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DMYTRYSHYN R.I.

A MULTIDIMENSIONAL GENERALIZATION OF THE RUTISHAUSER *QD*-ALGORITHM

In this paper the regular multidimensional C-fraction with independent variables, which is a generalization of regular C-fraction, is considered. An algorithm of calculation of the coefficients of the regular multidimensional C-fraction with independent variables correspondence to a given formal multiple power series is constructed. Necessary and sufficient conditions of the existence of this algorithm are established. The above mentioned algorithm is a multidimensional generalization of the Rutishauser *qd*-algorithm.

Key words and phrases: regular multidimensional C-fraction with independent variables, correspondence, multiple power series, algorithm.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: dmytryshynr@hotmail.com

INTRODUCTION

In constructing the branched continued fractions for a given formal multiple power series the concept of correspondence is used. Some general theory of correspondence for functions of one variables is developed in [15, pp. 148–160] (see also [11, pp. 241–274]) and some aspects of it for functions of several variables are considered in [7], [6, pp. 107–109]. As a result, different types of functional fractions are constructed in [1–6, 8–10, 12–14, 16].

In the present paper we construct and investigate an algorithm for the expansion of a given formal multiple power series into a corresponding regular multidimensional C-fraction with independent variables, which is a generalization of the regular C-fraction [15, p. 128-129]. It is a further expansion of the results obtained in [2].

1 CORRESPONDENCE

Let $\mathcal{L}$ be set of all formal multiple power series of the form

$$L(\mathbf{z}) = \sum_{|m(N)|\geq 0} c_{m(N)}\mathbf{z}^{m(N)}, \tag{1}$$

where $m(N) = m_1, m_2, \ldots, m_N$ is multiindex, $m_i \in \mathbb{Z}_+$, $1 \leq i \leq N$, $0(N) = 0, 0, \ldots, 0$, $|m(N)| = m_1 + m_2 + \cdots + m_N$, $c_{m(N)} \in \mathbb{C}$, $\mathbf{z}^{m(N)} = z_1^{m_1} z_2^{m_2} \cdot \ldots \cdot z_N^{m_N}$, $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^N$. Obviously, this set forms a ring with unity respect to the operations addition and multiplication of series. We define the mapping $\lambda : \mathcal{L} \to \mathbb{N}_0 \cup \{\infty\}$ as follows: $\lambda(L(\mathbf{z})) = \infty$, if $L(\mathbf{z}) \equiv 0$;

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$\lambda(L(\mathbf{z})) = n$, if $L(\mathbf{z}) \not\equiv 0$, where n is the smallest degree of homogeneous polynomial for which $c_{m(N)} \neq 0$, that is $n = |m(N)|$. We consider the sequence of rational functions

$$f_n(\mathbf{z}) = \frac{P_{m_n}(\mathbf{z})}{Q_{l_n}(\mathbf{z})}, \quad n \geq 1,$$

where $P_{m_n}(\mathbf{z})$, $Q_{l_n}(\mathbf{z})$ are polynomials of degrees m_n and l_n respectively, $\mathbf{z} \in \mathbb{C}^N$, moreover, $Q_{l_n}(0,0,\dots,0) \neq 0$.

The sequence $\{f_n(\mathbf{z})\}$ corresponds to series (1) at $\mathbf{z} = (0,0,\dots,0)$, if

$$\lim_{n\to+\infty} \lambda(L(\mathbf{z}) - L(f_n(\mathbf{z}))) = +\infty,$$

where $L(f_n(\mathbf{z}))$ is expansion of function $f_n(\mathbf{z})$ into Taylor series at $\mathbf{z} = (0,0,\dots,0)$. The order of correspondence of $f_n(\mathbf{z})$ is defined by the formula $\nu_n = \lambda(L(\mathbf{z}) - L(f_n(\mathbf{z})))$. This means that the expansion $f_n(\mathbf{z})$ into formal multiple power series coincides with $L(\mathbf{z})$ for all homogeneous polynomials to the degree $(\nu_n - 1)$ inclusively.

Let us introduce the following set of multiindices

$$\mathcal{J} = \{m(N) :\ m(N) = m_1, m_2, \dots, m_N,\ m_p \in \mathbb{Z}_+,\ 1 \leq p \leq N\}.$$

And now, let us define arithmetical operations on the set $\mathcal{J}$ componentwise. If

$$r(N) = r_1, r_2, \dots, r_N \in \mathcal{J}, \quad s(N) = s_1, s_2, \dots, s_N \in \mathcal{J}, \quad k \in \mathbb{Z}_+,$$

then

$$r(N) + s(N) = r_1 + s_1, r_2 + s_2, \dots, r_N + s_N, \quad kr(N) = kr_1, kr_2, \dots, kr_N.$$

We consider the regular multidimensional C-fraction with independent variables

$$\frac{a_0}{1 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \sum\limits_{i_k=1}^{i_{k-1}} \frac{a_{i(k)} z_{i_k}}{1}} = \frac{a_0}{1 + \sum\limits_{i_1=1}^{N} \cfrac{a_{i(1)} z_{i_1}}{1 + \sum\limits_{i_2=1}^{i_1} \cfrac{a_{i(2)} z_{i_2}}{1 + \ddots}}}, \tag{2}$$

where $i(k) = i_1, i_2, \dots, i_k$ is multiindex, $a_0 \neq 0$, $a_{i(k)} \neq 0$, $k \geq 1$, $1 \leq i_n \leq i_{n-1}$, $1 \leq n \leq k$, $i_0 = N$, $\mathbf{z} \in \mathbb{C}^N$.

Let $e_0 = 0, 0, \dots, 0$, $e_r = \delta_{r,1}, \delta_{r,2}, \dots, \delta_{r,N}$ be a multiindex, $\delta_{r,s}$ be a Kronecker symbol, $1 \leq r, s \leq N$. Let us introduce the following sets of multiindices

$$\begin{aligned} \mathcal{I} &= \{i(k) :\ i(k) = i_1, i_2, \dots, i_k,\ 1 \leq i_p \leq i_{p-1},\ 1 \leq p \leq k,\ k \geq 1,\ i_0 = N\}, \\ \mathcal{I}^* &= \{\mathbf{i}_{i(k)}^N :\ \mathbf{i}_{i(k)}^N = e_{i_1} + e_{i_2} + \dots + e_{i_k},\ i(k) \in \mathcal{I}\} \end{aligned}$$

and the mapping $\varphi : \mathcal{I} \to \mathcal{I}^*$, such that $\varphi(i(k)) = \mathbf{i}_{i(k)}^N$ for all $i(k) \in \mathcal{I}$ (we can show that the mapping φ is bijective).

Let $a_0 = b_0$, $a_{i(k)} = b_{\mathbf{i}_{i(k)}^N}$, $i(k) \in \mathcal{I}$, $\mathbf{i}_{i(k)}^N \in \mathcal{I}^*$. Then we write fraction (2) in the form

$$b_0 \left(1 + \mathop{\mathrm{D}}\limits_{k=1}^{\infty} \sum_{i_k=1}^{i_{k-1}} \frac{b_{\mathbf{i}_{i(k)}^N} z_{i_k}}{1}\right)^{-1}, \tag{3}$$

where $b_0 \neq 0$, $b_{\mathbf{i}^N_{i(k)}} \neq 0$, $\mathbf{i}^N_{i(k)} \in \mathcal{I}^*$, $\mathbf{z} \in \mathbb{C}^N$.

Let

$$g_n(\mathbf{z}) = b_0 \left(1 + \mathop{\mathrm{D}}_{k=1}^{n-1} \sum_{i_k=1}^{i_{k-1}} \frac{b_{\mathbf{i}^N_{i(k)}} z_{i_k}}{1}\right)^{-1}$$

be the nth approximant of regular multidimensional C-fraction with independent variables (3), $n \geq 1$.

The correspondence of fraction (3) to series (1) means that the sequence of approximants $\{g_n(\mathbf{z})\}$ corresponds to $L(\mathbf{z})$.

2 ALGORITHM

We shall construct and investigate the algorithm for the expansion of the formal multiple power series (1) into the corresponding regular multidimensional C-fraction with independent variables (3).

Let $c_{0(N)} \neq 0$ and

$$R_{e_0}(\mathbf{z}) = \sum_{|m(N)| \geq 0} \frac{c_{m(N)}}{c_{0(N)}} \mathbf{z}^{m(N)}.$$

Next, let

$$R'_{e_0}(\mathbf{z}) = \sum_{|m(N)| \geq 0} c^{(e_0)}_{m(N)} \mathbf{z}^{m(N)} \tag{4}$$

be reciprocal to series $R_{e_0}(\mathbf{z})$. The coefficient of FMPS (4) are uniquely determined by recurrent formulas

$$c^{(e_0)}_{m(N)} = -\sum_{|r(N)|=1}^{|m(N)|} c^{(e_0)}_{m(N)-r(N)} \frac{c_{r(N)}}{c_{0(N)}}, \quad m_j \geq 0,\ 1 \leq j \leq N,\ |m(N)| \geq 1, \tag{5}$$

where $c^{(e_0)}_{0(N)} = 1$, moreover, $c^{(e_0)}_{m(N)} = 0$, if here exist an index j, $1 \leq j \leq N$, such that $n_j < 0$.

By condition $c^{(e_0)}_{e_j} \neq 0$, $2 \leq j \leq N$, we write the series (4) in the form

$$R'_{e_0}(\mathbf{z}) = P_{e_1}(z_1) + \sum_{j=2}^{N} c^{(e_0)}_{e_j} z_j R_{e_j}(\mathbf{z}),$$

where

$$P_{e_1}(z_1) = \sum_{\substack{m_1=0 \\ m_j=0,\ 2 \leq j \leq N}}^{\infty} c^{(e_0)}_{m(N)} z_1^{m_1}, \quad R_{e_j}(\mathbf{z}) = \sum_{\substack{|r(N)| \geq 0 \\ r_i=0,\ j+1 \leq i \leq N}} \frac{c^{(e_0)}_{e_j+r(N)}}{c^{(e_0)}_{e_j}} \mathbf{z}^{r(N)}.$$

Then $L(\mathbf{z})$ can be written

$$L(\mathbf{z}) = \frac{c_{0(N)}}{P_{e_1}(z_1) + \sum_{j=2}^{N} c^{(e_0)}_{e_j} z_j R_{e_j}(\mathbf{z})}.$$

Let

$$\sum_{\substack{m_1=0 \\ m_j=0,\ 2 \leq j \leq N}}^{\infty} c_{m(N)} z_1^{m_1}$$

be a normal series (for the notion of normality of formal power series, see [15, pp. 185-190]). Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}}$, $i_p = 1$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table for $h = 0$:

$$\begin{array}{ccccccc}
 & q^{(0)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & & & & \\
e^{(1)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} & & e^{(0)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & & & & \\
 & q^{(1)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & q^{(0)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & & & \\
e^{(2)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} & & e^{(1)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & e^{(0)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & & \\
 & q^{(2)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & & q^{(1)}_{\mathbf{i}^N_{i(2)}+\mathbf{j}^N_{j(h)}} & \vdots & \ddots & \\
e^{(3)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} & & e^{(2)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & \vdots & & & \\
\vdots & q^{(3)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} & \vdots & & & & \\
 & \vdots & & & & &
\end{array} \tag{6}$$

the entries of which are defined by the initial conditions

$$e^{(n)}_{\mathbf{i}^N_{i(0)}+\mathbf{j}^N_{j(h)}} = 0, \quad q^{(n)}_{\mathbf{i}^N_{i(1)}+\mathbf{j}^N_{j(h)}} = \frac{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{m(N)+e_{i_1}+e_{j_h}}}{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{m(N)+e_{j_h}}}, \quad |m(N)| = m_{i_1} = n, \ \ n \ge 0, \tag{7}$$

moreover,

$$q^{(n)}_{\mathbf{i}^N_{i(1)}} = \frac{c_{m(N)+e_{i_1}}}{c_{m(N)}}, \quad |m(N)| = m_{i_1} = n, \ n \ge 0,$$

and the rhombus rule

$$\begin{aligned}
e^{(n)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} + q^{(n)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} &= q^{(n+1)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} + e^{(n+1)}_{\mathbf{i}^N_{i(r-1)}+\mathbf{j}^N_{j(h)}}, \quad r \ge 1, \ n \ge 0, \\
e^{(n)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} q^{(n)}_{\mathbf{i}^N_{i(r+1)}+\mathbf{j}^N_{j(h)}} &= q^{(n+1)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}} e^{(n+1)}_{\mathbf{i}^N_{i(r)}+\mathbf{j}^N_{j(h)}}, \quad r \ge 1, \ n \ge 0,
\end{aligned} \tag{8}$$

The procedure of calculation of the elements of table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) is called the Rutishauser qd-algorithm [15, p. 227].

We put $b_{\mathbf{i}^N_{i(2k-1)}} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}}$, $b_{\mathbf{i}^N_{i(2k)}} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}}$, $i_p = 1$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_1=0 \\ m_j=0,\, 2\le j\le N}}^{\infty} \frac{c_{m(N)}}{c_{0(N)}} z_1^{m_1} \sim \left(1 + \mathop{D}_{\substack{k=1 \\ i_p=1,\, 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}} z_1}{1} \right)^{-1}.$$

Here the symbol "$\sim$" means the correspondence between the series and the fraction. Moreover, according to Lemma 3 [4] we have

$$P_{e_1}(z_1) \sim 1 + \mathop{D}_{\substack{k=1 \\ i_p=1,\, 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}} z_1}{1},$$

since the series $P_{e_1}(z_1)$ is reciprocal to series

$$\sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} \frac{c_{m(N)}}{c_{0(N)}} z_1^{m_1}.$$

Thus we can write

$$L(\mathbf{z}) \sim \cfrac{c_{0(N)}}{1 + \underset{\substack{k=1\\ i_p=1,\ 1\le p\le k}}{\overset{\infty}{\mathrm{D}}} \cfrac{b_{\mathbf{i}^N_{i(k)}} z_1}{1} + \sum_{j=2}^{N} c_{e_j}^{(e_0)} z_j R_{e_j}(\mathbf{z})}.$$

Let l be an arbitrary natural number, moreover, $2 \le l \le N$. Next, let

$$\sum_{\substack{m_l=0\\ m_j=0,\ j\ne l,\ 1\le j\le N}}^{\infty} c_{m(N)} z_l^{m_l}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}}$, $i_p = l$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for $h = 0$.

We put $b'_{\mathbf{i}^N_{i(2k-1)}} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}}$, $b'_{\mathbf{i}^N_{i(2k)}} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}}$, $i_p = l$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_l=0\\ m_j=0,\ j\ne l,\ 1\le j\le N}}^{\infty} \frac{c_{m(N)} z_l^{m_l}}{c_{0(N)}} \sim \left(1 + \underset{\substack{k=1\\ i_p=l,\ 1\le p\le k}}{\overset{\infty}{\mathrm{D}}} \frac{b'_{\mathbf{i}^N_{i(k)}} z_l}{1}\right)^{-1}.$$

Since

$$c^{(e_0)}_{m(N)} = -\frac{c_{m(N)}}{c_{0(N)}} = b'_{\mathbf{i}^N_{i(1)}}, \quad m_l = 1,\ m_j = 0,\ j \ne l,\ 1 \le j \le N,\ i_1 = l,$$

then we put $b_{\mathbf{i}^N_{i(1)}} = b'_{\mathbf{i}^N_{i(1)}}$, $i_1 = l$.

Thus we can write

$$L(\mathbf{z}) \sim \cfrac{c_{0(N)}}{1 + \underset{\substack{k=1\\ i_p=1,\ 1\le p\le k}}{\overset{\infty}{\mathrm{D}}} \cfrac{b_{\mathbf{i}^N_{i(k)}} z_1}{1} + \sum_{j_1=2}^{N} b_{\mathbf{j}^N_{j(1)}} z_{j_1} R_{\mathbf{j}^N_{j(1)}}(\mathbf{z})}.$$

Again, let l be an arbitrary natural number, moreover, $2 \le l \le N$. Next, let

$$R'_{e_l}(\mathbf{z}) = \sum_{\substack{|m(N)|\ge 0\\ m_i=0,\ l+1\le i\le N}} c^{(e_l)}_{m(N)} \mathbf{z}^{m(N)} \tag{9}$$

be reciprocal to series $R_{e_l}(\mathbf{z})$. The coefficients of series (9) are uniquely determined by recurrent formulas for $m_i = 0,\ j_h + 1 \le i \le N,\ |m(N)| \ge 1$, and $\mathbf{j}^N_{j(h)} = e_l$

$$c^{(\mathbf{j}^N_{j(h)})}_{m(N)} = -\sum_{|r(N)|=1}^{|m(N)|} c^{(\mathbf{j}^N_{j(h)})}_{m(N)-r(N)} \frac{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{r(N)+e_{j_h}}}{c^{(\mathbf{j}^N_{j(h)}-e_{j_h})}_{e_{j_h}}}, \tag{10}$$

where $c^{(\mathbf{j}^N_{j(h)})}_{0(N)} = 1$, moreover, $c^{(\mathbf{j}^N_{j(h)})}_{n(N)} = 0$, if here exist an index p, $1 \le p \le N$, such that $n_p < 0$.

By condition $c^{(e_l)}_{e_j} \neq 0$, $2 \le j \le l$, we write the series (9) in the form

$$R'_{e_l}(\mathbf{z}) = P_{e_l+e_1}(z_1) + \sum_{j=2}^{l} c^{(e_l)}_{e_j} z_j R_{e_l+e_j}(\mathbf{z}),$$

where

$$P_{e_l+e_1}(z_1) = \sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} c^{(e_l)}_{m(N)} z_1^{m_1}, \quad R_{e_l+e_j}(\mathbf{z}) = \sum_{\substack{|r(N)|\ge 0\\ r_i=0,\ j+1\le i\le N}} \frac{c^{(e_l)}_{e_j+r(N)}}{c^{(e_l)}_{e_j}} \mathbf{z}^{r(N)}.$$

Then $R_{e_l}(\mathbf{z})$ can be written as follows

$$R_{e_l}(\mathbf{z}) = \left(P_{e_l+e_1}(z_1) + \sum_{j=2}^{l} c^{(e_l)}_{e_j} z_j R_{e_l+e_j}(\mathbf{z}) \right)^{-1}.$$

Let

$$\sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} c^{(e_0)}_{m(N)+e_l} z_1^{m_1}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $i_p = 1$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for $\mathbf{j}^N_{j(h)} = e_l$.

We put $b_{\mathbf{i}^N_{i(2k-1)}+e_l} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}+e_l}$, $b_{\mathbf{i}^N_{i(2k)}+e_l} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}+e_l}$, $i_p = l$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_1^{m_1} \sim \left(1 + \mathop{\mathrm{D}}_{\substack{k=1\\ i_p=1,\ 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}+e_l} z_1}{1} \right)^{-1}.$$

Since the series $P_{e_l+e_1}(z_1)$ is reciprocal to series

$$\sum_{\substack{m_1=0\\ m_j=0,\ 2\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_1^{m_1},$$

then according to Lemma 3 [4] we obtain

$$P_{e_l+e_1}(z_1) \sim 1 + \mathop{\mathrm{D}}_{\substack{k=1 \\ i_p=1,\, 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}+e_l} z_1}{1}.$$

Let t be an arbitrary natural number, moreover, $2 \le t \le l-1$. Next, let

$$\sum_{\substack{m_t=0 \\ m_j=0,\, j\neq t,\, 1\le j\le N}}^{\infty} c^{(e_0)}_{m(N)+e_l} z_t^{m_t}$$

be a normal series. Then according to Theorem 7.5 [15, pp. 228-229] there exist the real numbers $q^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $e^{(n)}_{\mathbf{i}^N_{i(k)}+e_l}$, $i_p = t$, $1 \le p \le k$, $k \ge 1$, $n \ge 0$, of qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8) for $\mathbf{j}^N_{j(h)} = e_l$.

We put $b'_{\mathbf{i}^N_{i(2k-1)}+e_l} = -q^{(0)}_{\mathbf{i}^N_{i(2k-1)}+e_l}$, $b'_{\mathbf{i}^N_{i(2k)}+e_l} = -e^{(0)}_{\mathbf{i}^N_{i(2k)}+e_l}$, $i_p = t$, $1 \le p \le k$, $k \ge 1$. According to Theorem 7.7 [15, pp. 230-231]

$$\sum_{\substack{m_t=0 \\ m_j=0,\, j\neq t,\, 1\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_t^{m_t} \sim \left(1 + \mathop{\mathrm{D}}_{\substack{k=1 \\ i_p=t,\, 1\le p\le k}}^{\infty} \frac{b'_{\mathbf{i}^N_{i(k)}+e_l} z_t}{1}\right)^{-1}.$$

Since the series $P_{e_l+e_r}(z_t)$ is reciprocal to series

$$\sum_{\substack{m_t=0 \\ m_j=0,\, j\neq t,\, 1\le j\le N}}^{\infty} \frac{c^{(e_0)}_{m(N)+e_l}}{c^{(e_0)}_{e_l}} z_t^{m_t},$$

then according to Lemma 3 [4] we obtain

$$P_{e_l+e_r}(z_t) \sim 1 + \mathop{\mathrm{D}}_{\substack{k=1 \\ i_p=t,\, 1\le p\le k}}^{\infty} \frac{b'_{\mathbf{i}^N_{i(k)}+e_l} z_t}{1}.$$

Since

$$c^{(e_l)}_{e_t} = -\frac{c^{(e_0)}_{e_l+e_t}}{c^{(e_0)}_{e_l}} = -\frac{c_{0(N)} c_{e_l+e_t} - c^2_{e_l}}{c_{e_l} c_{0(N)}} = b'_{e_l+e_t}, \quad c^{(e_l)}_{e_l} = -\frac{c^{(e_0)}_{2e_l}}{c^{(e_0)}_{e_l}} = -\frac{c_{0(N)} c_{2e_l} - c^2_{e_l}}{c_{e_l} c_{0(N)}} = b'_{2e_l},$$

than we put $b_{e_l+e_t} = b'_{e_l+e_t}$, $b_{2e_l} = b'_{2e_l}$.

Thus we can write

$$L(\mathbf{z}) \sim \cfrac{c_{0(N)}}{Q_{\mathbf{j}^N_{j(0)}}(z_1) + \sum\limits_{j_1=2}^{N} \cfrac{b_{\mathbf{j}^N_{j(1)}} z_{j_1}}{Q_{\mathbf{j}^N_{j(1)}}(z_1) + \sum\limits_{j_2=2}^{j_1} b_{\mathbf{j}^N_{j(2)}} z_{j_2} R_{\mathbf{j}^N_{j(2)}}(\mathbf{z})}},$$

where

$$Q_{\mathbf{j}^N_{j(h)}}(z_1) = 1 + \mathop{\mathrm{D}}_{\substack{k=1\\ i_p=1,\ 1\le p\le k}}^{\infty} \frac{b_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}} z_1}{1}, \quad h \ge 0,$$

moreover, $j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$, if $h \ge 1$.

Next, computing the coefficients

$$c^{(\mathbf{j}^N_{j(h)})}_{m(N)}, \quad m_i = 0,\ j_h + 1 \le i \le N,\ |m(N)| \ge 1,\ j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*,$$

by recurrent formulas (10) and continuing process of iteration under the conditions that the series

$$\sum_{\substack{m_l=0\\ m_i=0,\ i\neq l,\ 1\le i\le N}}^{\infty} c_{m(N)} z_l^{m_l}, \qquad \sum_{\substack{m_t=0\\ m_i=0,\ i\neq t,\ 1\le i\le N}}^{\infty} c^{(e_0)}_{m(N)+e_p} z_t^{m_t}, \qquad \sum_{\substack{m_r=0\\ m_i=0,\ i\neq r,\ 1\le i\le N}}^{\infty} c^{(\mathbf{j}^N_{j(h)})}_{m(N)+e_{j_h}} z_r^{m_r}, \tag{11}$$

where $1 \le l \le N,\ 1 \le t \le p-1,\ 2 \le p \le N,\ 1 \le r \le j_h - 1,\ j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$, are normal, for series (1) we obtain fraction (3), where $c_0 = c_{0(N)}$, $b_{\mathbf{i}^N_{i(2k-1)}+\mathbf{j}^N_{j(h)}} = -q^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $b_{\mathbf{i}^N_{i(2k)}+\mathbf{j}^N_{j(h)}} = -e^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $i_p = n,\ 1 \le p \le k,\ 1 \le n \le j_h - 1,\ k \ge 1,\ j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$ (the numbers $q^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $e^{(0)}_{\mathbf{i}^N_{i(k)}+\mathbf{j}^N_{j(h)}}$, $i_p = n,\ 1 \le p \le k,\ 1 \le n \le j_h - 1,\ k \ge 1,\ j_r \neq 1,\ 1 \le r \le h,\ \mathbf{j}^N_{j(h)} \in \mathcal{I}^*$, are the diagonal elements of the qd-table (6) the entries of which are defined by the initial conditions (7) and the rhombus rule (8)).

Thus, if the coefficients of the formal multiple power series (1) are given, then the recurrent algorithm of calculation of the coefficients of the regular multidimensional C-fraction with independent variables (3) is constructed. This algorithm is a multidimensional generalization of Rutishauser qd-algorithm [15, p. 227]. The correspondence of fraction (3) to series (1) can be proved by a scheme proposed in [5].

Hence, the following theorem holds:

Theorem. *The regular multidimensional C-fraction with independent variables* (3) *corresponds to the given formal multiple power series* (1) *if and only if the formal power series* (11) *are normal.*

We remark that some examples of functions of two variables represented by regular two-dimensional C-fractions with independent variables are given in [2].

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Розглядається регулярний багатовимірний C-дріб з нерівнозначними змінними, який є узагальненням регулярного C-дробу. Побудовано алгоритм обчислення коефіцієнтів багатовимірного C-дробу з нерівнозначними змінними, відповідного заданому формальному кратному степеневому ряду, який є узагальненням *qd*-алгоритму Рутисхаузера. Встановлено необхідні та достатні умови існування такого алгоритму.

Ключові слова і фрази: регулярний багатовимірний C-дріб з нерівнозначними змінними, відповідність, кратний степеневий ряд, алгоритм.