal{H}_\tau^{\widehat{\otimes}n}$ is densely and continuously embedded into the space $\mathcal{H}_{ext}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$

$$|f^{(n)}|_{ext}^2 \leq n!c(\tau)^n|f^{(n)}|_\tau^2. \tag{8}$$

In what follows, it will be convenient to assume that the *indexes τ such that $\mathcal{H}_\tau$ is not continuously embedded into $\mathcal{H}_{\tau'}$, are removed from T.*

Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, f^{(n)} \in \mathcal{D}^{\widehat{\otimes}n}, N_f \in \mathbb{Z}_+\} \subset (L^2)$. Accept on default $q \in \mathbb{Z}_+$, $\tau \in T$; set $\mathcal{H}_\tau^{\widehat{\otimes}0} := \mathbb{R}$; and define scalar products $(\cdot,\cdot)_{\tau,q}$ on $\mathcal{P}_W$ by setting for

$$f = \sum_{n=0}^{N_f} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, \ g = \sum_{n=0}^{N_g} :\langle \circ^{\otimes n}, g^{(n)}\rangle : \in \mathcal{P}_W$$

$$(f,g)_{\tau,q} := \sum_{n=0}^{\min(N_f,N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes}n}}. \tag{9}$$

Let $\|\cdot\|_{\tau,q}$ be the corresponding norms, i.e., $\|f\|_{\tau,q} = \sqrt{(f,f)_{\tau,q}}$. In order to verify the well-posedness of this definition, i.e., that formula (9) defines *scalar*, and not just quasiscalar products, we note that if for $f \in \mathcal{P}_W$ $\|f\|_{\tau,q} = 0$ then by (9) for each coefficient $f^{(n)}$ of f $|f^{(n)}|_\tau = 0$ and therefore by (8) $|f^{(n)}|_{ext} = 0$. So, in this case $f = 0$ in (L^2).

Definition. *We define Kondratiev spaces of nonregular test functions $(\mathcal{H}_\tau)_q$ as completions of $\mathcal{P}_W$ with respect to the norms $\|\cdot\|_{\tau,q}$, and set*

$$(\mathcal{H}_\tau) := \underset{q\in\mathbb{Z}_+}{\mathrm{pr\,lim}}(\mathcal{H}_\tau)_q, \quad (\mathcal{D}) := \underset{q\in\mathbb{Z}_+,\tau\in T}{\mathrm{pr\,lim}}(\mathcal{H}_\tau)_q.$$

As is easy to see, $f \in (\mathcal{H}_\tau)_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f^{(n)}\rangle :, \ f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n} \tag{10}$$

(the series converges in $(\mathcal{H}_\tau)_q$), with

$$\|f\|_{\tau,q}^2 := \|f\|_{(\mathcal{H}_\tau)_q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_\tau^2 < \infty; \tag{11}$$

and for $f, g \in (\mathcal{H}_\tau)_q$

$$(f,g)_{(\mathcal{H}_\tau)_q} = \sum_{n=0}^{\infty} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_\tau^{\widehat{\otimes}n}},$$

where $f^{(n)}, g^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$ are the kernels from decompositions (10) for f and g respectively (since for each $n \in \mathbb{Z}_+$ $\mathcal{H}_\tau^{\widehat{\otimes}n} \subseteq \mathcal{H}_{ext}^{(n)}$, for $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$ $:\langle \circ^{\otimes n}, f^{(n)}\rangle :$ is a well defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_\tau)$ ($f \in (\mathcal{D})$) if and only if f can be presented in form (10) and norm (11) is finite for each $q \in \mathbb{Z}_+$ (for each $q \in \mathbb{Z}_+$ and each $\tau \in T$).

Proposition. ([21]) *For each $\tau \in T$ there exists $q_0 = q_0(\tau) \in \mathbb{Z}_+$ such that for each $q \in \mathbb{N}_{q_0} := \{q_0, q_0+1, \cdots\}$ the space $(\mathcal{H}_\tau)_q$ is densely and continuously embedded into (L^2).*

In view of this proposition for $\tau \in T$ and $q \geq q_0(\tau)$ one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_\tau)_q \supset (\mathcal{H}_\tau) \supset (\mathcal{D}), \tag{12}$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \operatorname{ind\,lim}_{q\to\infty}(\mathcal{H}_{-\tau})_{-q}$ and $(\mathcal{D}') = \operatorname{ind\,lim}_{q\to\infty,\tau\in T}(\mathcal{H}_{-\tau})_{-q}$ are the spaces dual of $(\mathcal{H}_\tau)_q$, $(\mathcal{H}_\tau)$ and $(\mathcal{D})$ with respect to (L^2).

Definition. *Chain* (12) *is called a nonregular rigging of the space (L^2). The negative spaces of this chain $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$ and $(\mathcal{D}')$ are called Kondratiev spaces of nonregular generalized functions.*

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}'^{(m)} \supset \mathcal{H}_{-\tau}^{(m)} \supset \mathcal{H}_{ext}^{(m)} \supset \mathcal{H}_\tau^{\widehat{\otimes} m} \supset \mathcal{D}^{\widehat{\otimes} m}, \tag{13}$$

$m \in \mathbb{Z}_+$ (for $m = 0$ $\mathcal{D}^{\widehat{\otimes} 0} = \mathcal{H}_\tau^{\widehat{\otimes} 0} = \mathcal{H}_{ext}^{(0)} = \mathcal{H}_{-\tau}^{(0)} = \mathcal{D}'^{(0)} = \mathbb{R}$), where $\mathcal{H}_{-\tau}^{(m)}$ and $\mathcal{D}'^{(m)} = \operatorname{ind\,lim}_{\tau\in T}\mathcal{H}_{-\tau}^{(m)}$ are the spaces dual of $\mathcal{H}_\tau^{\widehat{\otimes} m}$ and $\mathcal{D}^{\widehat{\otimes} m}$ with respect to $\mathcal{H}_{ext}^{(m)}$. The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [22]).

Proposition. ([21]) *There exists a system of generalized functions*

$$\left\{ :\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle: \in (\mathcal{H}_{-\tau})_{-q} \mid F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)},\ m \in \mathbb{Z}_+ \right\}$$

such that

1) for $F_{ext}^{(m)} \in \mathcal{H}_{ext}^{(m)} \subset \mathcal{H}_{-\tau}^{(m)}$ $:\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle:$ is a Wick monomial that was defined in Subsection 1.2;

2) any generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ can be presented as a series

$$F = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle:,\ F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}, \tag{14}$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$, i.e.,

$$\|F\|^2_{-\tau,-q} := \|F\|^2_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm}|F_{ext}^{(m)}|^2_{\mathcal{H}_{-\tau}^{(m)}} < \infty; \tag{15}$$

and, vice versa, any series (14) *with finite norm* (15) *is a generalized function from $(\mathcal{H}_{-\tau})_{-q}$ (i.e., such a series converges in $(\mathcal{H}_{-\tau})_{-q}$);*

3) for $F, G \in (\mathcal{H}_{-\tau})_{-q}$ the scalar product has a form

$$(F, G)_{(\mathcal{H}_{-\tau})_{-q}} = \sum_{m=0}^{\infty} 2^{-qm}(F_{ext}^{(m)}, G_{ext}^{(m)})_{\mathcal{H}_{-\tau}^{(m)}},$$

where $F_{ext}^{(m)}, G_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$ are the kernels from decompositions (14) *for F and G respectively;*

4) the dual pairing between $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_\tau)_q$ that is generated by the scalar product in (L^2), has the form

$$\langle\langle F, f\rangle\rangle_{(L^2)} = \sum_{m=0}^{\infty} m!\langle F_{ext}^{(m)}, f^{(m)}\rangle_{ext}, \tag{16}$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$ and $f^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m}$ are the kernels from decompositions (14) *and* (10) *for F and f respectively, $\langle\cdot,\cdot\rangle_{ext}$ denotes the dual pairings between elements of negative and positive spaces from chains* (13)*, these pairings are generated by the scalar products in $\mathcal{H}_{ext}^{(m)}$.*

It is clear that $F \in (\mathcal{H}_{-\tau})$ ($F \in (\mathcal{D}')$) if and only if F can be presented in form (14) and norm (15) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

1.4 Stochastic derivatives and integrals

First, following [24], we recall the notion of the Hida stochastic derivative on the spaces of nonregular test functions, and of the extended stochastic integral on the spaces of nonregular generalized functions. Decomposition (5) for elements of (L^2) defines an isometric isomorphism (a generalized Wiener-Itô-Sigal isomorphism) $\mathbf{I} : (L^2) \to \bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$, where $\bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$ is a weighted extended Fock space (cf. [26]): for $F \in (L^2)$ of form (5) $\mathbf{I}F = (F^{(0)}, F^{(1)}, \ldots, F^{(n)}, \ldots) \in \bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}$. Let $\mathbf{1} : \mathcal{H} \to \mathcal{H}$ be the identity operator. Then the operator $\mathbf{I} \otimes \mathbf{1} : (L^2) \otimes \mathcal{H} \to (\bigoplus_{n=0}^{\infty} n!\mathcal{H}_{ext}^{(n)}) \otimes \mathcal{H} \cong \bigoplus_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H})$ is an isometric isomorphism between the spaces $(L^2) \otimes \mathcal{H}$ and $\bigoplus_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H})$. It is clear that for arbitrary $n \in \mathbb{Z}_+$ and $F_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ a vector $(\underbrace{0, \ldots, 0}_{n}, F_\cdot^{(n)}, 0, \ldots)$ belongs to $\bigoplus_{n=0}^{\infty} n!(\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H})$. Set

$$:\langle \circ^{\otimes n}, F_\cdot^{(n)} \rangle: \stackrel{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1}(\underbrace{0, \ldots, 0}_{n}, F_\cdot^{(n)}, 0, \ldots) \in (L^2) \otimes \mathcal{H}. \tag{17}$$

By the construction elements $:\langle \circ^{\otimes n}, F_\cdot^{(n)} \rangle:$, $n \in \mathbb{Z}_+$, form an orthogonal basis in the space $(L^2) \otimes \mathcal{H}$ in the sense that any $F \in (L^2) \otimes \mathcal{H}$ can be presented as

$$F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F_\cdot^{(n)} \rangle:, \quad F_\cdot^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$$

(the series converges in $(L^2) \otimes \mathcal{H}$), with $\|F\|^2_{(L^2)\otimes\mathcal{H}} = \sum_{n=0}^{\infty} n!|F_\cdot^{(n)}|^2_{\mathcal{H}_{ext}^{(n)}\otimes\mathcal{H}} < \infty$. Since, as is easily seen, the restriction of the generalized Wiener-Itô-Sigal isomorphism $\mathbf{I}$ to the space $(\mathcal{H}_\tau)_q$ is an isometric isomorphism between $(\mathcal{H}_\tau)_q$ and a weighted Fock space $\bigoplus_{n=0}^{\infty} (n!)^2 2^{qn} \mathcal{H}_\tau^{\widehat{\otimes} n}$ (cf. [26]), and, of course, the restriction of the identity operator on $\mathcal{H}$ to the space $\mathcal{H}_\tau$ is the identity operator on $\mathcal{H}_\tau$, for arbitrary $n \in \mathbb{Z}_+$ and $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau \subset \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}$ we have $:\langle \circ^{\otimes n}, f_\cdot^{(n)} \rangle: \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$. Moreover, elements $:\langle \circ^{\otimes n}, f_\cdot^{(n)} \rangle:$, $f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau$, $n \in \mathbb{Z}_+$, form orthogonal bases (in the above-described sense) in the spaces $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$.

Definition. *For $g \in (\mathcal{H}_\tau)_q$ we define a Hida stochastic derivative $\partial_\cdot g \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ by the formula*

$$\partial_\cdot g := \sum_{n=0}^{\infty} (n+1) :\langle \circ^{\otimes n}, g^{(n+1)}(\cdot) \rangle:, \tag{18}$$

where $g^{(n+1)} \in \mathcal{H}_\tau^{\widehat{\otimes} n+1}$, $n \in \mathbb{Z}_+$, are the kernels from decomposition (10) *for g considered as elements of $\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau$.*

Since (see (11))

$$\begin{aligned} \|\partial_\cdot g\|^2_{(\mathcal{H}_\tau)_q\otimes\mathcal{H}_\tau} &= \sum_{n=0}^{\infty} ((n+1)!)^2 2^{qn} |g^{(n+1)}(\cdot)|^2_{\mathcal{H}_\tau^{\widehat{\otimes} n}\otimes\mathcal{H}_\tau} \\ &= 2^{-q} \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |g^{(n+1)}|^2_\tau \le 2^{-q} \|g\|^2_{\tau,q}, \end{aligned} \tag{19}$$

this definition is well posed and, moreover, the Hida stochastic derivative

$$\partial_\cdot : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \tag{20}$$

is a linear *continuous* operator. It is shown in [24] that this derivative is (generated by) the restriction to $(\mathcal{H}_\tau)_q$ of the Hida stochastic derivative on (L^2). We note also that the restrictions of derivative (20) to $(\mathcal{H}_\tau)$ and to $(\mathcal{D})$ generate linear continuous operators $\partial_\cdot : (\mathcal{H}_\tau) \to (\mathcal{H}_\tau) \otimes \mathcal{H}_\tau := \mathrm{pr}\lim_{q\in\mathbb{Z}_+}(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ and $\partial_\cdot : (\mathcal{D}) \to (\mathcal{D}) \otimes \mathcal{D} := \mathrm{pr}\lim_{q\in\mathbb{Z}_+,\tau\in T}(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ respectively.

Definition. *We define an extended stochastic integral*

$$\int \circ(u)\widehat{d}L_u : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})_{-q} \tag{21}$$

as a linear continuous operator adjoint to Hida stochastic derivative (20)*: for* $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$

$$\int F(u)\widehat{d}L_u := \partial_\cdot^* F \in (\mathcal{H}_{-\tau})_{-q}, \tag{22}$$

i.e., for arbitrary $g \in (\mathcal{H}_\tau)_q$ $\langle\langle \int F(u)\widehat{d}L_u, g\rangle\rangle_{(L^2)} = \langle\langle F(\cdot), \partial_\cdot g\rangle\rangle_{(L^2)\otimes\mathcal{H}}$.

It is shown in [24] that integral (21) is an extension of the extended Skorohod stochastic integral on $(L^2) \otimes \mathcal{H}$.

By analogy one can define linear continuous operators $\int \circ(u)\widehat{d}L_u : (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})$ and $\int \circ(u)\widehat{d}L_u : (\mathcal{D}') \otimes \mathcal{D}' \to (\mathcal{D}')$, where $(\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau} := \mathrm{ind}\lim_{q\to\infty}(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$, $(\mathcal{D}') \otimes \mathcal{D}' := \mathrm{ind}\lim_{q\to\infty,\tau\in T}(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$.

In contrast to formula (18) for the Hida stochastic derivative, formula (22) for integral (21) is inconvenient for calculations. Therefore let us write out a representation for this integral in terms of orthogonal bases in the spaces of nonregular generalized functions.

First we note that, as in the case of the spaces $(\mathcal{H}_{-\tau})_{-q}$, it follows from the general duality theory that there exists a system of orthogonal in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ generalized functions $\{:\langle\circ^{\otimes m}, F_{ext,\cdot}^{(m)}\rangle: \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \mid F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau},\ m \in \mathbb{Z}_+\}$ such that for $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H} \subset \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$ $:\langle\circ^{\otimes m}, F_{ext,\cdot}^{(m)}\rangle:$ is given by (17); and any generalized function $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ can be presented as a convergent in $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ series

$$F(\cdot) = \sum_{m=0}^{\infty} :\langle\circ^{\otimes m}, F_{ext,\cdot}^{(m)}\rangle:,\ F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}, \tag{23}$$

now

$$\|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau}} = \sum_{m=0}^{\infty} 2^{-qm}|F_{ext,\cdot}^{(m)}|^2_{\mathcal{H}_{-\tau}^{(m)}\otimes\mathcal{H}_{-\tau}} < \infty. \tag{24}$$

Consider a family of chains

$$\mathcal{D}'^{\widehat{\otimes} m} \supset \mathcal{H}_{-\tau}^{\widehat{\otimes} m} \supset \mathcal{H}^{\widehat{\otimes} m} \supset \mathcal{H}_{\tau}^{\widehat{\otimes} m} \supset \mathcal{D}^{\widehat{\otimes} m},\ m \in \mathbb{Z}_+ \tag{25}$$

(as is well known (e.g., [4]), $\mathcal{H}_{-\tau}^{\widehat{\otimes} m}$ and $\mathcal{D}'^{\widehat{\otimes} m} = \mathrm{ind}\lim_{\tau\in T}\mathcal{H}_{-\tau}^{\widehat{\otimes} m}$ are the spaces dual of $\mathcal{H}_{\tau}^{\widehat{\otimes} m}$ and $\mathcal{D}^{\widehat{\otimes} m}$ respectively; in the case $m = 0$ all spaces from chain (25) are equal to $\mathbb{R}$). Since the spaces of test functions in chains (25) and (13) coincide, there exists a family of natural isomorphisms

$$U_m : \mathcal{D}'^{(m)} \to \mathcal{D}'^{\widehat{\otimes} m},\ m \in \mathbb{Z}_+,$$

such that for all $F_{ext}^{(m)} \in \mathcal{D}'^{(m)}$ and $f^{(m)} \in \mathcal{D}^{\widehat{\otimes} m}$

$$\langle F_{ext}^{(m)}, f^{(m)} \rangle_{ext} = \langle U_m F_{ext}^{(m)}, f^{(m)} \rangle. \tag{26}$$

It is easy to see that the restrictions of U_m to $\mathcal{H}_{-\tau}^{(m)}$ are isometric isomorphisms between the spaces $\mathcal{H}_{-\tau}^{(m)}$ and $\mathcal{H}_{-\tau}^{\widehat{\otimes} m}$.

Remark. *As we saw above, $\mathcal{H}_{ext}^{(1)} = \mathcal{H}$, and therefore in the case $m = 1$ chains* (25) *and* (13) *coincide. Thus $U_1 = \mathbf{1}$ is the identity operator on $\mathcal{D}'^{(1)} = \mathcal{D}'$. In the case $m = 0$ U_0 is, obviously, the identity operator on $\mathbb{R}$.*

Proposition. ([24]) *Let $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$. The extended stochastic integral can be presented in the form*

$$\int F(u)\widehat{d}L_u = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m+1}, \widehat{F}_{ext}^{(m)} \rangle :, \tag{27}$$

where

$$\widehat{F}_{ext}^{(m)} := U_{m+1}^{-1}\{Pr[(U_m \otimes \mathbf{1})F_{ext,\cdot}^{(m)}]\} \in \mathcal{H}_{-\tau}^{(m+1)}, \tag{28}$$

Pr is the symmetrization operator (more exactly, the orthoprojector acting for each $m \in \mathbb{Z}_+$ from $\mathcal{H}_{-\tau}^{\widehat{\otimes} m} \otimes \mathcal{H}_{-\tau}$ to $\mathcal{H}_{-\tau}^{\widehat{\otimes} m+1}$), $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$, $m \in \mathbb{Z}_+$, are the kernels from decomposition (23) *for F.*

Remark. *Sometimes it can be convenient to introduce the Hida stochastic derivative and the extended stochastic integral as linear continuous operators acting from $(\mathcal{H}_\tau)_q$ to $(\mathcal{H}_\tau)_q \otimes \mathcal{H}$ and from $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}$ to $(\mathcal{H}_{-\tau})_{-q}$ respectively, this case is described in detail in [21].*

Unfortunately, in contrast to the Hida stochastic derivative, the extended stochastic integral with respect to a Lévy process cannot be naturally restricted to the spaces of nonregular test functions. More precisely, for $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ $\int f(u)\widehat{d}L_u$ not necessary a nonregular test function (one can show that for $\tau \in T$ and $q \in \mathbb{Z}_+$ such that $q > \log_2 c(\tau)$, where $c(\tau) > 0$ from estimate (8), if $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ then $\int f(u)\widehat{d}L_u \in (L^2)$; and for q sufficiently large this integral is a *regular* test function [21]). Nevertheless, one can introduce on each space of nonregular test functions a linear operator that has properties quite analogous to the properties of the extended stochastic integral. Now we'll introduce such operators (which will be called *generalized stochastic integrals*) and consider them in detail.

Let $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$. Using the above-described orthogonal basis in this space, we can write

$$f(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_\cdot^{(n)} \rangle :, \; f_\cdot^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau \tag{29}$$

(the series converges in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$), in this case

$$\|f\|_{(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f_\cdot^{(n)}|_{\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau}^2 < \infty. \tag{30}$$

Definition. *We define a generalized stochastic integral*

$$\mathbb{I} : (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau \to (\mathcal{H}_\tau)_q \tag{31}$$

as a linear continuous operator given for $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau$ by the formula

$$\mathbb{I}(f) := \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \hat{f}^{(n)} \rangle: \tag{32}$$

(cf. (27)*), where $\hat{f}^{(n)} := Pr f_{\cdot}^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n+1}$ are the orthoprojections onto $\mathcal{H}_\tau^{\widehat{\otimes} n+1}$ (the symmetrizations by all variables) of the kernels $f_{\cdot}^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau$ from decomposition* (29) *for f.*

Since (see (11), (32) and (30))

$$\begin{aligned}\|\mathbb{I}(f)\|_{\tau,q}^2 &= \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |\hat{f}^{(n)}|_\tau^2 \leq 2^q \sum_{n=0}^{\infty} (n!)^2 2^{(q+1)n} [(n+1)^2 2^{-n}] |f_{\cdot}^{(n)}|^2_{\mathcal{H}_\tau^{\widehat{\otimes} n} \otimes \mathcal{H}_\tau} \\ &\leq 9 \cdot 2^{q-2} \|f\|^2_{(\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau},\end{aligned}$$

this definition is well posed. It is clear that the restriction of the operator $\mathbb{I}$ to the space $(\mathcal{H}_\tau) \otimes \mathcal{H}_\tau$ (respectively to the space $(\mathcal{D}) \otimes \mathcal{D}$) is a linear continuous operator acting from $(\mathcal{H}_\tau) \otimes \mathcal{H}_\tau$ to $(\mathcal{H}_\tau)$ (respectively from $(\mathcal{D}) \otimes \mathcal{D}$ to $(\mathcal{D})$).

The Hida stochastic derivative, in turn, has no a natural extension to the spaces of nonregular generalized functions (the kernels from decompositions (14) for elements of $(\mathcal{H}_{-\tau})_{-q}$ belong to the spaces $\mathcal{H}_{-\tau}^{(m)}$, $m \in \mathbb{Z}_+$, and for elements of these spaces it is impossible "to separate a variable"). Nevertheless, one can define a natural analog of this derivative (a *generalized Hida derivative*) on each of the above-mentioned spaces as an operator adjoint to $\mathbb{I}$.

Definition. *We define a generalized Hida derivative*

$$\widetilde{\partial}_{\cdot} : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau} \tag{33}$$

as a linear continuous operator adjoint to generalized stochastic integral (31) *($\widetilde{\partial}_{\cdot} := \mathbb{I}^*$), i.e., for all $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau$*

$$\langle\langle \widetilde{\partial}_{\cdot} F, f(\cdot) \rangle\rangle_{(L^2) \otimes \mathcal{H}} = \langle\langle F, \mathbb{I}(f) \rangle\rangle_{(L^2)}. \tag{34}$$

By analogy one can define linear continuous operators $\widetilde{\partial}_{\cdot} : (\mathcal{H}_{-\tau}) \to (\mathcal{H}_{-\tau}) \otimes \mathcal{H}_{-\tau}$ and $\widetilde{\partial}_{\cdot} : (\mathcal{D}') \to (\mathcal{D}') \otimes \mathcal{D}'$. We note also that since operators (33) and (31) are continuous, $\widetilde{\partial}_{\cdot}^* = \mathbb{I}^{**} = \mathbb{I}$ and $\widetilde{\partial}_{\cdot}^{**} = \mathbb{I}^* = \widetilde{\partial}_{\cdot}$.

In order to make calculations with derivative (33), let us obtain a representation for this operator in terms of orthogonal bases in the spaces of nonregular generalized functions.

Proposition. *Let $F \in (\mathcal{H}_{-\tau})_{-q}$. Then*

$$\widetilde{\partial}_{\cdot} F = \sum_{m=0}^{\infty} (m+1) :\langle \circ^{\otimes m}, F_{ext}^{(m+1)}(\cdot) \rangle: \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau} \tag{35}$$

(cf. (18)*), where*

$$F_{ext}^{(m+1)}(\cdot) := (U_m^{-1} \otimes \mathbf{1})(U_{m+1} F_{ext}^{(m+1)})(\cdot) \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}, \tag{36}$$

here $F_{ext}^{(m+1)} \in \mathcal{H}_{-\tau}^{(m+1)}$ are the kernels from decomposition (14) *for F.*

Proof. Using (34), (14), (32), (16), (26), (36) and (29), for $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_\tau)_{q+1} \otimes \mathcal{H}_\tau$ we obtain

$$\begin{aligned}
\langle\langle \widetilde{\partial}.F, f\rangle\rangle_{(L^2)\otimes\mathcal{H}} &= \langle\langle F, \mathbb{I}(f)\rangle\rangle_{(L^2)} = \langle\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, F_{ext}^{(m)}\rangle :, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n+1}, \hat{f}^{(n)}\rangle :\rangle\rangle_{(L^2)} \\
&= \sum_{m=0}^{\infty} (m+1)! \langle F_{ext}^{(m+1)}, \hat{f}^{(m)}\rangle_{\mathcal{H}_{ext}^{(m+1)}} = \sum_{m=0}^{\infty} (m+1)! \langle U_{m+1} F_{ext}^{(m+1)}, Prf_{\cdot}^{(m)}\rangle_{\mathcal{H}^{\widehat{\otimes} m+1}} \\
&= \sum_{m=0}^{\infty} (m+1)! \langle (U_{m+1} F_{ext}^{(m+1)})(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}^{\widehat{\otimes} m}\otimes\mathcal{H}} \\
&= \sum_{m=0}^{\infty} m!(m+1) \langle (U_m^{-1} \otimes \mathbf{1})(U_{m+1} F_{ext}^{(m+1)})(\cdot), f_{\cdot}^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} \\
&= \langle\langle \sum_{m=0}^{\infty} (m+1) :\langle \circ^{\otimes m}, (U_m^{-1} \otimes \mathbf{1})(U_{m+1} F_{ext}^{(m+1)})(\cdot)\rangle :, \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, f_{\cdot}^{(n)}\rangle :\rangle\rangle_{(L^2)\otimes\mathcal{H}} \\
&= \langle\langle \sum_{m=0}^{\infty} (m+1) :\langle \circ^{\otimes m}, F_{ext}^{(m+1)}(\cdot)\rangle :, f\rangle\rangle_{(L^2)\otimes\mathcal{H}},
\end{aligned} \tag{37}$$

whence the result follows. □

Sometimes it can be necessary to define a generalized stochastic integral by formula (32) as a linear unbounded operator

$$\mathbb{I} : (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \to (\mathcal{H}_\tau)_q \tag{38}$$

with the domain

$$\mathrm{dom}(\mathbb{I}) := \{f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau : \|\mathbb{I}(f)\|_{\tau,q}^2 = \sum_{n=0}^{\infty} ((n+1)!)^2 2^{q(n+1)} |\hat{f}^{(n)}|_\tau^2 < \infty\}. \tag{39}$$

Since set (39) is dense in $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$, one can define now a corresponding generalized Hida derivative as an unbounded operator adjoint to operator (38):

$$\widetilde{\partial}. := \mathbb{I}^* : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}. \tag{40}$$

The domain of operator (40) by definition consists of $F \in (\mathcal{H}_{-\tau})_{-q}$ such that $(\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \supset \mathrm{dom}(\mathbb{I}) \ni f \mapsto \langle\langle F, \mathbb{I}(f)\rangle\rangle_{(L^2)}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $H \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ such that $\langle\langle F, \mathbb{I}(f)\rangle\rangle_{(L^2)} = \langle\langle H, f\rangle\rangle_{(L^2)\otimes\mathcal{H}}$. But by definition of $\widetilde{\partial}.$ we have $H = \widetilde{\partial}.F$ and therefore the domain of operator (40) can be described by the condition $\widetilde{\partial}.F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$. Since for $f \in \mathrm{dom}(\mathbb{I})$ and $F \in \mathrm{dom}(\widetilde{\partial}.)$ calculation (37) is, obviously, valid, $\widetilde{\partial}.F$ has form (35). So, the domain of operator (40) can be described as follows:

$$\begin{aligned}
\mathrm{dom}(\widetilde{\partial}.) = \{F \in (\mathcal{H}_{-\tau})_{-q} : \|\widetilde{\partial}.F\|_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau}}^2 &= \sum_{m=0}^{\infty} 2^{-qm} (m+1)^2 |F_{ext}^{(m+1)}(\cdot)|_{\mathcal{H}_{-\tau}^{(m)}\otimes\mathcal{H}_{-\tau}}^2 \\
&= \sum_{m=0}^{\infty} 2^{-qm} (m+1)^2 |F_{ext}^{(m+1)}|_{\mathcal{H}_{-\tau}^{(m+1)}}^2 < \infty\}
\end{aligned} \tag{41}$$

(see (36)).

Proposition. *Generalized stochastic integral* (38) *and generalized Hida derivative* (40) *are mutually adjoint and, in particular, closed operators.*

Proof. Since set (41) is dense in $(\mathcal{H}_{-\tau})_{-q}$, the operator $\widetilde{\partial}^*_{\cdot} = \mathbb{I}^{**} : (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau \to (\mathcal{H}_\tau)_q$ is well defined as a linear unbounded operator with the domain that consists of $f \in (\mathcal{H}_\tau)_q \otimes \mathcal{H}_\tau$ such that $(\mathcal{H}_{-\tau})_{-q} \supset \mathrm{dom}(\widetilde{\partial}_{\cdot}) \ni F \mapsto \langle\!\langle \widetilde{\partial}_{\cdot} F, f \rangle\!\rangle_{(L^2)\otimes\mathcal{H}}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $h \in (\mathcal{H}_\tau)_q$ such that $\langle\!\langle \widetilde{\partial}_{\cdot} F, f \rangle\!\rangle_{(L^2)\otimes\mathcal{H}} = \langle\!\langle F, h \rangle\!\rangle_{(L^2)}$. But by (40) $h = \mathbb{I}(f)$ and therefore the domain of $\widetilde{\partial}^*_{\cdot}$ can be described by the condition $\mathbb{I}(f) \in (\mathcal{H}_\tau)_q$. Compareing this condition with (39) one can conclude that $\mathrm{dom}(\widetilde{\partial}^*_{\cdot}) = \mathrm{dom}(\mathbb{I})$, therefore $\widetilde{\partial}^*_{\cdot} = \mathbb{I}^{**} = \mathbb{I}$. The equality $\mathbb{I}^* = \widetilde{\partial}_{\cdot}$ is a definition of $\widetilde{\partial}_{\cdot}$. □

2 OPERATORS OF STOCHASTIC DIFFERENTIATION

2.1 The case of bounded operators

As we said above, just as the Hida stochastic derivative, the operators of stochastic differentiation on (L^2) [8, 9] cannot be naturally continued to the spaces of nonregular generalized functions (because the kernels from decompositions (14) for elements of $(\mathcal{H}_{-\tau})_{-q}$ belong to too wide spaces). Nevertheless, one can introduce on these spaces natural analogs of the above-mentioned operators. These analogs have properties similar to the properties of operators of stochastic differentiation, and can be accepted as operators of stochastic differentiation on the spaces of nonregular generalized functions. In order to give an exact definition of the just now mentioned operators, we need a preparation.

Let $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$, $n, m \in \mathbb{N}$, $m > n$. We define a generalized partial pairing $\langle F_{ext}^{(m)}, f^{(n)} \rangle_{ext} \in \mathcal{H}_{-\tau}^{(m-n)}$ by setting for any $g^{(m-n)} \in \mathcal{H}_\tau^{\widehat{\otimes}m-n}$

$$\langle \langle F_{ext}^{(m)}, f^{(n)} \rangle_{ext}, g^{(m-n)} \rangle_{ext} = \langle F_{ext}^{(m)}, f^{(n)} \widehat{\otimes} g^{(m-n)} \rangle_{ext}. \tag{42}$$

Since by the generalized Cauchy-Bunyakovsky inequality

$$|\langle F_{ext}^{(m)}, f^{(n)} \widehat{\otimes} g^{(m-n)} \rangle_{ext}| \le |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)}} |f^{(n)} \widehat{\otimes} g^{(m-n)}|_\tau \le |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)}} |f^{(n)}|_\tau |g^{(m-n)}|_\tau,$$

this definition is well posed and

$$|\langle F_{ext}^{(m)}, f^{(n)} \rangle_{ext}|_{\mathcal{H}_{-\tau}^{(m-n)}} \le |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)}} |f^{(n)}|_\tau. \tag{43}$$

Definition. *Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes}n}$. We define (the analog of) the operator of stochastic differentiation*

$$(\widetilde{D}^n \circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q-1}$$

as a linear continuous operator that is given by the formula

$$\begin{aligned}(\widetilde{D}^n F)(f^{(n)}) :&= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} {:}\langle \circ^{\otimes m-n}, \langle F_{ext}^{(m)}, f^{(n)} \rangle_{ext} \rangle{:} \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} {:}\langle \circ^{\otimes m}, \langle F_{ext}^{(m+n)}, f^{(n)} \rangle_{ext} \rangle{:},\end{aligned} \tag{44}$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$ are the kernels from decomposition (14) *for $F \in (\mathcal{H}_{-\tau})_{-q}$.*

Since (see (15), (44) and (43))

$$\|(\widetilde{D}^n F)(f^{(n)})\|^2_{-\tau,-q-1} = \sum_{m=0}^{\infty} 2^{(-q-1)m} \frac{((m+n)!)^2}{(m!)^2} |\langle F^{(m+n)}_{ext}, f^{(n)}\rangle_{ext}|^2_{\mathcal{H}^{(m)}_{-\tau}}$$
$$\leq |f^{(n)}|^2_\tau 2^{qn} \sum_{m=0}^{\infty} 2^{-q(m+n)} |F^{(m+n)}_{ext}|^2_{\mathcal{H}^{(m+n)}_{-\tau}} \Big[2^{-m}\frac{((m+n)!)^2}{(m!)^2}\Big] \leq |f^{(n)}|^2_\tau 2^{qn} C(n) \|F\|^2_{-\tau,-q},$$

where $C(n) := \max_{m\in\mathbb{Z}_+} [2^{-m}\frac{((m+n)!)^2}{(m!)^2}] \leq \max_{m\in\mathbb{Z}_+}[2^{-m}(m+n)^{2n}] < \infty$, this definition is well posed. It is clear that the operator $(\widetilde{D}^n\circ)(f^{(n)})$ can be naturally continued to a linear continuous operator on the space $(\mathcal{H}_{-\tau})$ (or $(\mathcal{D}')$).

Let us consider main properties of the operator $\widetilde{D}^n$.

Theorem. *1) For $k_1,\dots,k_m \in \mathbb{N}$, $f_j^{(k_j)} \in \mathcal{H}_\tau^{\widehat{\otimes} k_j}$, $j \in \{1,\dots,m\}$,*

$$(\widetilde{D}^{k_m}(\cdots(\widetilde{D}^{k_2}((\widetilde{D}^{k_1}\circ)(f_1^{(k_1)})))(f_2^{(k_2)})\cdots))(f_m^{(k_m)}) = (\widetilde{D}^{k_1+\cdots+k_m}\circ)(f_1^{(k_1)}\widehat{\otimes}\cdots\widehat{\otimes} f_m^{(k_m)}).$$

2) For each $F \in (\mathcal{H}_{-\tau})_{-q}$ the kernels $F^{(n)}_{ext} \in \mathcal{H}^{(n)}_{-\tau}$, $n \in \mathbb{N}$, from decomposition (14) *can be presented in the form*

$$F^{(n)}_{ext} = \frac{1}{n!}\mathbb{E}(\widetilde{D}^n F),$$

i.e., for each $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ $\langle F^{(n)}_{ext}, f^{(n)}\rangle_{ext} = \frac{1}{n!}\mathbb{E}((\widetilde{D}^n F)(f^{(n)}))$, here $\mathbb{E}\circ := \langle\!\langle \circ, 1\rangle\!\rangle_{(L^2)}$ is a generalized expectation.

3) The adjoint to $\widetilde{D}^n$ operator has the form

$$(\widetilde{D}^n g)(f^{(n)})^* = \sum_{m=0}^{\infty} :\langle \circ^{m+n}, f^{(n)}\widehat{\otimes} g^{(m)}\rangle: \in (\mathcal{H}_\tau)_q, \tag{45}$$

where $g \in (\mathcal{H}_\tau)_{q+1}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, and $g^{(m)} \in \mathcal{H}_\tau^{\widehat{\otimes} m}$ are the kernels from decomposition (10) *for g.*

Proof. 1) The proof consists in the application of the mathematical induction method.

2) Using (44) and (16) we obtain

$$\mathbb{E}((\widetilde{D}^n F)(f^{(n)})) = \langle\!\langle (\widetilde{D}^n F)(f^{(n)}), 1\rangle\!\rangle_{(L^2)} = n!\langle F^{(n)}_{ext}, f^{(n)}\rangle_{ext}.$$

3) Since (see (11), (10))

$$\|\sum_{m=0}^{\infty} :\langle \circ^{m+n}, f^{(n)}\widehat{\otimes} g^{(m)}\rangle:\|^2_{\tau,q} = \sum_{m=0}^{\infty} ((m+n)!)^2 2^{q(m+n)} |f^{(n)}\widehat{\otimes} g^{(m)}|^2_\tau$$
$$\leq |f^{(n)}|^2_\tau 2^{qn} \sum_{m=0}^{\infty} (m!)^2 2^{(q+1)m} |g^{(m)}|^2_\tau \Big[2^{-m}\frac{((m+n)!)^2}{(m!)^2}\Big] \leq |f^{(n)}|^2_\tau 2^{qn} C(n) \|g\|^2_{\tau,q+1} < \infty$$

(here $C(n) = \max_{m\in\mathbb{Z}_+}[2^{-m}\frac{((m+n)!)^2}{(m!)^2}]$ as above), the right hand side of (45) is well defined as an element of $(\mathcal{H}_\tau)_q$. Further, using (44), (10), (16) and (42), for $F \in (\mathcal{H}_{-\tau})_{-q}$ of form (14) we obtain

$$\begin{aligned}
&\langle\!\langle F,(\widetilde{D}^n g)(f^{(n)})^*\rangle\!\rangle_{(L^2)} = \langle\!\langle(\widetilde{D}^n F)(f^{(n)}),g\rangle\!\rangle_{(L^2)}\\
&= \langle\!\langle \sum_{m=0}^{\infty}\frac{(m+n)!}{m!}:\langle\circ^{\otimes m},\langle F_{ext}^{(m+n)},f^{(n)}\rangle_{ext}\rangle:,\sum_{k=0}^{\infty}:\langle\circ^{\otimes k},g^{(k)}\rangle:\rangle\!\rangle_{(L^2)}\\
&= \sum_{m=0}^{\infty}(m+n)!\langle\langle F_{ext}^{(m+n)},f^{(n)}\rangle_{ext},g^{(m)}\rangle_{ext} = \sum_{m=0}^{\infty}(m+n)!\langle F_{ext}^{(m+n)},f^{(n)}\widehat{\otimes}g^{(m)}\rangle_{ext}\\
&= \langle\!\langle \sum_{k=0}^{\infty}:\langle\circ^{\otimes k},F_{ext}^{(k)}\rangle:,\sum_{m=0}^{\infty}:\langle\circ^{\otimes m+n},f^{(n)}\widehat{\otimes}g^{(m)}\rangle:\rangle\!\rangle_{(L^2)}\\
&= \langle\!\langle F,\sum_{m=0}^{\infty}:\langle\circ^{\otimes m+n},f^{(n)}\widehat{\otimes}g^{(m)}\rangle:\rangle\!\rangle_{(L^2)},
\end{aligned} \tag{46}$$

whence the result follows. □

Now we consider in more detail the case $n=1$. Denote $\widetilde{D} := \widetilde{D}^1$.

Theorem. *1) For all $g \in (\mathcal{H}_\tau)_{q+1}$ and $f^{(1)} \in \mathcal{H}_\tau$*

$$(\widetilde{D}g)(f^{(1)})^* = \mathbb{I}(g\otimes f^{(1)}) \in (\mathcal{H}_\tau)_q. \tag{47}$$

2) For all $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f^{(1)} \in \mathcal{H}_\tau$

$$(\widetilde{D}F)(f^{(1)}) = \langle\widetilde{\partial}.F,f^{(1)}(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q-1}, \tag{48}$$

where $\langle\widetilde{\partial}.F,f^{(1)}(\cdot)\rangle$ is a partial pairing, i.e., the unique element of $(\mathcal{H}_{-\tau})_{-q-1}$ such that for arbitrary $g \in (\mathcal{H}_\tau)_{q+1}$ $\langle\!\langle\langle\widetilde{\partial}.F,f^{(1)}(\cdot)\rangle,g\rangle\!\rangle_{(L^2)} = \langle\!\langle\widetilde{\partial}.F,g\otimes f^{(1)}(\cdot)\rangle\!\rangle_{(L^2)\otimes\mathcal{H}}$.

Remark. *Similarly to the proof of the fact that the generalized partial pairing $\langle\cdot,\cdot\rangle_{ext}$ is well posed and satisfies estimate* (43)*, one can easily show that a partial pairing is well posed and satisfies a generalized Cauchy-Bunyakovsky inequality (in our case this inequality has the form $\|\langle\widetilde{\partial}.F,f^{(1)}(\cdot)\rangle\|_{-\tau,-q-1} \le \|\widetilde{\partial}.F\|_{(\mathcal{H}_{-\tau})_{-q-1}\otimes\mathcal{H}_{-\tau}}|f^{(1)}|_\tau$).*

Proof. 1) The result follows from representation (45) with $n=1$ and the definition of an operator $\mathbb{I}$ (see (32)).

2) Taking into account (47) and (34), for all $g \in (\mathcal{H}_\tau)_{q+1}$ we obtain

$$\begin{aligned}
\langle\!\langle(\widetilde{D}F)(f^{(1)}),g\rangle\!\rangle_{(L^2)} &= \langle\!\langle F,(\widetilde{D}g)(f^{(1)})^*\rangle\!\rangle_{(L^2)} = \langle\!\langle F,\mathbb{I}(g\otimes f^{(1)})\rangle\!\rangle_{(L^2)}\\
&= \langle\!\langle\widetilde{\partial}.F,g\otimes f^{(1)}(\cdot)\rangle\!\rangle_{(L^2)\otimes\mathcal{H}} = \langle\!\langle\langle\widetilde{\partial}.F,f^{(1)}(\cdot)\rangle,g\rangle\!\rangle_{(L^2)},
\end{aligned}$$

whence the result follows. □

Remark. *Formally substituting in* (48) *$f^{(1)} = \delta_t$, $t \in \mathbb{R}_+$ (here δ_t is the Dirac delta-function concentrated at t; the substitution is formal because $\delta_t \notin \mathcal{H}_\tau$), we obtain a* formal *equality $\widetilde{\partial}_t\circ = (\widetilde{D}\circ)(\delta_t)$ (whence $\widetilde{\partial}.\circ = (\widetilde{D}\circ)(\delta.)$). In this connection we note that for the Hida stochastic derivative $\partial.$ and the operator of stochastic differentiation D on the spaces of nonregular test functions, for each $t \in \mathbb{R}_+$ $\partial_t\circ = (D\circ)(\delta_t)$ [24]; the formal analog of the last equality is valid on spaces that belong to the regular rigging of (L^2) [8].*

In some applications of the Gaussian analysis (in particular, in the Malliavin calculus) an important role belongs to the commutator between the extended stochastic integral and the operator of stochastic differentiation (see, e.g., [1]). Analogs of this commutator are calculated in the Meixner analysis [19, 20] and on the spaces of regular test and generalized functions of the Lévy analysis [8, 9]. Unfortunately, it is impossible to calculate a direct analog of the above-mentioned commutator on the spaces of nonregular test functions of the Lévy analysis: as we saw above, the extended stochastic integral cannot be naturally restricted to these spaces. Nevertheless, there exists an analog of this integral on the just now mentioned spaces — the generalized stochastic integral $\mathbb{I}$. So, now it is natural to calculate the commutator between $\mathbb{I}$ and the operator of stochastic differentiation, this commutator is calculated in [24]. On the spaces of nonregular generalized functions of the Lévy analysis the extended stochastic integral with respect to a Lévy process is well defined, and the role of the operator of stochastic differentiation belongs to the operator $\widetilde{D}$. So, it is natural to calculate the commutator between the above-mentioned integral and $\widetilde{D}$. In order to do this, let us introduce operators of stochastic differentiation on the spaces $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ (this notion is intuitively clear and can be used without an additional explanation, but we prefer to give an exact definition).

As above, we begin with a preparation. Let $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$, $g_{\cdot}^{(m)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau}$. We define

$$f^{(n)} \overline{\otimes} g_{\cdot}^{(m)} := (Pr \otimes \mathbf{1})(f^{(n)} \otimes g_{\cdot}^{(m)}) \in \mathcal{H}_{\tau}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\tau}, \tag{49}$$

where $Pr \otimes \mathbf{1}$ is the operator of symmetrization "by $n+m$ variables, except the variable $\cdot$" or, which is the same, the orthoprojector acting from $\mathcal{H}_{\tau}^{\widehat{\otimes} n} \otimes \mathcal{H}_{\tau}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau}$ to $\mathcal{H}_{\tau}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\tau}$ (of course, this operator depends on n and m, but we simplify the nonation). It is clear that

$$|f^{(n)} \overline{\otimes} g_{\cdot}^{(m)}|_{\mathcal{H}_{\tau}^{\widehat{\otimes} n+m} \otimes \mathcal{H}_{\tau}} \leq |f^{(n)}|_{\mathcal{H}_{\tau}^{\widehat{\otimes} n}} |g_{\cdot}^{(m)}|_{\mathcal{H}_{\tau}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau}}, \tag{50}$$

and for $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$, $g^{(m)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} m}$, $h^{(1)} \in \mathcal{H}_{\tau}$

$$f^{(n)} \overline{\otimes} (g^{(m)} \otimes h^{(1)}) = (f^{(n)} \widehat{\otimes} g^{(m)}) \otimes h^{(1)}. \tag{51}$$

Let $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$, $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$, $n, m \in \mathbb{N}$, $m \geq n$. We define a generalized partial pairing $\langle F_{ext,\cdot}^{(m)}, f^{(n)} \rangle_{EXT} \in \mathcal{H}_{-\tau}^{(m-n)} \otimes \mathcal{H}_{-\tau}$ by setting for arbitrary $g_{\cdot}^{(m-n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} m-n} \otimes \mathcal{H}_{\tau}$

$$\langle \langle F_{ext,\cdot}^{(m)}, f^{(n)} \rangle_{EXT}, g_{\cdot}^{(m-n)} \rangle_{\mathcal{H}_{ext}^{(m-n)} \otimes \mathcal{H}} = \langle F_{ext,\cdot}^{(m)}, f^{(n)} \overline{\otimes} g_{\cdot}^{(m-n)} \rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}}. \tag{52}$$

Since by the generalized Cauchy-Bunyakovsky inequality and (50)

$$\begin{aligned} |\langle F_{ext,\cdot}^{(m)}, f^{(n)} \overline{\otimes} g_{\cdot}^{(m-n)} \rangle_{\mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}}| &\leq |F_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}} |f^{(n)} \overline{\otimes} g_{\cdot}^{(m-n)}|_{\mathcal{H}_{\tau}^{\widehat{\otimes} m} \otimes \mathcal{H}_{\tau}} \\ &\leq |F_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}} |f^{(n)}|_{\mathcal{H}_{\tau}^{\widehat{\otimes} n}} |g_{\cdot}^{(m-n)}|_{\mathcal{H}_{\tau}^{\widehat{\otimes} m-n} \otimes \mathcal{H}_{\tau}}, \end{aligned}$$

this definition is well posed and

$$|\langle F_{ext,\cdot}^{(m)}, f^{(n)} \rangle_{EXT}|_{\mathcal{H}_{-\tau}^{(m-n)} \otimes \mathcal{H}_{-\tau}} \leq |F_{ext,\cdot}^{(m)}|_{\mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}} |f^{(n)}|_{\mathcal{H}_{\tau}^{\widehat{\otimes} n}}. \tag{53}$$

Remark. *Let* $F_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m)}$, $H^{(1)} \in \mathcal{H}_{-\tau}$*;* $g^{(m-n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} m-n}$, $h^{(1)} \in \mathcal{H}_{\tau}$. *For* $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$ *by* (52), (51) *and* (42) *we obtain*

$$\begin{aligned}
&\langle\langle F_{ext}^{(m)} \otimes H^{(1)}(\cdot), f^{(n)}\rangle_{EXT}, g^{(m-n)} \otimes h^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}} \\
&= \langle F_{ext}^{(m)} \otimes H^{(1)}(\cdot), f^{(n)}\overline{\otimes}(g^{(m-n)} \otimes h^{(1)}(\cdot))\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} \\
&= \langle F_{ext}^{(m)} \otimes H^{(1)}(\cdot), (f^{(n)}\widehat{\otimes} g^{(m-n)}) \otimes h^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} = \langle F_{ext}^{(m)}, f^{(n)}\widehat{\otimes} g^{(m-n)}\rangle_{\mathcal{H}_{ext}^{(m)}}\langle H^{(1)}, h^{(1)}\rangle_{\mathcal{H}} \\
&= \langle\langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext}, g^{(m-n)}\rangle_{\mathcal{H}_{ext}^{(m-n)}}\langle H^{(1)}, h^{(1)}\rangle_{\mathcal{H}} \\
&= \langle\langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext} \otimes H^{(1)}(\cdot), g^{(m-n)} \otimes h^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m-n)}\otimes\mathcal{H}}.
\end{aligned}$$

Since the set $\{g^{(m-n)} \otimes h^{(1)} : g^{(m-n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} m-n},\ h^{(1)} \in \mathcal{H}_{\tau}\}$ *is total in the space* $\mathcal{H}_{\tau}^{\widehat{\otimes} m-n} \otimes \mathcal{H}_{\tau}$, *we can conclude that*

$$\langle F_{ext}^{(m)} \otimes H^{(1)}, f^{(n)}\rangle_{EXT} = \langle F_{ext}^{(m)}, f^{(n)}\rangle_{ext} \otimes H^{(1)} \tag{54}$$

in the space $\mathcal{H}_{-\tau}^{(m-n)} \otimes \mathcal{H}_{-\tau}$.

Definition. *Let* $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$. *We define a linear continuous operator*

$$(\widetilde{\mathbf{D}}^n\circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau} \to (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau}$$

by setting for $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$

$$\begin{aligned}
(\widetilde{\mathbf{D}}^n F(\cdot))(f^{(n)}) :&= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} {:}\langle \circ^{\otimes m-n}, \langle F_{ext,\cdot}^{(m)}, f^{(n)}\rangle_{EXT}\rangle{:} \\
&\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} {:}\langle \circ^{\otimes m}, \langle F_{ext,\cdot}^{(m+n)}, f^{(n)}\rangle_{EXT}\rangle{:},
\end{aligned} \tag{55}$$

where $F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$ *are the kernels from decomposition* (23) *for* F.

Since (see (24), (55) and (53))

$$\begin{aligned}
\|(\widetilde{\mathbf{D}}^n F(\cdot))(f^{(n)})\|^2_{(\mathcal{H}_{-\tau})_{-q-1}\otimes\mathcal{H}_{-\tau}} &= \sum_{m=0}^{\infty} 2^{(-q-1)m} \frac{((m+n)!)^2}{(m!)^2} |\langle F_{ext,\cdot}^{(m+n)}, f^{(n)}\rangle_{EXT}|^2_{\mathcal{H}_{-\tau}^{(m)}\otimes\mathcal{H}_{-\tau}} \\
&\le |f^{(n)}|_{\tau}^2 2^{qn} \sum_{m=0}^{\infty} 2^{-q(m+n)} |F_{ext,\cdot}^{(m+n)}|^2_{\mathcal{H}_{-\tau}^{(m+n)}\otimes\mathcal{H}_{-\tau}} \left[2^{-m}\frac{((m+n)!)^2}{(m!)^2}\right] \\
&\le |f^{(n)}|_{\tau}^2 2^{qn} C(n) \|F\|^2_{(\mathcal{H}_{-\tau})_{-q}\otimes\mathcal{H}_{-\tau}},
\end{aligned}$$

where, as above, $C(n) = \max_{m\in\mathbb{Z}_+}[2^{-m}\frac{((m+n)!)^2}{(m!)^2}]$, this definition is well posed.

Remark. *Let* $F \in (\mathcal{H}_{-\tau})_{-q}$, $H^{(1)} \in \mathcal{H}_{-\tau}$. *Using* (55), (54) *and* (44) *one can easily show that for each* $n \in \mathbb{N}$ *and* $f^{(n)} \in \mathcal{H}_{\tau}^{\widehat{\otimes} n}$

$$(\widetilde{\mathbf{D}}^n F \otimes H^{(1)})(f^{(n)}) = (\widetilde{D}^n F)(f^{(n)}) \otimes H^{(1)} \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau}.$$

Denote $\widetilde{\mathbf{D}} := \widetilde{\mathbf{D}}^1$.

Theorem. *For all* $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}_{-\tau}$ *and* $f^{(1)} \in \mathcal{H}_\tau$

$$(\widetilde{D}\int F(u)\widehat{d}L_u)(f^{(1)}) = \int(\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u + \int F(u)f^{(1)}(u)du \in (\mathcal{H}_{-\tau})_{-q-1}, \qquad (56)$$

here $\int(\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u := \int J(u)\widehat{d}L_u$*, where* $J(\cdot) := (\widetilde{\mathbf{D}}F(\cdot))(f^{(1)}) \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}_{-\tau}$*;* $\int F(u)f^{(1)}(u)du$ *is a generalized Pettis integral, i.e.,*

$$\int F(u)f^{(1)}(u)du \equiv \langle F(\cdot), f^{(1)}(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q} \subset (\mathcal{H}_{-\tau})_{-q-1}$$

($\langle F(\cdot), f^{(1)}(\cdot)\rangle$ *is a partial pairing).*

Proof. Using (27) and (44) we obtain

$$(\widetilde{D}\int F(u)\widehat{d}L_u)(f^{(1)}) = \sum_{m=0}^{\infty}(m+1){:}\langle \circ^{\otimes m}, \langle \widehat{F}_{ext}^{(m)}, f^{(1)}\rangle_{ext}\rangle{:},$$

where $\widehat{F}_{ext}^{(m)} \in \mathcal{H}_{-\tau}^{(m+1)}$ are the kernels from decomposition (27) (which is decomposition (14) for $\int F(u)\widehat{d}L_u$), i.e., $\widehat{F}_{ext}^{(m)}$ are given by formula (28) ($F_{ext,\cdot}^{(m)} \in \mathcal{H}_{-\tau}^{(m)} \otimes \mathcal{H}_{-\tau}$ in (28) are the kernels from decomposition (23) for F). On the other hand, by (55), (27) and (28)

$$\int(\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u = \sum_{m=0}^{\infty} m{:}\langle \circ^{\otimes m}, U_m^{-1}\{Pr[(U_{m-1}\otimes\mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}\rangle{:}.$$

Let $g = \sum_{k=0}^{\infty}{:}\langle\circ^{\otimes k}, g^{(k)}\rangle{:} \in (\mathcal{H}_\tau)_{q+1}$, $g^{(k)} \in \mathcal{H}_\tau^{\widehat{\otimes} k}$ (see (10)). By (16) we have

$$\langle\!\langle(\widetilde{D}\int F(u)\widehat{d}L_u)(f^{(1)}), g\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m!(m+1)\langle\langle \widehat{F}_{ext}^{(m)}, f^{(1)}\rangle_{ext}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}},$$

$$\langle\!\langle\int(\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u, g\rangle\!\rangle_{(L^2)} = \sum_{m=0}^{\infty} m!m\langle U_m^{-1}\{Pr[(U_{m-1}\otimes\mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}}.$$

Further, since for each m $g^{(m)}$ belongs to the *symmetric* tensor power of $\mathcal{H}_\tau$, by (26), (52) and (49)

$$\begin{aligned}
&m\langle U_m^{-1}\{Pr[(U_{m-1}\otimes\mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}}\\
&= m\langle(U_{m-1}\otimes\mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}, g^{(m)}\rangle_{\mathcal{H}^{\otimes m}}\\
&= m\langle(U_{m-1}\otimes\mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}, g^{(m)}(\cdot)\rangle_{\mathcal{H}^{\widehat{\otimes} m-1}\otimes\mathcal{H}} = m\langle\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}, g^{(m)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m-1)}\otimes\mathcal{H}}\\
&= m\langle F_{ext,\cdot}^{(m)}, f^{(1)}\overline{\otimes} g^{(m)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} = \langle F_{ext,\cdot}^{(m)}(\cdot_1,\dots,\cdot_m), f^{(1)}(\cdot_1)\otimes g^{(m)}(\cdot_2,\dots,\cdot_m,\cdot)\\
&+ f^{(1)}(\cdot_2)\otimes g^{(m)}(\cdot_3,\dots,\cdot_m,\cdot_1,\cdot) + \dots + f^{(1)}(\cdot_m)\otimes g^{(m)}(\cdot_1,\dots,\cdot_{m-1},\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}};
\end{aligned}$$

and by (42), (28), (26), the symmetry of $f^{(1)}\widehat{\otimes} g^{(m)}$ and $g^{(m)}$, and the last calculation

$$\begin{aligned}
&(m+1)\langle\langle \widehat{F}_{ext}^{(m)}, f^{(1)}\rangle_{ext}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}} = (m+1)\langle \widehat{F}_{ext}^{(m)}, f^{(1)}\widehat{\otimes} g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m+1)}}\\
&= (m+1)\langle(U_m\otimes\mathbf{1})F_{ext,\cdot}^{(m)}, f^{(1)}\widehat{\otimes} g^{(m)}\rangle_{\mathcal{H}^{\otimes m+1}} = (m+1)\langle(U_m\otimes\mathbf{1})F_{ext,\cdot}^{(m)}, (f^{(1)}\widehat{\otimes} g^{(m)})(\cdot)\rangle_{\mathcal{H}^{\widehat{\otimes} m}\otimes\mathcal{H}}\\
&= (m+1)\langle F_{ext,\cdot}^{(m)}, (f^{(1)}\widehat{\otimes} g^{(m)})(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} = \langle F_{ext,\cdot}^{(m)}(\cdot_1,\dots,\cdot_m), g^{(m)}(\cdot_1,\dots,\cdot_m)\otimes f^{(1)}(\cdot)\\
&+ f^{(1)}(\cdot_1)\otimes g^{(m)}(\cdot_2,\dots,\cdot_m,\cdot) + f^{(1)}(\cdot_2)\otimes g^{(m)}(\cdot_3,\dots,\cdot_m,\cdot_1,\cdot)\\
&+ \dots + f^{(1)}(\cdot_m)\otimes g^{(m)}(\cdot_1,\dots,\cdot_{m-1},\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} = \langle F_{ext,\cdot}^{(m)}, g^{(m)}\otimes f^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}}\\
&+ m\langle U_m^{-1}\{Pr[(U_{m-1}\otimes\mathbf{1})\langle F_{ext,\cdot}^{(m)}, f^{(1)}\rangle_{EXT}]\}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}}.
\end{aligned}$$

Later, by (23), the construction of a pairing in a tensor product of chains (e.g., [4]), (29) and the definition of a partial pairing

$$\begin{aligned}\sum_{m=0}^{\infty} m!\langle F_{ext,\cdot}^{(m)}, g^{(m)} \otimes f^{(1)}(\cdot)\rangle_{\mathcal{H}_{ext}^{(m)}\otimes\mathcal{H}} &= \langle\!\langle F(\cdot), \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, g^{(m)} \otimes f^{(1)}(\cdot)\rangle :\rangle\!\rangle_{(L^2)\otimes\mathcal{H}} \\ &= \langle\!\langle F(\cdot), g \otimes f^{(1)}(\cdot)\rangle\!\rangle_{(L^2)\otimes\mathcal{H}} = \langle\!\langle \langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}}, g\rangle\!\rangle_{(L^2)},\end{aligned} \tag{57}$$

where $\langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}} \equiv \langle F(\cdot), f^{(1)}(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q} \subset (\mathcal{H}_{-\tau})_{-q-1}$ is a partial pairing.

So, for arbitrary $g \in (\mathcal{H}_\tau)_{q+1}$

$$\langle\!\langle (\widetilde{D} \int F(u)\widehat{d}L_u)(f^{(1)}), g\rangle\!\rangle_{(L^2)} = \langle\!\langle \int (\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u, g\rangle\!\rangle_{(L^2)} + \langle\!\langle \langle F(\cdot), f^{(1)}(\cdot)\rangle, g\rangle\!\rangle_{(L^2)},$$

from where (56) follows. □

Remark. *As follows from* (57)*, the definition of a partial pairing, and* (16)*, for* $g = \sum_{k=0}^{\infty} :\langle \circ^{\otimes k}, g^{(k)}\rangle : \in (\mathcal{H}_\tau)_q$

$$\begin{aligned}\langle\!\langle \langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}}, g\rangle\!\rangle_{(L^2)} &= \sum_{m=0}^{\infty} m!\langle \langle F_{ext,\cdot}^{(m)}, f^{(1)}(\cdot)\rangle_{\mathcal{H}}, g^{(m)}\rangle_{\mathcal{H}_{ext}^{(m)}} \\ &= \langle\!\langle \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \langle F_{ext,\cdot}^{(m)}, f^{(1)}(\cdot)\rangle_{\mathcal{H}}\rangle :, g\rangle\!\rangle_{(L^2)},\end{aligned}$$

from where $\langle F(\cdot), f^{(1)}(\cdot)\rangle_{\mathcal{H}} = \sum_{m=0}^{\infty} :\langle \circ^{\otimes m}, \langle F_{ext,\cdot}^{(m)}, f^{(1)}(\cdot)\rangle_{\mathcal{H}}\rangle :$ *in* $(\mathcal{H}_{-\tau})_{-q}$.

Remark. *One can easily show that the restriction of an operator* $(\widetilde{\mathbf{D}}^n\circ)(f^{(n)})$*,* $n \in \mathbb{N}$*,* $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$*, to the space* $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}$ *can be interpreted as a linear continuous operator acting from* $(\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}$ *to* $(\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}$*. Let us consider the extended stochastic integral* $\int_\Delta \circ(u)\widehat{d}L_u := \int \circ(u)1_\Delta(u)\widehat{d}L_u : (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H} \to (\mathcal{H}_{-\tau})_{-q}$*,* $\Delta \in \mathcal{B}(\mathbb{R}_+)$ *— the Borel* σ*-algebra (this integral satisfies* (27) *with kernels* (28)*, see [21] for a detailed presentation). By analogy with the proof of the last theorem one can show that for all* $F \in (\mathcal{H}_{-\tau})_{-q} \otimes \mathcal{H}$ *and* $f^{(1)} \in \mathcal{H}_\tau$

$$(\widetilde{D} \int_\Delta F(u)\widehat{d}L_u)(f^{(1)}) = \int_\Delta (\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u + \int_\Delta F(u)f^{(1)}(u)du \in (\mathcal{H}_{-\tau})_{-q-1},$$

where $\int_\Delta (\widetilde{\mathbf{D}}F(u))(f^{(1)})\widehat{d}L_u := \int_\Delta J(u)\widehat{d}L_u$*,* $J(\cdot) := (\widetilde{\mathbf{D}}F(\cdot))(f^{(1)}) \in (\mathcal{H}_{-\tau})_{-q-1} \otimes \mathcal{H}$*;*

$$\int_\Delta F(u)f^{(1)}(u)du := \int F(u)f^{(1)}(u)1_\Delta(u)du \equiv \langle F(\cdot), f^{(1)}(\cdot)1_\Delta(\cdot)\rangle \in (\mathcal{H}_{-\tau})_{-q} \subset (\mathcal{H}_{-\tau})_{-q-1}$$

is a partial pairing.

As is easily seen, the results of this subsection hold true (up to obvious modifications) if we consider the operators of stochastic differentiation on the space $(\mathcal{H}_{-\tau})$ or $(\mathcal{D}')$.

Remark. *As is known [1], in the classical Gaussian white noise analysis the operator of stochastic differentiation is a differentiation with respect to a so-called Wick product. This result holds true in the so-called Gamma-analysis [17] and in a more general Meixner analysis. In forthcoming papers we'll obtain similar results on spaces of test and generalized functions of the Lévy white noise analysis.*

2.2 The case of unbounded operators

Similarly to the analysis on spaces of regular test and generalized functions [9, 8], sometimes it can be necessary to consider $(\widetilde{D}^n\circ)(f^{(n)})$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$, as a linear operator acting in $(\mathcal{H}_{-\tau})_{-q}$. Let us accept a corresponding definition.

Definition. *Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$. We define the operator of stochastic differentiation*

$$(\widetilde{D}^n\circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q} \tag{58}$$

with the domain

$$\begin{aligned} \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})) := \{F \in (\mathcal{H}_{-\tau})_{-q} : \\ \|(\widetilde{D}^n F)(f^{(n)})\|^2_{-\tau,-q} = \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |\langle F_{ext}^{(m+n)}, f^{(n)}\rangle_{ext}|^2_{\mathcal{H}_{-\tau}^{(m)}} < \infty\} \end{aligned} \tag{59}$$

(here $F_{ext}^{(m+n)} \in \mathcal{H}_{-\tau}^{(m+n)}$ are the kernels from decomposition (14) *for F) by formula* (44)*.*

Proposition. *Operator of stochastic differentiation* (58) *with domain* (59) *is closed.*

Proof. Let us show that there exists a second adjoint to $(\widetilde{D}^n\circ)(f^{(n)})$ operator $(\widetilde{D}^n\circ)(f^{(n)})^{**} = (\widetilde{D}^n\circ)(f^{(n)})$ (it is well known that an adjoint operator is closed). Since, obviously, the domain of operator (58) is a dense set in $(\mathcal{H}_{-\tau})_{-q}$, the adjoint operator $(\widetilde{D}^n\circ)(f^{(n)})^* : (\mathcal{H}_\tau)_q \to (\mathcal{H}_\tau)_q$ is well defined. By definition, $g \in \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^*)$ if and only if $(\mathcal{H}_{-\tau})_{-q} \supset \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})) \ni F \mapsto \langle\langle(\widetilde{D}^n F)(f^{(n)}), g\rangle\rangle_{(L^2)}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $h \in (\mathcal{H}_\tau)_q$ such that $\langle\langle(\widetilde{D}^n F)(f^{(n)}), g\rangle\rangle_{(L^2)} = \langle\langle F, h\rangle\rangle_{(L^2)}$. But by calculation (46) h has form (45), therefore

$$\begin{aligned} \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^*) := \{g \in (\mathcal{H}_\tau)_q : \\ \|(\widetilde{D}^n F)(f^{(n)})^*\|^2_{\tau,q} = \sum_{m=0}^{\infty} ((m+n)!)^2 2^{q(m+n)} |f^{(n)}\widehat{\otimes} g^{(m)}|^2_\tau < \infty\} \end{aligned}$$

(see (11)), this set is dense in $(\mathcal{H}_\tau)_q$, hence the operator $(\widetilde{D}^n\circ)(f^{(n)})^{**} : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q}$ is well defined. Now it remains to show that

$$\mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^{**}) = \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})). \tag{60}$$

By definition, $F \in \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^{**})$ if and only if $(\mathcal{H}_\tau)_q \supset \mathrm{dom}((\widetilde{D}^n\circ)(f^{(n)})^*) \ni g \mapsto \langle\langle F, (\widetilde{D}^n g)(f^{(n)})^*\rangle\rangle_{(L^2)}$ is a linear continuous functional. By properties of Hilbert equipments the last is possible if and only if there exists $H \in (\mathcal{H}_{-\tau})_{-q}$ such that $\langle\langle F, (\widetilde{D}^n g)(f^{(n)})^*\rangle\rangle_{(L^2)} = \langle\langle H, g\rangle\rangle_{(L^2)}$. It is clear that H has form (44), therefore equality (60) follows from (59). $\square$

Remark. *Let*

$$A_n := \{F \in (\mathcal{H}_{-\tau})_{-q} : \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |F_{ext}^{(m+n)}|^2_{\mathcal{H}_{-\tau}^{(m+n)}} < \infty\},\ n \in \mathbb{N},$$

here $F_{ext}^{(m+n)} \in \mathcal{H}_{-\tau}^{(m+n)}$ are the kernels from decomposition (14) *for F. For each $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$ we define the operator of stochastic differentiation*

$$(\widehat{D}^n\circ)(f^{(n)}) : (\mathcal{H}_{-\tau})_{-q} \to (\mathcal{H}_{-\tau})_{-q} \tag{61}$$

with the domain A_n *by formula* (44) *with* $\widehat{D}^n$ *instead of* $\widetilde{D}^n$. *It follows from the just proved proposition that this operator is closable (its closure is operator* (58)*). Moreover, for each* $F \in A_n$ *the operator* $(\widehat{D}^n F)(\circ) : \mathcal{H}_\tau^{\widehat{\otimes} n} \to (\mathcal{H}_{-\tau})_{-q}$ *is linear bounded (and, therefore, continuous): by* (44)*,* (15) *and* (43) *for each* $f^{(n)} \in \mathcal{H}_\tau^{\widehat{\otimes} n}$

$$\|(\widehat{D}^n F)(f^{(n)})\|^2_{-\tau,-q} = \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |\langle F_{ext}^{(m+n)}, f^{(n)} \rangle_{ext}|^2_{\mathcal{H}_{-\tau}^{(m)}}$$
$$\leq |f^{(n)}|^2_\tau \sum_{m=0}^{\infty} 2^{-qm} \frac{((m+n)!)^2}{(m!)^2} |F_{ext}^{(m+n)}|^2_{\mathcal{H}_{-\tau}^{(m+n)}}.$$

It is clear that the results of Subsection 2.1 hold true (up to obvious modifications) for operators (58) and (61).

REFERENCES

[1] Benth F.E. *The Gross derivative of generalized random variables*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 1999, **2** (3), 381–396. doi:10.1142/S0219025799000229

[2] Benth F.E., Di Nunno G., Lokka A., Oksendal B., Proske F. *Explicit representation of the minimal variance portfolio in markets driven by Lévy processes*. Math. Finance 2003, **13** (1), 55–72. doi:10.1111/1467-9965.t01-1-00005

[3] Berezansky Yu. M., Kondratiev Yu.G. Spectral Methods in Infinite-Dimensional Analysis. Kluwer Academic Publishers, Netherlands, 1995.

[4] Berezansky Yu. M., Sheftel Z.G., Us G.F. Functional Analysis. Birkhäuser Verlag, Basel–Boston–Berlin, 1996.

[5] Bertoin J. Lévy Processes. Cambridge University Press, Cambridge, 1996.

[6] Di Nunno G., Oksendal B., Proske F. Malliavin Calculus for Lévy Processes with Applications to Finance. Universitext. Springer-Verlag, Berlin, 2009.

[7] Di Nunno G., Oksendal B., Proske F. *White noise analysis for Lévy processes*. J. Funct. Anal. 2004, **206** (1), 109–148. doi:10.1016/S0022-1236(03)00184-8

[8] Dyriv M.M., Kachanovsky N.A. *On operators of stochastic differentiation on spaces of regular test and generalized functions of Lévy white noise analysis*. Carpathian Math. Publ. 2014, **6** (2), 212–229. doi:10.15330/cmp.6.2.212-229

[9] Dyriv M.M., Kachanovsky N.A. *Operators of stochastic differentiation on spaces of regular test and generalized functions in the Lévy white noise analysis*. Research Bull. Nat. Tech. Univ. Ukraine "Kyiv Polytechnic Institute" 2014, (4), 36–40.

[10] Dyriv M.M., Kachanovsky N.A. *Stochastic integrals with respect to a Levy process and stochastic derivatives on spaces of regular test and generalized functions*. Research Bull. Nat. Tech. Univ. Ukraine "Kyiv Polytechnic Institute" 2013, (4), 27–30.

[11] Gelfand I.M., Vilenkin N.Ya. Generalized Functions. Academic Press, New York, London, 1964.

[12] Gihman I.I., Skorohod A.V. Theory of Random Processes. Nauka, Moscow, 1973.

[13] Holden H., Oksendal B., Uboe J., Zhang T.-S. Stochastic Partial Differential Equations—a Modeling. White Noise Functional Approach. Birkhäuser, Boston, 1996.

[14] Itô K. *Spectral type of the shift transformation of differential processes with stationary increments*. Trans. Am. Math. Soc. 1956, **81** (2), 253–263. doi:10.2307/1992916

[15] Kabanov Yu.M. *Extended stochastic integrals*. Teorija Verojatnostej i ee Pril. 1975, **20** (4), 725–737.

[16] Kabanov Yu.M., Skorohod A.V. Extended stochastic integrals. In: Proc. School-Symposium "Theory Stoch. Proc.", Druskininkai, Lietuvos Respublika, November 25–30, 1974, Inst. Phys. Math., Vilnus, 1975, 123–167.

[17] Kachanovsky N.A. *A generalized Malliavin derivative connected with the Poisson- and Gamma-measures*. Methods Funct. Anal. Topol. 2003, **9** (3), 213–240.

[18] Kachanovsky N.A. *A generalized stochastic derivative on the Kondratiev-type space of regular generalized functions of Gamma white noise*. Methods Funct. Anal. Topol. 2006, **12** (4), 363–383.

[19] Kachanovsky N.A. *Generalized stochastic derivatives on a space of regular generalized functions of Meixner white noise*. Methods Funct. Anal. Topol. 2008, **14** (1), 32–53.

[20] Kachanovsky N.A. *Generalized stochastic derivatives on parametrized spaces of regular generalized functions of Meixner white noise*. Methods Funct. Anal. Topol. 2008, **14** (4), 334–350.

[21] Kachanovsky N.A. *Extended stochastic integrals with respect to a Lévy process on spaces of generalized functions*. Math. Bull. Shevchenko Sci. Soc. 2013, **10**, 169–188.

[22] Kachanovsky N.A. *On an extended stochastic integral and the Wick calculus on the connected with the generalized Meixner measure Kondratiev-type spaces*. Methods Funct. Anal. Topol. 2007, **13** (4), 338–379.

[23] Kachanovsky N.A. *On extended stochastic integrals with respect to Lévy processes*. Carpathian Math. Publ. 2013, **5** (2), 256–278. doi:10.15330/cmp.5.2.256-278

[24] Kachanovsky N.A. *Operators of stochastic differentiation on spaces of nonregular test functions of Lévy white noise analysis*. Methods Funct. Anal. Topol. 2015, **21** (4), 336–360.

[25] Kachanovsky N.A. *Bounded operators of stochastic differentiation on spaces of nonregular generalized functions in the Lévy white noise analysis*. Research Bull. National Tech. Univ. Ukraine "Kyiv Polytechnic Institute" 2016, (4), in print.

[26] Kachanovsky N.A., Tesko V.A. *Stochastic integral of Hitsuda-Skorokhod type on the extended Fock space*. Ukrainian Math. J. 2009, **61** (6), 873–907. doi:10.1007/s11253-009-0257-2 (translation of Ukrain. Mat. Zh. 2009, **61** (6), 733–764. (in Ukrainian))

[27] Lytvynov E.W. *Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes*. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 2003, **6** (1), 73–102.

[28] Meyer P.A. Quantum Probability for Probabilists. In: Lect. Notes in Math., 1538. Springer–Verlag, Berlin, 1993.

[29] Nualart D. Schoutens W. *Chaotic and predictable representations for Lévy processes*. Stochastic Process. Appl. 2000, **90** (1), 109–122. doi:10.1016/S0304-4149(00)00035-1

[30] Protter P. Stochastic Integration and Differential Equations. Springer, Berlin, 1990.

[31] Sato K. Lévy Processes and Infinitely Divisible Distributions. In: Cambridge University Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 1999.

[32] Schoutens W. Stochastic Processes and Orthogonal Polynomials. In: Lect. Notes in Statist., 146. Springer–Verlag, New York, 2000.

[33] Skorohod A.V. Integration in Hilbert Space. Springer-Verlag, Berlin–New York–Heidelberg, 1974.

[34] Skorohod A.V. *On a generalization of a stochastic integral*. Teorija Verojatnostej i ee Pril. 1975, **20** (2), 223–238.

[35] Solé J.L., Utzet F., Vives J. Chaos expansions and Malliavin calculus for Lévy processes. In: Stoch. Anal. and Appl., Abel Symposium 2. Springer, Berlin, 2007, 595–612.

[36] Surgailis D. *On L^2 and non-L^2 multiple stochastic integration*. In: Lect. Notes in Control and Information Sciences, 36. Springer-Verlag., Berlin–Heidelberg, 1981. 212–226.

[37] Ustunel A.S. An Introduction to Analysis on Wiener Space. In: Lect. Notes in Math., 1610. Springer–Verlag, Berlin, 1995.

Received 17.01.2016

Качановський М.О. *Оператори стохастичного диференціювання на просторах нерегулярних узагальнених функцій аналізу білого шуму Леві* // Карпатські матем. публ. — 2016. — Т.8, №1. — С. 83–106.

Оператори стохастичного диференціювання, які тісно пов'язані з розширеним стохастичним інтегралом Скорохода та зі стохастичною похідною Хіди, грають важливу роль у класичному (гауссівському) аналізі білого шуму. Зокрема, ці оператори можна використовувати для вивчення деяких властивостей розширеного стохастичного інтеграла та розв'язків стохастичних рівнянь з нелінійностями віківського типу.

Протягом останніх років оператори стохастичного диференціювання були уведені та вивчені, зокрема, у межах майкснерівського аналізу білого шуму, так само як і на просторах регулярних основних і узагальнених функцій та на просторах нерегулярних основних функцій аналізу білого шуму Леві. У цій статті ми робимо наступний природний крок: уводимо та вивчаємо оператори стохастичного диференціювання на просторах нерегулярних узагальнених функцій аналізу білого шуму Леві (тобто на просторах узагальнених функцій, які належать так званому нерегулярному оснащенню простору квадратично інтегровних за мірою білого шуму Леві функцій). При цьому використовується литвинівське узагальнення властивості хаотичного розкладу. Дослідження цієї статті можна розглядати як внесок у подальший розвиток аналізу білого шуму Леві.

Ключові слова і фрази: оператор стохастичного диференціювання, стохастична похідна, розширений стохастичний інтеграл, процес Леві.