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INVERSE CAUCHY PROBLEM FOR FRACTIONAL TELEGRAPH EQUATION WITH DISTRIBUTIONS

The inverse Cauchy problem for the fractional telegraph equation

$$u_t^{(\alpha)} - r(t)u_t^{(\beta)} + a^2(-\Delta)^{\gamma/2}u = F_0(x)g(t), \quad (x,t) \in \mathrm{R}^n \times (0,T],$$

with given distributions in the right-hand sides of the equation and initial conditions is studied. Our task is to determinate a pair of functions: a generalized solution u (continuous in time variable in general sense) and unknown continuous minor coefficient $r(t)$. The unique solvability of the problem is established.

Key words and phrases: generalized function, fractional derivative, inverse problem, Green vector-function.

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INTRODUCTION

The existence and uniqueness theorems were proved, and the representation (in terms of the Green function) of classical solution of a time- and a time-space-fractional Cauchy problem was obtained, for example, in [1, 3–5, 14]. The unique solvability of a time-space-fractional Cauchy problem in spaces of distributions was proved in [8, 10].

Inverse problems for such equations arise in many branches of science and engineering. The inverse boundary value problems for determination of a leading coefficient, or a part of the right-hand side, or an order of a diffusion-wave equation, or an unknown boundary condition, were studied, for example, in [2, 6, 11, 12, 15].

In the present paper we prove the existence and uniqueness of a solution (u, r) of the inverse Cauchy problem

$$u_t^{(\alpha)} - r(t)u_t^{(\beta)} + a^2(-\Delta)^{\gamma/2}u = F_0(x)g(t), \quad (x,t) \in \mathrm{R}^n \times (0,T], \tag{1}$$

$$u(x,0) = F_1(x), \quad u_t(x,0) = F_2(x), \ x \in \mathrm{R}^n, \tag{2}$$

$$\big(u(\cdot,t), \varphi_0(\cdot)\big) = F(t), \quad t \in (0,T] \tag{3}$$

with the Riemann-Liouville fractional derivatives $u_t^{(\alpha)}, u_t^{(\beta)}$, where F_0, F_1, F_2 are given distributions, F, g, φ_0 are given smooth functions, the symbol (f, φ) stands for the value of the distribution f on the test function φ, a^2 is a positive constant, $(-\Delta)^{\gamma/2}u$ is defined with the use of the Fourier transform as follows

$$F[(-\Delta)^{\gamma/2}u] = |\lambda|^\gamma F[u],$$

and the following assumption holds:

(L) $\qquad \alpha \in (1,2), \beta \in (0,1), \gamma > \alpha, \quad \min\{n, 2, \gamma\} > (n-1)/2.$

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1 NOTATIONS AND AUXILIARY RESULTS

Denote the set of natural numbers by symbol N. Let $Z_+ := N \cup \{0\}$, $Q := R^n \times (0, T]$, $n \in N$. Let $\mathcal{E}(R^n) := C^\infty(R^n)$ and $\mathcal{D}(R^n)$ be the space of infinitely differentiable functions compactly supported in R^n. $\mathcal{D}(\bar{Q})$ is the space of infinitely differentiable functions having compact supports with respect to space variables and such that $(\frac{\partial}{\partial t})^k v|_{t=T} = 0$, $k \in Z_+$, $\mathcal{D}^k(R^n)$ is the space of functions from $C^k(R^n)$ having compact supports, $\|\varphi\|_{D^k(R^n)} := \max\limits_{|\kappa|\le k} \max\limits_{x\in supp\varphi} |D^\kappa \varphi(x)|$, where $\kappa = (\kappa_1, \ldots, \kappa_n)$, $\kappa_j \in Z_+$, $j \in \{1, \ldots, n\}$, $|\kappa| = \kappa_1 + \cdots + \kappa_n$, $D^\kappa\varphi(x) := \frac{\partial^{|\kappa|}\varphi(x)}{\partial x_1^{\kappa_1} \ldots \partial x_n^{\kappa_n}}$, while $\mathcal{D}'(R^n)$, $\mathcal{E}'(R^n)$ and $\mathcal{D}'(\bar{Q})$ are spaces of linear continuous functionals (distributions) over $\mathcal{D}(R^n)$, $\mathcal{E}(R^n)$ and $\mathcal{D}(\bar{Q})$, respectively. Note that $\mathcal{E}'(R^n)$ is the space of generalized functions with compact supports. Let

$$\mathcal{D}'_+(R) := \{f \in \mathcal{D}'(R) : f = 0, \forall t < 0\},$$
$$\mathcal{D}'_C(Q) = \{v \in \mathcal{D}'(\bar{Q}) : (v(\cdot, t), \varphi(\cdot)) \in C(0, T] \text{ for all } \varphi \in \mathcal{D}(R^n)\}.$$

We denote by $f * g$ the convolution of the generalized functions f and g, and use the function

$$f_\lambda(t) = \begin{cases} \frac{\theta(t) t^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0, \\ f'_{1+\lambda}(t), & \lambda \le 0, \end{cases}$$

where $\Gamma(z)$ is the gamma-function, $\theta(t)$ is the Heaviside function. Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$.

Recall that the Riemann-Liouville derivative of order $\beta > 0$ is defined as

$$v_t^{(\beta)}(x, t) = f_{-\beta}(t) * v(x, t),$$

and the Caputo fractional derivative is defined in [3] by

$$D_t^\beta v(x, t) = \frac{1}{\Gamma(1-\beta)} \Big[\frac{\partial}{\partial t} \int_0^t \frac{v(x, \tau)}{(t-\tau)^\beta} d\tau - \frac{v(x, 0)}{t^\beta}\Big], \quad \beta \in (0, 1),$$

$$D_t^\beta v(x, t) = \frac{1}{\Gamma(2-\beta)} \Big[\frac{\partial}{\partial t} \int_0^t \frac{v_\tau(x, \tau)}{(t-\tau)^{\beta-1}} d\tau - \frac{v_t(x, 0)}{(t-\tau)^{\beta-1}}\Big], \quad \beta \in (1, 2).$$

Denote by

$$C_{\alpha,\gamma}(Q) := \{v \in C(Q) : (-\Delta)^{\gamma/2} v, D_t^\alpha v \in C(Q)\},$$
$$C_{\alpha,\gamma}(\bar{Q}) := \{v \in C_{\alpha,\gamma}(Q) \mid v, v_t \in C(\bar{Q})\},$$
$$(Lv)(x, t) := v_t^{(\alpha)}(x, t) + a^2(-\Delta)^{\gamma/2} v(x, t),$$
$$(L^{reg} v)(x, t) := D_t^\alpha v(x, t) + a^2(-\Delta)^{\gamma/2} v(x, t),$$
$$(\widehat{L} v)(x, t) := f_{-\alpha}(t) \hat{*} v(x, t) + a^2(-\Delta)^{\gamma/2} v(x, t), \quad (x, t) \in Q,$$

where $f_{-\alpha}(t) \hat{*} v(x, t) = \big(f_{-\alpha}(\tau), v(x, t+\tau)\big)$, $v \in \mathcal{D}(\bar{Q})$. The Green formula holds [8]:

$$\int_Q v(y, \tau)(\widehat{L}\psi)(y, \tau) dy d\tau = \int_Q (L^{reg} v)(y, \tau)\psi(y, \tau) dy d\tau$$

$$- \int_{R^n} v(y, 0) dy \int_0^T f_{2-\alpha}(\tau)\psi_\tau(y, \tau) d\tau + \int_{R^n} v_t(y, 0) dy \int_0^T f_{2-\alpha}(\tau)\psi(y, \tau) d\tau,$$

for all $v \in C_{\alpha,\gamma}(\bar{Q})$, $\psi \in \mathcal{D}(\bar{Q})$.

Assumptions:

(A1) $F_0, F_1, F_2 \in \mathcal{E}'(\mathrm{R}^n)$, $t^{\varepsilon} g(t)$ is a continuous function on $[0,T]$ for some $\varepsilon \in (0, \alpha/2)$;

(A2) $F, F^{(\beta)} \in C(0,T]$, $\inf\limits_{t\in(0,T]} |F^{(\beta)}(t)| = f = const > 0$, $t^{\varepsilon} F^{(\alpha)}(t)$ is a continuous function on $[0,T]$ for some $\varepsilon \in (0,\alpha/2)$, $\varphi_0 \in \mathcal{D}(\mathrm{R}^n)$.

Definition 1. *A pair of functions $(u,r) \in \mathcal{D}'_C(Q) \times C(0,T]$ satisfying the identity*

$$(u, \widehat{L}\psi) = \int\limits_0^T g(t)\big(F_0(\cdot), \psi(\cdot,t)\big)dt + \int\limits_0^T r(t)\big(u_t^{(\beta)}(\cdot,t), \psi(\cdot,t)\big)dt + \sum_{j=1}^{2}\big(F_j(x) f_{j-\alpha}(t), \psi(x,t)\big) \quad (4)$$

for all $\psi \in \mathcal{D}(\bar{Q})$ and the condition (3) *is called a solution of the problem* (1)–(3).

We use the Green function method to prove the solvability of this problem.

Definition 2. *A vector-function $\big(G_0(x,t), G_1(x,t), G_2(x,t)\big)$ such that under rather regular g_0, g_1, g_2 the function*

$$u(x,t) = \int\limits_0^t d\tau \int\limits_{\mathrm{R}^n} G_0(x-y, t-\tau) g_0(y,\tau)dy + \sum_{j=1}^{2} \int\limits_{\mathrm{R}^n} G_j(x-y,t) g_j(y)dy, \quad (x,t) \in \bar{Q} \quad (5)$$

is a classical (from $C_{\alpha,\gamma}(\bar{Q})$) solution of the Cauchy problem

$$L^{reg} u(x,t) = g_0(x,t), \quad (x,t) \in Q,$$
$$u(x,0) = g_1(x),\ u_t(x,0) = g_2(x), \quad x \in \mathrm{R}^n,$$

is called a Green vector-function of this problem.

Denote by

$$(\widehat{\mathcal{G}}_0 \varphi)(y,\tau) := \int\limits_\tau^T \int\limits_{\mathrm{R}^n} G_0(x-y, t-\tau)\varphi(x,t)dxdt,$$

$$(\widehat{\mathcal{G}}_j \varphi)(y) := \int\limits_0^T \int\limits_{\mathrm{R}^n} G_j(x-y,t)\varphi(x,t)dxdt, \quad j = 1,2.$$

Lemma 1 ([8]). *The following relations hold:*

$$G_j(x,t) = \big(f_{j-\alpha}(\tau), G_0(x, t-\tau)\big), \quad (x,t) \in Q, \quad j = 1,2, \quad (6)$$

$$(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau) = \psi(y,\tau), \quad (y,\tau) \in \bar{Q},$$
$$(\widehat{\mathcal{G}}_j(\widehat{L}\psi))(y) = \big(f_{j-\alpha}(\tau), \psi(y,\tau)\big), \quad y \in \mathrm{R}^n, \quad j = 1,2, \text{ for all } \psi \in \mathcal{D}(\bar{Q}). \quad (7)$$

Lemma 2 ([1,4]). *The Green vector-function of the Cauchy problem* (1), (2) *exists.*

We also use the notations

$$(\widehat{G_j}\varphi)(y,t) := \int\limits_{\mathrm{R}^n} G_j(x-y,t)\varphi(x)\, dx, \quad j = 0,1,2.$$

Lemma 3. *For all* $k \in \mathrm{Z}_+$*, multi-index* κ*,* $|\kappa| = k$*,* $\varphi \in \mathcal{D}(\mathrm{R}^n)$ *we have*

$$D_y^\kappa(\widehat{G_j}\varphi) \in C(Q), \qquad j = 0, 1, 2,$$

and for all $\varepsilon \in (0, 1)$ *the following estimates hold:*

$$\begin{aligned} \left|D_y^\kappa(\widehat{G_0}\varphi)(y,t)\right| &\le c_k t^{\alpha-\varepsilon-1}\|\varphi\|_{\mathcal{D}^k(\mathrm{R}^n)}, \\ \left|D_y^\kappa(\widehat{G_1}\varphi)(y,t)\right| &\le c_k(1 + |\ln t|)\|\varphi\|_{\mathcal{D}^k(\mathrm{R}^n)}, \\ \left|D_y^\kappa(\widehat{G_2}\varphi)(y,t)\right| &\le c_k\|\varphi\|_{\mathcal{D}^k(\mathrm{R}^n)}, \quad (y,t) \in Q. \end{aligned}$$

Hereinafter b_i, c_i*,* $i \in \mathrm{Z}_+$*, are positive constants.*

Proof. Lemma can be proved with the use of the estimates of the Green vector-function components, which were obtained in [8] by using the properties of the H-functions of Fox [7,13]. □

Theorem 1. *Assume that (L), (A1) hold. Then there exists a unique solution* $u \in \mathcal{D}'_C(Q)$ *of the problem* (1), (2) *with* $r(t) = 0$, $t \in [0, T]$. *It is defined by*

$$\Big(u(\cdot,t), \varphi(\cdot)\Big) = h_\varphi(t) \ \text{ for all } \varphi \in \mathcal{D}(\mathrm{R}^n),\ t \in (0, T], \tag{8}$$

where

$$h_\varphi(t) = \sum_{j=1}^{2}\Big(F_j(\cdot), (\widehat{G}_j\varphi)(\cdot,t)\Big) + \int_0^t g(\tau)\Big(F_0(\cdot), (\widehat{G}_0\varphi)(\cdot, t-\tau)\Big)d\tau, \ \ t \in (0, T].$$

Proof. A distribution from $\mathcal{E}'(\mathrm{R}^n)$ has a finite order of the singularity. So, there exist $k_0, k_1, k_2 \in \mathrm{Z}_+$ and the functions $g_{0\kappa}, g_{1\kappa}, g_{2\kappa} \in L_1(\mathrm{R}^n)$ such that

$$\big(F_j, \varphi\big) = \sum_{|\kappa|\le k_j}\int_{\mathrm{R}^n} g_{j\kappa}(y)D^\kappa\varphi(y)dy \quad \text{ for all } \varphi \in \mathcal{D}(\mathrm{R}^n), \quad j = 0, 1, 2. \tag{9}$$

It means that $s(F_j) \le k_j$, $j = 0, 1, 2$.

Using (9) and Lemma 3, similarly to [9], we show that the function (8) belongs to $\mathcal{D}'_C(Q)$, and using (7), show that it satisfies the equality (4) with $r(t) = 0$, $t \in [0, T]$. The uniqueness of a solution can be proved as in [9]. □

2 The existence and uniqueness theorems for the inverse problem

As we know from the Theorem 1, under assumptions (L), (A1) the solution $u \in \mathcal{D}'_C(Q)$ of the Cauchy problem (1), (2) satisfies the equation

$$\big(u(\cdot,t), \varphi(\cdot)\big) = h_\varphi(t) + \int_0^t r(\tau)\Big(u_t^{(\beta)}(\cdot,t), (\widehat{G}_0\varphi)(\cdot, t-\tau)\Big)d\tau, \ \ \varphi \in \mathcal{D}(\mathrm{R}^n), t \in (0, T], \tag{10}$$

and $h_\varphi \in C(0, T]$ for all $\varphi \in \mathcal{D}(\mathrm{R}^n)$. Conversely, any solution $u \in \mathcal{D}'_C(Q)$ of (10) is the solution of the problem (1), (2).

From the equation (1) we obtain

$$\big(u_t^{(\alpha)}(\cdot,t),\varphi_0(\cdot)\big)=a^2\big(u(\cdot,t),(-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)+r(t)\big(u_t^{(\beta)}(\cdot,t),\varphi_0\big)+g(t)\big(F_0,\varphi_0\big).$$

Using (3) and (A2) find

$$r(t)=\big[F^{(\alpha)}(t)-a^2\big(u(\cdot,t),(-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)-g(t)\big(F_0,\varphi_0\big)\big][F^{(\beta)}(t)]^{-1},\quad t\in(0,T]. \tag{11}$$

Denote by $H(u,t)$ the right-hand side of (11), substitute it in (10) instead of $r(t)$. We obtain the nonlinear operator equation

$$\big(u(\cdot,t),\varphi(\cdot)\big)=h_\varphi(t)+\int_0^t H(u,\tau)\big(u(\cdot,t),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)d\tau,\ \ \varphi\in\mathcal{D}(\mathrm{R}^n),\ t\in(0,T], \tag{12}$$

relatively unknown function $u\in\mathcal{D}'{}_C(Q)$. Conversely, if $u\in\mathcal{D}'{}_C(Q)$ is a solution of (12), r is defined by (11) then, by the Theorem 1, the pair (u,r) satisfies the problem (1)–(3). So, under assumptions (L), (A1), (A2) a pair $(u,r)\in\mathcal{D}'_C(Q)\times C(0,T]$ is a solution of the problem (1)–(3) if and only if the function $u\in\mathcal{D}'{}_C(Q)$ is a solution of (12) and $r(t)$ is defined by (11).

Theorem 2. *Assume that (L), (A1), (A2) hold. Then there exist $T^*\in(0,T]$ ($Q^*=\mathrm{R}^n\times(0,T^*]$, respectively) and the solution $(u,r)\in\mathcal{D}'_C(Q^*)\times C(0,T^*]$ of the problem (1)–(3): the function u is a solution of (12), r is defined by (11).*

Proof. By the Theorem 1 the right-hand side of (12) is continuous on $(0,T]$. It is enough to prove the solvability of the equation (12) in $\mathcal{D}'_C(Q)$. Using (9) and Lemma 3, for all $\varepsilon\in(0,1)$, $\varphi\in\mathcal{D}^K(\mathrm{R}^n)$ with $K\in\mathrm{Z}_+$, $K\geq\max\{k_0,k_1,k_2\}$, where $s(F_j)\leq k_j$, $j=0,1,2$, we obtain

$$t^\varepsilon\Big|\int_0^t g(\tau)\big(F_0(\cdot),(\widehat{\mathcal{G}}_0\varphi)(\cdot,t,\tau)\big)d\tau\Big|\leq b_0t^\alpha\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}, \tag{13}$$

$$t^\varepsilon|h_\varphi(t)|\leq\big[t^\alpha b_0+b_1\big]\ \|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}. \tag{14}$$

Let $\mathrm{R}>0$, $\varepsilon\in(0,\alpha/2)$,

$$M_{R,\varepsilon}=M_{R,\varepsilon}(Q)=\Big\{v\in\mathcal{D}'{}_C(Q):\|v\|_\varepsilon=\sup_{t\in(0,T]}\ \sup_{\varphi\in D^K(\mathrm{R}^n)}\frac{t^\varepsilon\big|\big(v(\cdot,t),\varphi(\cdot)\big)\big|}{\|\varphi\|_{D^K(\mathrm{R}^n)}}\leq R\Big\}.$$

Define the operator $P:\mathcal{D}'_C(Q)\to\mathcal{D}'_C(Q)$ as follows

$$\big((Pv)(\cdot,t),\varphi(\cdot)\big)=h_\varphi(t)+\int_0^t H(v,\tau)\big(v(\cdot,t),(\widehat{G}_0\varphi)(\cdot,t-\tau)\big)d\tau,\quad\varphi\in D^K(\mathrm{R}^n). \tag{15}$$

We use the Banach principle to prove the solvability of the equation (12), that is

$$u=Pu,\quad u\in M_{R,\varepsilon}(Q)\subset\mathcal{D}'_C(Q).$$

At the beginning we show that there exist $R>0$, $T^*\in(0,T]$, $Q^*=\mathrm{R}^n\times(0,T^*]$ and $M^*_{R,\varepsilon}=M_{R,\varepsilon}(Q^*)$ such that $P:M^*_{R,\varepsilon}\to M^*_{R,\varepsilon}$.

For every $v \in M_{R,\varepsilon}$ we have

$$\tau^\varepsilon \big|\big(v(\cdot,\tau), a^2(-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)\big| \le R\|(-\Delta)^{\gamma/2}\varphi_0\|_{\mathcal{D}^K(\mathrm{R}^n)} := b_2 R,$$

and therefore

$$\tau^\varepsilon |H(v,\tau)| \le \frac{B+b_2R}{f}, \text{ where } B = \sup_{\tau\in(0,T]} \tau^\varepsilon |F^{(\alpha)}(\tau) - g(\tau)(F_0,\varphi_0)|.$$

From here, taking into account (13), (14) and Lemma 3, for all $v \in M_{R,\varepsilon}$, $\varphi \in \mathcal{D}(\mathrm{R}^n)$ we obtain

$$\begin{aligned}\frac{t^\varepsilon\big|\big((Pv)(\cdot,t),\varphi(\cdot)\big)\big|}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} &\le t^\alpha b_0 + b_1 + \frac{(B+b_2R)R}{f}\int_0^t \frac{\|(\widehat{\mathcal{G}}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}\tau^{-\varepsilon}d\tau}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} \\ &\le t^\alpha b_0 + b_1 + \frac{(B+b_2R)R}{f}\int_0^t c_K (t-\tau)^{\alpha-\varepsilon-1}\tau^{-\varepsilon}d\tau \\ &\le t^{\alpha-2\varepsilon}(q_0R^2 + q_1R + q_2) + b_1,\end{aligned}$$

where q_j ($j \in \{0,1,2\}$) are positive constants.

To realize the inequality

$$t^{\alpha-2\varepsilon}(q_0R^2 + q_1R + q_2) + b_1 \le R \text{ for all } t \in [0,T^*] \tag{16}$$

with some $T^* \in (0,T]$, we will at first choose $R \ge 2b_1$ and $t_0 \in (0,T]$ such that

$$q_2 t^{\alpha-2\varepsilon} + b_1 \le R/2 \text{ for all } t \in [0,t_0].$$

Then (16) follows from the inequality

$$(q_0+q_1)t^{\alpha-2\varepsilon}R \le \frac{1}{2} \text{ for all } t \in [0,T^*] \tag{17}$$

for some $R \ge \max\{1, 2b_1\}$, where $T^* = \min\{t_0, 1/[2(q_0+q_1)R]^{1/(\alpha-2\varepsilon)}\}$. We have proved the existence R, T^* such that $P : M^*_{R,\varepsilon} \to M^*_{R,\varepsilon}$.

Now we show that P is the contraction operator on $M^*_{R,\varepsilon}$. For $v_1, v_2 \in M^*_{R,\varepsilon}$, $\varphi \in \mathcal{D}(\mathrm{R}^n)$ and $t \in [0,T^*]$ we have

$$\begin{aligned}&\frac{t^\varepsilon\big|\big((Pv_1)(\cdot,t) - \big((Pv_2)(\cdot,t),\varphi(\cdot)\big)\big|}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} = \frac{t^\varepsilon}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}}\int_0^t \Big| H(v_2,\tau)\big(v_1(\cdot,t) - v_2(\cdot,t), (\widehat{G}_0\varphi)(\cdot,t-\tau)\big) \\ &+ \big(H(v_1,\tau) - H(v_2,\tau)\big)\big(v_1(\cdot,t), (\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\Big| d\tau \\ &\le \frac{(B+b_2R)t^\varepsilon}{f}\int_0^t \frac{\big|\big(v_1(\cdot,t) - v_2(\cdot,t), (\widehat{G}_0\varphi)(\cdot,t-\tau)\big)\big|}{\|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}} \frac{\|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}}\tau^{-\varepsilon}d\tau \\ &+ \frac{a^2t^\varepsilon R\|(-\Delta)^{\gamma/2}\varphi_0\|_{\mathcal{D}^K(\mathrm{R}^n)}}{f}\int_0^t \frac{\big|\big(v_1(\cdot,\tau) - v_2(\cdot,\tau), (-\Delta)^{\gamma/2}\varphi_0(\cdot)\big)\big|}{\|(-\Delta)^{\gamma/2}\varphi_0\|_{\mathcal{D}^K(\mathrm{R}^n)}} \frac{\|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} d\tau \\ &\le \frac{(B+2b_2R)}{f}\cdot \|v_1 - v_2\|_\varepsilon \frac{\int_0^t \|(\widehat{G}_0\varphi)(\cdot,t-\tau)\|_{\mathcal{D}^K(\mathrm{R}^n)}\tau^{-\varepsilon}d\tau}{\|\varphi\|_{\mathcal{D}^K(\mathrm{R}^n)}} \le (2q_0R + q_1)t^{\alpha-2\varepsilon}\|v_1 - v_2\|_\varepsilon.\end{aligned}$$

If $(-\Delta)^{\gamma/2}\varphi_0(x) \equiv 0$, $x \in \mathbb{R}^n$, then $\big(v_1(\cdot,t) - v_2(\cdot,t), (-\Delta)^{\gamma/2}\varphi_0(\cdot)\big) = 0$ for all $t \in [0, T^*]$, and the factor 2 is absent in the obtained expression.

For $t \in [0, T^*]$ we have

$$(2q_0R + q_1)t^{\alpha-2\varepsilon} \leq \frac{2q_0R + q_1}{2(q_0+q_1)R} \leq \frac{2q_0 + q_1}{2(q_0+q_1)} < 1.$$

So, P is the contraction operator on $M_{R,\varepsilon}(Q^*)$, and by the Banach theorem we obtain the solvability of the equation (12) in $M^*_{R,\varepsilon} \subset \mathcal{D}'_C(Q^*)$. □

Theorem 3. *Under conditions* $F^{(\beta)} \in C(0,T]$, $\inf\limits_{t\in(0,T]} |F^{(\beta)}(t)| \neq 0$ *a solution* $(u, r) \in \mathcal{D}'_C(Q) \times C(0,T]$ *of the problem (1)–(3) is unique.*

Proof. Take two solutions (u_1, r_1), $(u_2, r_2) \in \mathcal{D}'_C(Q) \times C(0,T]$ of the problem (1)–(3) and substitute them in (1), (2). Putting $u = u_1 - u_2$, $r = r_1 - r_2$ obtain the Cauchy problem for the equation

$$u_t^{(\alpha)} = a^2(-\Delta)^{\gamma/2}u + r_2u_t^{(\beta)} + r{u_1}_t^{(\beta)} \tag{18}$$

with zero initial conditions. By the definition of solution

$$\big(u, \widehat{L}\psi\big) = \int_0^T \Big[r_2(t)\big(u_t^{(\beta)}(\cdot,t), \psi(\cdot,t)\big) + r(t)\big({u_1}_t^{(\beta)}(\cdot,t), \psi(\cdot,t)\big)\Big]dt \;\text{ for all }\; \psi \in \mathcal{D}(\bar{Q}).$$

According to [8], for each $\varrho \in \mathcal{D}(\bar{Q})$ there exists $\psi = \widehat{\mathcal{G}}_0\varrho \in \mathcal{D}(\bar{Q}_0)$ such that $\widehat{L}\psi = \varrho$ in Q. Then for each $\varrho \in \mathcal{D}(\bar{Q})$ we have

$$\int_0^T \Big(u(\cdot,t), \varrho(\cdot,t)\Big)dt = \int_0^T \Big(r_2(t)u_t^{(\beta)}(\cdot,t) + r(t){u_1}_t^{(\beta)}(\cdot,t), (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\Big)dt. \tag{19}$$

From the over-determination condition (3), by using (11), we find

$$a^2\Big(u(z,t), (-\Delta)^{\gamma/2}\varphi_0(z)\Big) = -r(t)F^{(\beta)}(t), \quad t \in (0,T], \tag{20}$$

and then, from (19), for all $\varrho \in \mathcal{D}(\bar{Q})$ we obtain the equation

$$\int_0^T \Big(u_t^{(\beta)}(\cdot,t), \varrho(\cdot,t) - r_2(t)(\widehat{\mathcal{G}}_0\varrho)(\cdot,t) + \frac{(-\Delta)^{\gamma/2}\varphi_0(\cdot)w_\varrho(t)}{F^{(\beta)}(t)}\Big)dt = 0, \tag{21}$$

where

$$\begin{aligned} w_\varrho(t) &= a^2\big({u_1}_t^{(\beta)}(\cdot,t), (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\big) \\ &= a^2\big(f_{-\beta}(t) * u_1(\cdot,t), (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\big) = a^2\big(u_1(\cdot,t), f_{-\beta}(t) \hat{*} (\widehat{\mathcal{G}}_0\varrho)(\cdot,t)\big) \end{aligned}$$

is the known function from $C(0,T]$,

$$\varrho(\cdot,t) - r_2(t)(\widehat{\mathcal{G}}_0\varrho)(\cdot,t) + \frac{(-\Delta)^{\gamma/2}\varphi_0(\cdot)w_\varrho(t)}{F^{(\beta)}(t)} \in \mathcal{D}(\mathbb{R}^n), \quad t \in (0,T]$$

is the continuous function in $t \in (0, T]$. So, for each $\varphi \in \mathcal{D}(\mathrm{R}^n)$, $\mu \in \mathcal{D}(0, T]$, $\mu(T) = 0$ there exists a unique solution $\varrho \in \mathcal{D}(\bar{Q})$ of the second type Volterra integral equation

$$\varrho(x,t) - r_2(t)(\widehat{\mathcal{G}}_0\varrho)(x,t) + \frac{(-\Delta)^{\gamma/2}\varphi_0(x)w_\varrho(t)}{F^{(\beta)}(t)} = \varphi(x)\mu(t), \quad (x,t) \in \bar{Q},$$

with integrable kernel. Then (21) implies that

$$\int_0^T \Big(u_t^{(\beta)}(\cdot,t), \varphi(\cdot)\Big)\mu(t)dt = 0 \ \text{ for all } \ \varphi \in \mathcal{D}(\mathrm{R}^n), \ \ \mu \in \mathcal{D}(0,T], \ \ \mu(T) = 0.$$

By the Dubua-Rejmon lemma we obtain

$$\big(u_t^{(\beta)}(\cdot,t), \varphi(\cdot)\big) = 0 \ \text{ for all } \ \varphi \in \mathcal{D}(\mathrm{R}^n), \ \ t \in (0,T].$$

Therefore, $u_t^{(\beta)} = 0$, i.e. $f_{-\beta}(t) * u(x,t) = 0$, i.e. $f_\beta(t) * f_{-\beta}(t) * u(x,t) = 0$, i.e. $u = 0$ in $\mathcal{D}'_C(Q)$, and (20) implies that $r(t) = 0$, $t \in (0, T]$. □

3 CONCLUSIONS

The inverse Cauchy problem for a time-space-fractional telegraph equation with given distributions in the right-hand sides has been studied. We have determinated a generalized solution u of direct Cauchy problem and unknown, depending on time variable, continuous minor coefficient r of the equation. The existence of a solution $(u,r) \in \mathcal{D}'_C(Q^*) \times C(0,T^*]$ is obtained for some $T^* \in (0,T]$. The uniqueness of a solution $(u,r) \in \mathcal{D}'_C(Q) \times C(0,T]$ is obtained for arbitrary $T > 0$.

Let $\mathcal{D}'_C(\bar{Q}) = \{v \in \mathcal{D}'(\bar{Q}) : (v(\cdot,t), \varphi(\cdot)) \in C[0,T] \ \text{ for all } \ \varphi \in \mathcal{D}(\mathrm{R}^n)\}$. The Green vector-function of the Cauchy problem for the operator $D_t^\alpha - A(x,D)$, where $A(x,D)$ is an elliptic differential expression of the second order with infinitely differentiable coefficients, has the exponential descending at infinity. So, unlike the case of the proposed problem (1)–(3), under assumptions $F_0, F_1, F_2 \in \mathcal{E}'(\mathrm{R}^n)$, $g \in C[0,T]$, $F, F^{(\beta)}, F^{(\alpha)} \in C[0,T]$, $F^{(\beta)}(t) \neq 0$, $t \in [0,T]$ and the compatibility conditions

$$(F_1, \varphi_0) = F(0), \quad (F_2, \varphi_0) = F'(0),$$

there exist $T^* \in (0,T]$ and the solution $(u,r) \in \mathcal{D}'_C(\bar{Q}^*) \times C[0,T^*]$ of the problem (1)–(3) with the operator $-A(x,D)$ instead of $a^2(-\Delta)^{\gamma/2}$.

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Досліджуємо обернену задачу Коші для рівняння

$$u_t^{(\alpha)} - r(t)u_t^{(\beta)} + a^2(-\Delta)^{\gamma/2}u = F_0(x)g(t), \quad (x,t) \in \mathrm{R}^n \times (0,T],$$

з дробовими похідними та заданими узагальненими функціями в правих частинах рівняння і початкових умов. Наше завдання полягає у визначенні пари функцій: узагальненого розв'язку u (неперервного за часом в узагальненому сенсі) та невідомого молодшого коефіцієнта $r(t)$. У статті встановлено однозначну розв'язність задачі.

Ключові слова і фрази: узагальнена функція, дробова похідна, обернена задача, вектор функція Гріна.