ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2016, 8 (2), 239–250
doi:10.15330/cmp.8.2.239-250

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2016, Т.8, №2, С.239–250

DOBUSHOVSKYY M.S., SHEREMETA M.M.

# ANALOGUES OF WHITTAKER'S THEOREM FOR LAPLACE-STIELTJES INTEGRALS

Lower estimates on a sequence for the maximum of the integrand of Laplace-Stieltjes integrals are found. Using these estimates we obtained analogues of Whittaker's theorem for entire functions given by lacunary power series.

*Key words and phrases:* Laplace-Stieltjes integral, maximum of integrand, Whittaker's theorem.

Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: mdobush19@gmail.com (Dobushovskyy M.S.), m_m_sheremeta@list.ru (Sheremeta M.M.)

## INTRODUCTION

For an entire function

$$g(z)=\sum_{n=0}^{\infty} a_n z^{\lambda_n}, \quad z=re^{i\theta}, \tag{1}$$

let $M_g(r)=\max\{|g(z)|:\ |z|=r\}$ and $\varrho=\overline{\lim\limits_{r\to+\infty}}\dfrac{\ln\ln M_g(r)}{\ln r}$, $\lambda=\underline{\lim\limits_{r\to+\infty}}\dfrac{\ln\ln M_g(r)}{\ln r}$ be the order and the lower order of $g$ correspondingly. J.M. Whittaker [1] has proved that $\lambda\le\varrho\beta$, where $\beta=\underline{\lim\limits_{n\to+\infty}}(\ln\lambda_n)/\ln\lambda_{n+1}$. For an analytic in $\{z:\ |z|<1\}$ function (1) of the order $\varrho_0=\overline{\lim\limits_{r\uparrow 1}}\dfrac{\ln\ln M_g(r)}{-\ln(1-r)}$ and the lower order $\lambda_0=\underline{\lim\limits_{r\uparrow 1}}\dfrac{\ln\ln M_g(r)}{-\ln(1-r)}$ L.R. Sons [2] tried to prove that $\lambda_0+1\le(\varrho_0+1)\beta$. In [3] this result is disproved and it is showed that $\lambda_0\le\varrho_0\beta$, i. e. absolute analogue of Whittaker's theorem is valid. Moreover, in [3] it is obtained analogues of Whittaker's theorem for Dirichlet series $\sum\limits_{n=0}^{\infty}a_ne^{\lambda_n s}$, $s=\sigma+it$, with an arbitrary abscissa of the absolute convergence $\sigma_a=A\in(-\infty,+\infty]$, where $0=\lambda_0<\lambda_n\uparrow+\infty$, $n\to\infty$.

Here we investigate similar problems for Laplace-Stieltjes integrals.

## 1 MAIN RESULTS

Let $V$ be the class of all nonnegative nondecreasing unbounded continuous on the right functions $F$ on $[0,+\infty)$. We say that $F\in V(l)$ if $F\in V$ and $F(x)-F(x-0)\le l<+\infty$ for all $x\ge 0$.

For a nonnegative function $f$ on $[0,+\infty)$ the integral

$$I(\sigma)=\int_0^{\infty} f(x)e^{x\sigma}dF(x), \quad \sigma\in\mathbb{R}, \tag{2}$$

УДК 517.5
2010 *Mathematics Subject Classification:* 33B50, 44A10.

is called of Laplace-Stieltjes [4]. Integral (1) is a direct generalisation of the ordinary Laplace integral $I(\sigma) = \int_0^\infty f(x)e^{x\sigma}dx$ and of the Dirichlet series $\sum\limits_{n=0}^{\infty} a_n e^{\lambda_n\sigma}$ with nonnegative coefficients $a_n$ and exponents $\lambda_n$, $0 \le \lambda_n \uparrow +\infty$, $n \to \infty$, if we choose $F(x) = n(x) = \sum\limits_{\lambda_n \le x} 1$ and $f(\lambda_n) = a_n \ge 0$ for all $n \ge 0$. The maximal therm of this Dirichlet series is defined by formula $\mu(\sigma) = \max\{a_n e^{\lambda_n\sigma} : n \ge 0\}$.

By $\Omega(A)$ we denote the class of all positive unbounded on $(-\infty, A)$ functions $\Phi$ such that the derivative $\Phi'$ is positive continuously differentiable and increasing to $+\infty$ on $(-\infty, A)$. From now on, we denote by $\varphi$ the inverse function to $\Phi'$, and let $\Psi(x) = x - \Phi(x)/\Phi'(x)$ be the function associated with $\Phi$ in the sense of Newton. It is clear that the function $\varphi$ is continuously differentiable and increasing to $A$ on $(0, +\infty)$. The function $\Psi$ is [4–6] continuously differentiable and increasing to $A$ on $(-\infty, A)$.

For $\Phi \in \Omega(A)$ and $0 < a < b < +\infty$ we put

$$G_1(a, b, \Phi) = \frac{ab}{b-a}\int_a^b \frac{\Phi(\varphi(t)}{t^2}dt, \quad G_2(a, b, \Phi) = \Phi\left(\frac{1}{b-a}\int_a^b \varphi(t)dt\right).$$

It is known [5] that $G_1(a, b, \Phi) < G_2(a, b, \Phi)$, and in [3] the following Lemma is proved.

**Lemma 1.** *Let $(x_k)$ be an increasing to $+\infty$ sequence of positive numbers, $\Phi \in \Omega(A)$ and $\mu_D(\sigma)$ be the maximal term of formal Dirichlet series*

$$D(s) = \sum_{k=1}^{\infty} \exp\{-x_k\Psi(\varphi(x_k)) + sx_k\}, \quad s = \sigma + it.$$

*Then*

$$\overline{\lim_{\sigma\uparrow A}} \frac{\ln \mu_D(\sigma)}{\Phi(\sigma)} = 1, \quad \overline{\lim_{\sigma\uparrow A}} \frac{\ln\ln \mu_D(\sigma)}{\ln \Phi(\sigma)} = 1, \tag{3}$$

$$\underline{\lim_{\sigma\uparrow A}} \frac{\ln \mu_D(\sigma)}{\Phi(\sigma)} = \underline{\lim_{k\to\infty}} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)} \tag{4}$$

*and if*

$$\ln \mu_D(\sigma) + \left(\frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1\right)\ln \Phi(\sigma) \ge 0, \quad \sigma \in [\sigma_0, A), \tag{5}$$

*then*

$$\underline{\lim_{\sigma\uparrow A}} \frac{\ln\ln \mu_D(\sigma)}{\ln \Phi(\sigma)} = \underline{\lim_{k\to\infty}} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \tag{6}$$

It is clear that integral (2) either converges for all $\sigma \in \mathbb{R}$ or diverges for all $\sigma \in \mathbb{R}$ or there exists a number $\sigma_c$ such that integral (2) converges for $\sigma < \sigma_c$ and diverges for $\sigma > \sigma_c$. In the latter case the number $\sigma_c$ is called abscissa of the convergence of integral (2). If integral (2) converges for all $\sigma \in \mathbb{R}$ then we put $\sigma_c = +\infty$, and if it diverges for all $\sigma \in \mathbb{R}$ then we put $\sigma_c = -\infty$.

Let

$$\mu(\sigma, I) = \sup\{f(x)e^{x\sigma} : x \ge 0\}, \quad \sigma \in \mathbb{R},$$

be the maximum of the integrand. Then either $\mu(\sigma, I) < +\infty$ for all $\sigma \in \mathbb{R}$ or $\mu(\sigma, I) = +\infty$ for all $\sigma \in \mathbb{R}$ or there exists a number $\sigma_\mu$ such that $\mu(\sigma, I) < +\infty$ for all $\sigma < \sigma_\mu$ and $\mu(\sigma, I) = +\infty$

for for all $\sigma > \sigma_\mu$. By analogy the number $\sigma_\mu$ is called abscissa of maximum of the integrand. It is well known ( [4]) that if $F \in V$ and $\ln F(x) = o(x)$ as $x \to +\infty$ then $\sigma_c \geq \sigma_\mu$.

For each Dirichlet series $\sigma_c \leq \sigma_\mu$. In general case this inequality can be not executed. We will say in this connection as in [4] that a nonnegative function $f$ has regular variation in regard to $F$ if there exist $a \geq 0$, $b \geq 0$ and $h > 0$ such that for all $x \geq a$

$$\int\limits_{x-a}^{x+b} f(t)dF(t) \geq hf(x). \tag{7}$$

In [4] it is proved that if $F \in V$ and $f$ has regular variation in regard to $F$ then $\sigma_c \leq \sigma_\mu$. We need also the following lemma.

**Lemma 2** ( [4]). *Let $\sigma_\mu = A \in (-\infty, +\infty]$ and $\Phi \in \Omega(A)$. In order that $\ln \mu(\sigma, I) \leq \Phi(\sigma)$ for all $\sigma \in [\sigma_0, A)$, it is necessary and sufficient that $\ln f(x) \leq -x\Psi(\varphi(x))$ for all $x \geq x_0$.*

Let $L$ be the class of all positive continuous functions $\alpha$ increasing to $+\infty$ on $(x_0, +\infty)$, $x_0 \geq -\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$, and $\alpha \in L_{si}$ if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$.

Using Lemmas 1 and 2 first we will prove the following theorem.

**Theorem 1.** *Let $\sigma_\mu = +\infty$, $\Phi \in \Omega(+\infty)$, $\ln \mu(\sigma, I) \leq \Phi(\sigma)$ for all $\sigma \geq \sigma_0$ and $X = (x_k)$ be a some sequence of positive numbers increasing to $+\infty$. Suppose that $f$ is a nonincreasing function. Then:*

*1) if either $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ and $\Phi \in L^0$, or $x_{k+1} - x_k \leq H < +\infty$ for all $k \geq 0$, or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ and $\Phi \in L^0$, then*

$$\varlimsup_{\sigma\to+\infty} \frac{\ln \mu(\sigma, I)}{\Phi(\sigma)} \leq \varlimsup_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}; \tag{8}$$

*2) if*

$$\ln \sigma + \left(\frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1\right) \ln \Phi(\sigma) \geq q > -\infty, \quad \sigma \geq \sigma_0, \tag{9}$$

*and either $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ and $\ln \Phi \in L^0$, or $\ln f(x_k) \leq a \ln f(x_{k+1})$, $0 < a < 1$, and $\ln \Phi \in L_{si}$, or $x_{k+1} - x_k \leq H < +\infty$ for all $k \geq 0$, or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ and $\Phi \in L^0$ or $x_{k+1} \leq Ax_k$ for all $k \geq 0$ and $\ln \Phi \in L_{si}$ then*

$$\varlimsup_{\sigma\to+\infty} \frac{\ln \ln \mu(\sigma, I)}{\ln \Phi(\sigma)} \leq \varlimsup_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \tag{10}$$

*Proof.* At first we remark that in view of the condition $\sigma_\mu = +\infty$ we have $f(x) \to 0$ as $x \to +\infty$ and $\sigma = o(\ln \mu(\sigma, I))$ as $\sigma \to +\infty$. Now, we put $x_0 = 0$ and $\mu(\sigma, I; X) = \max\{f(x_k)e^{\sigma x_k} : k \geq 0\}$. Clearly,

$$\ln \mu(\sigma, I) = \sup_{x\geq 0}(\ln f(x) + \sigma x) \geq \sup_{k\geq 0}(\ln f(x_k) + \sigma x_k) = \ln \mu(\sigma, I, X). \tag{11}$$

Therefore, $\ln\mu(\sigma, I; X) \le \Phi(\sigma)$ for all $\sigma \ge \sigma_0$ and by Lemma 2 $\ln f(x_k) \le -x_k\Psi(\varphi(x_k))$ for all $k \ge k_0$. Hence it follows that $\ln\mu(\sigma, I; X) \le \ln\mu_D(r)$ for $\sigma \ge \sigma_0$. Therefore, by Lemma 1 from (4) we obtain

$$\varliminf_{\sigma\to+\infty} \frac{\ln\mu(\sigma, I; X)}{\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}. \tag{12}$$

On the other hand for $\sigma > 0$

$$\ln\mu(\sigma, I) = \max_{k\ge 0} \sup_{x_k \le x < x_{k+1}} (\ln f(x) + x\sigma) \le \max_{k\ge 0}(\ln f(x_k) + x_{k+1}\sigma). \tag{13}$$

If $\ln f(x_k) = (1 + o(1)) \ln f(x_{k+1})$ as $k \to \infty$ then for every $\varepsilon > 0$ we have $\ln f(x_k) \le (\ln f(x_{k+1}))/(1+\varepsilon)$ for all $k \ge k_0 = k_0(\varepsilon)$. Therefore,

$$\begin{aligned}
&\max_{k>0}(\ln f(x_k) + x_{k+1}\sigma) \\
&\quad = \max\left\{\max_{k\le k_0}(\ln f(x_k) + x_{k+1}\sigma), \max_{k\ge k_0}\left(\frac{\ln f(x_k)}{\ln f(x_{k+1})}\ln f(x_{k+1}) + x_{k+1}\sigma\right)\right\} \\
&\quad \le \max\left\{O(\sigma), \max_{k\ge k_0}\left(\frac{\ln f(x_{k+1})}{1+\varepsilon} + x_{k+1}\sigma\right)\right\} \\
&\quad \le \frac{1}{1+\varepsilon}\max_{k\ge 0}\left(\ln f(x_{k+1}) + x_{k+1}\sigma(1+\varepsilon)\right) + O(\sigma), \quad \sigma \to +\infty.
\end{aligned}$$

Hence and from (13) it follows that $\ln\mu(\sigma, I) \le \ln\mu\left(\sigma(1+\varepsilon), I; X\right)$ for $\sigma \ge \sigma_0^*$. Thus,

$$\begin{aligned}
\varliminf_{r\to+\infty} \frac{\ln\mu(\sigma, I)}{\Phi(\sigma)} &\le \varliminf_{\sigma\to+\infty} \frac{\ln\mu(\sigma(1+\varepsilon), I; X)}{\Phi(\sigma)} \\
&\le \varliminf_{r\to+\infty} \frac{\ln\mu(\sigma, I; X)}{\Phi(\sigma)} \varlimsup_{\sigma\to+\infty} \frac{\Phi(\sigma(1+\varepsilon))}{\Phi(\sigma)} \le A(\varepsilon) \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)},
\end{aligned} \tag{14}$$

where $A(\varepsilon) = \varlimsup\limits_{r\to+\infty} \dfrac{\Phi(\sigma(1+\varepsilon))}{\Phi(\sigma)}$. For $\Phi \in L^0$ in [7] is proved that $A(\varepsilon) \searrow 1$ as $\varepsilon \downarrow 0$. Therefore, (14) implies (8).

If $x_{k+1} = (1 + o(1))x_k$ as $k \to \infty$ then for arbitrary $\varepsilon > 0$ from (13) it follows that

$$\ln\mu(\sigma, I) \le \ln\mu\left(\sigma(1+\varepsilon), I; X\right) + O(\sigma), \quad \sigma_0^*(\varepsilon) \le \sigma \to +\infty,$$

whence in view of the condition $\Phi \in L^0$ as above we obtain (8).

If $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ then from (13) we have

$$\ln\mu(\sigma, I) \le \max_{k\ge 0}(\ln f(x_{k+1}) + x_k\sigma + \ln f(x_k) - \ln f(x_{k+1})) \le \ln\mu(\sigma, I; X) + \text{const}, \tag{15}$$

that is in view of (12)

$$\varliminf_{r\to+\infty} \frac{\ln\mu(\sigma, I)}{\Phi(\sigma)} \le \varliminf_{\sigma\to+\infty} \frac{\ln\mu(\sigma, I; X)}{\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}. \tag{16}$$

Finally, if $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ then from (13) follows that

$$\ln\mu(\sigma, I) \le \max_{k\ge 0}(\ln f(x_k) + x_k\sigma + \sigma(x_{k+1} - x_k)) \le \ln\mu(\sigma, I; X) + H\sigma, \tag{17}$$

that is in view of (12) we obtain again (16). The first part of Theorem 1 is proved.

Now we will prove the second part. Since $\ln \sigma = o(\ln \mu(\sigma, I))$ as $\sigma \to +\infty$, condition (9) follows from (5).

If either $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ or $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ then from either (16), or (17) in view of (12) and Lemma 1 we obtain

$$\underline{\lim}_{\sigma\to+\infty} \frac{\ln\ln \mu(\sigma, I)}{\ln \Phi(\sigma)} \le \underline{\lim}_{\sigma\to+\infty} \frac{\ln\ln \mu(\sigma, I; X)}{\ln \Phi(\sigma)} \le \underline{\lim}_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}.$$

If either $\ln f(x_k) \le (1+o(1)) \ln f(x_{k+1})$ or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ as $x \to +\infty$ then as above from (13) we have $\ln\ln \mu(\sigma, I) \le \ln\ln \mu\left(\sigma(1+\varepsilon), I; X\right)$ for every $\varepsilon > 0$ and all $\sigma \ge \sigma_0(\varepsilon)$, whence (10) follows in view of the condition $\ln \Phi \in L^0$.

If $\ln f(x_k) \le a \ln f(x_{k+1})$, $0 < a < 1$, then from (13) we have

$$\ln \mu(\sigma, I) \le a \max_{k\ge0}(\ln f(x_{k+1}) + x_{k+1}\sigma/a) = a \ln \mu(\sigma/a, I; X);$$

and since $\ln \Phi \in L_{si}$, we obtain

$$\underline{\lim}_{\sigma\to+\infty} \frac{\ln\ln \mu(\sigma, I)}{\ln \Phi(\sigma)} \le \underline{\lim}_{r\to+\infty} \frac{\ln\ln \mu(\sigma/a, I; X)}{\ln \Phi(\sigma/a)} \overline{\lim}_{r\to+\infty} \frac{\ln \Phi(\sigma/a)}{\ln \Phi(\sigma)} \le \underline{\lim}_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}.$$

If $x_{k+1} \le Ax_k$ for all $k \ge 0$ then $\ln \mu(\sigma, I) \le \ln \mu\left(A\sigma, I; X\right) + O(\sigma)$ as $\sigma \to +\infty$, whence in view of the condition $\ln \Phi \in L_{si}$ we obtain (10). The proof of Theorem 1 is complete. □

Now we consider the case $\sigma_\mu = 0$. Let $\hat{L}$ be the class of all positive continuous on $(\sigma_0, 0)$, $\sigma_0 \ge -\infty$, functions $\beta$, increasing to $+\infty$. We say that $\beta \in \hat{L}^0$ if $\beta \in \hat{L}$ and $\beta((1+o(1))\sigma) = (1+o(1))\beta(\sigma)$ as $\sigma \uparrow 0$, and $\beta \in \hat{L}_{si}$ if $\beta(c\sigma) = (1+o(1))\beta(\sigma)$ as $\sigma \uparrow 0$ for each $c \in (0, +\infty)$.

**Lemma 3.** *Let* $\beta \in \hat{L}$ *and* $B(\delta) = \overline{\lim}_{\sigma\uparrow0} \dfrac{\beta(\sigma/(1+\delta))}{\beta(\sigma)}$ $(\delta > 0)$. *In order that* $\beta \in \hat{L}^0$, *it is necessary and sufficient that* $B(\delta) \to 1$ *as* $\delta \downarrow 0$.

*Proof.* Suppose that $\beta \in \hat{L}^0$ but $B(\delta) \not\to 1$ as $\delta \downarrow 0$. Since the function $B(\delta)$ is nondecreasing, there exists $\lim_{\delta\downarrow0} B(\delta) = b^* > 1$, that is $B(\delta) \ge b^* > 1$. We choose an arbitrary sequence $(\delta_n) \downarrow 0$. For every $\delta_n$ there exists a sequence $(\sigma_{n,k}) \uparrow 0$ such that $\beta((1+\delta_n)\sigma_{n,k}) \ge b\beta(\sigma_{n,k})$, $1 < b < b^*$. We put $\sigma_1 = \sigma_{1,1}$ and $\sigma_n = \min\{\sigma_{n,k} \ge \sigma_{n-1} : k \ge n-1\}$ and construct a function $\gamma(\sigma) \to 0$, $\sigma \uparrow 0$, such that $\gamma(\sigma_n) = \delta_n$. Then $\beta(\sigma_n/(1+\gamma(\sigma_n))) = \beta(\sigma_n/(1+\delta_n)) \ge b\beta(\sigma_n)$. In view of definition of $\hat{L}^0$ it is impossible.

On the contrary, let $B(\delta) \to 1$ as $\delta \downarrow 0$ but $\beta \notin \hat{L}^0$. Then there exists a function $\gamma(\sigma) \to 0$, $\sigma \uparrow 0$, and sequence $(\sigma_n) \uparrow 0$, $n \to \infty$, such that $\lim_{n\to\infty} \beta(\sigma_n/(1+\gamma(\sigma_n))/\beta(\sigma_n) = a \ne 1$. Clearly, $a < 1$ provided $\gamma(\sigma_n) < 0$ and $a > 1$ provided $\gamma(\sigma_n) > 0$. We examine, for example, the second case. Let $\delta > 0$ be an arbitrary number. Then $\gamma(\sigma_n) < \delta$ for $n \ge n_0$ and

$$B(\delta) = \overline{\lim}_{\sigma\uparrow0} \frac{\beta(\sigma/(1+\delta))}{\beta(\sigma)} \ge \overline{\lim}_{n\to\infty} \frac{\beta(\sigma_n/(1+\delta))}{\beta(\sigma_n)} \ge \overline{\lim}_{n\to\infty} \frac{\beta(\sigma_n/(1+\gamma(\sigma_n)))}{\beta(\sigma_n)} = a > 1,$$

which is impossible. Lemma 3 is proved. □

**Theorem 2.** *Let $\sigma_\mu = 0$, $\Phi \in \Omega(0)$, $\ln \mu(\sigma, I) \le \Phi(\sigma)$ for all $\sigma \ge \sigma_0$ and $X = (x_k)$ be some sequence $X = (x_k)$ of positive numbers increasing to $+\infty$. Suppose that $f(x) \nearrow +\infty$ as $x \to +\infty$. Then:*

1) *if either $\ln f(x_{k+1}) - \ln f(x_k) \le H$ or $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$, or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ and $\Phi \in \hat{L}^0$, or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ and $\Phi \in \hat{L}^0$, or $x_{k+1} \le Ax_k$ for $k \ge 0$ and $\Phi \in \hat{L}_{si}$ then*
$$\varliminf_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I)}{\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}, \tag{18}$$

2) *if*
$$\left(\frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1\right) \ln \Phi(\sigma) \ge q > -\infty, \quad \sigma \in [\sigma_0, 0), \tag{19}$$
$$\varliminf_{\sigma\uparrow 0} \frac{\ln \ln \mu(\sigma, I)}{\ln \Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{\ln G_1(x_k, x_{k+1}, \Phi)}{\ln G_2(x_k, x_{k+1}, \Phi)}. \tag{20}$$

*Proof.* As above let $\mu(\sigma, I; X) = \max\{f(x_k)e^{\sigma x_k} : k \ge 0\}$. Clearly, (11) holds. Therefore, $\ln \mu(\sigma, I; X) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_0, 0)$ and by Lemma 2 $\ln f(x_k) \le -x_k \Psi(\varphi(x_k))$ for all $k \ge k_0$, that is $\ln \mu(\sigma, I; X) \le \ln \mu_D(r)$ for $\sigma \ge \sigma_0$. Therefore, by Lemma 1
$$\varliminf_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I; X)}{\Phi(\sigma)} \le \varliminf_{k\to\infty} \frac{G_1(x_k, x_{k+1}, \Phi)}{G_2(x_k, x_{k+1}, \Phi)}. \tag{21}$$

On the other hand for $\sigma < 0$ now we have
$$\ln \mu(\sigma, I) = \max_{k\ge 0} \sup_{x_k \le x < x_{k+1}} (\ln f(x) + x\sigma) \le \max_{k\ge 0} (\ln f(x_{k+1}) + x_k\sigma). \tag{22}$$

Therefore, if either $\ln f(x_{k+1}) - \ln f(x_k) \le H$ or $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ hence we obtain either $\ln \mu(\sigma, I) \le \ln \mu(\sigma, I; X) + H$ or $\ln \mu(\sigma, I) \le \ln \mu(\sigma, I; X) + H\sigma$, whence
$$\varliminf_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I)}{\Phi(\sigma)} \le \varliminf_{\sigma\uparrow 0} \frac{\ln \mu(\sigma, I; X)}{\Phi(\sigma)}. \tag{23}$$

Inequalities (21) and (23) imply (18).

If either $x_{k+1} = (1+o(1))x_k$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ then from (23) as in the proof of Theorem 1 for every $\varepsilon > 0$ we have correspondingly $\ln \mu(\sigma, I) \le \ln \mu(\sigma/(1+\varepsilon), I; X)$ and $\ln \mu(\sigma, I) \le (1+\varepsilon) \ln \mu(\sigma/(1+\varepsilon), I; X)$ for $\sigma \in [\sigma_0(\varepsilon), 0)$, whence in view of condition $\ln \Phi \in \hat{L}^0$, of Lemma 3 and of the arbitrariness of $\varepsilon$ we obtain (23) and, thus, (18) holds.

Finally, if $x_{k+1} \le Ax_k$ for $k \ge 0$ then $\ln \mu(\sigma, I) \le \ln \mu(\sigma/A, I; X)$, whence in view of condition $\Phi \in \hat{L}_{si}$ we obtain again (23). The first part of Theorem 2 is proved.

For the proof of the second part we remark that from the condition $f(x) \nearrow +\infty$ as $x \to +\infty$ it follows that $\ln \mu(\sigma, I) \uparrow +\infty$ as $\sigma \uparrow 0$. Therefore, (19) implies (5). We remark also that if either $\ln f(x_{k+1}) - \ln f(x_k) \le H$ or $x_{k+1} - x_k \le H < +\infty$ for all $k \ge 0$ or $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$ and $\ln \Phi \in \hat{L}^0$ or $x_{k+1} = (1+o(1))x_k$ as $k \to \infty$ and $\ln \Phi \in \hat{L}^0$ or $x_{k+1} \le Ax_k$ for $k \ge 0$ and $\ln \Phi \in \hat{L}_{si}$ then from the inequalities obtained above we get (20). If $\ln f(x_{k+1}) \le A \ln f(x_k)$ for $k \ge 0$ then from (21) we obtain the inequality $\ln \mu(\sigma, I) \le A \ln \mu(\sigma/A, I; X)$, whence in view of the condition $\ln \Phi \in \hat{L}_{si}$ inequality (20) follows. The proof of Theorem 2 is complete. □

## 2 ANALOGUES OF WHITTAKER'S THEOREM

Examing the other scale of growth from Theorems 1 and 2 gives us a possible to get the series of results for Laplace-Stieltjes integrals. Here we will be stopped only for two cases which more frequent at meet in mathematical works. The most used characteristics of growth for integrals (2) with $\sigma_c = +\infty$ (by analogy with Dirichlet series) are $R$-order $\varrho_R[I]$, lower $R$-order $\lambda_R[I]$ and (if $\varrho_R[I] \in (0, +\infty)$) $R$-type $T_R[I]$, lower $R$-type $t_R[I]$, which are defined by formulas

$$\varrho_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln \ln I(\sigma)}{\sigma}, \quad \lambda_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln \ln I(\sigma)}{\sigma},$$
$$T_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln I(\sigma)}{\exp\{\sigma\varrho_R[I]\}}, \quad t_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln I(\sigma)}{\exp\{\sigma\varrho_R[I]\}}.$$

We will show that in this formulas $\ln I(\sigma)$ can be replaced by $\ln \mu(\sigma, I)$ and will use the following Lemmas for this purpose.

**Lemma 4** ( [4, 8]). *Let $F \in V$, $f$ has regular variation in regard to $F$ and either $\sigma_\mu = +\infty$ or $\sigma_\mu = 0$ and $\varliminf_{x\to+\infty} f(x) = +\infty$. Then $\ln \mu(\sigma, I) \le (1 + o(1)) \ln I(\sigma)$ as $\sigma \uparrow \sigma_\mu$.*

**Lemma 5** ( [4, 9]). *Let $F \in V$, $\sigma_\mu = +\infty$ and $\varlimsup_{x\to+\infty} (\ln F(x))/x = \tau < +\infty$. Then $I(\sigma) \le \mu(\sigma + \tau + \varepsilon, I)$ for every $\varepsilon > 0$ and all $\sigma \ge \sigma(\varepsilon)$.*

It is easy to check that these lemmas imply the following statement.

**Proposition 1.** *Let $F \in V$, $f$ has regular variation in regard to $F$ and $\sigma_\mu = +\infty$. If $\ln F(x) = O(x)$ as $x \to +\infty$ then*

$$\varrho_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln \ln \mu(\sigma, I)}{\sigma}, \quad \lambda_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln \ln \mu(\sigma, I)}{\sigma}, \tag{24}$$

*and if $\ln F(x) = o(x)$ as $x \to +\infty$ then*

$$T_R[I] = \varlimsup_{\sigma\to+\infty} \frac{\ln \mu(\sigma, I)}{\exp\{\sigma\varrho_R[I]\}}, \quad t_R[I] = \varliminf_{\sigma\to+\infty} \frac{\ln \mu(\sigma, I)}{\exp\{\sigma\varrho_R[I]\}}. \tag{25}$$

Using Theorem 1 and Proposition 1 we prove the following theorem.

**Theorem 3.** *Let $F \in V$, $\sigma_\mu = +\infty$ and $X = (x_k)$ be some sequence of positive numbers increasing to $+\infty$. Suppose that $f$ is a nonincreasing function and has regular variation in regard to $F$.*

*If $\ln F(x) = O(x)$ as $x \to +\infty$ and $\ln f(x_k) = (1 + o(1)) \ln f(x_{k+1})$ as $k \to \infty$ then*

$$\lambda_R[I] \le \beta\varrho_R[I], \quad \beta = \varliminf_{k\to\infty} \frac{\ln x_k}{\ln x_{k+1}}. \tag{26}$$

*If $\ln F(x) = o(x)$ as $x \to +\infty$ and $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$ then*

$$t_R[I] \le T_R[I] \frac{\gamma}{1-\gamma} \exp\left\{1 + \frac{\gamma \ln \gamma}{1-\gamma}\right\} \ln \frac{1}{\gamma}, \quad \gamma = \varliminf_{k\to\infty} \frac{x_k}{x_{k+1}}. \tag{27}$$

*Proof.* From (24) and (25) for every $\varepsilon$ and all $\sigma \geq \sigma_0(\varepsilon)$ we have accordingly $\ln \mu(\sigma, I) \leq \exp\{(\varrho_R[I]+\varepsilon)\sigma\}$ and $\ln \mu(\sigma, I) \leq (T_R[I]+\varepsilon)\exp\{\varrho_R[I]\sigma\}$. We choose $\Phi \in \Omega(+\infty)$ such that $\Phi(\sigma) = Te^{\varrho\sigma}$ for $\sigma \geq \sigma_0(\varepsilon)$, where either $\varrho = \varrho_R[I]+\varepsilon$ and $T = 1$ or $\varrho = \varrho_R[I]$ and $T = T_R[I]+\varepsilon$. Then $\ln \mu(\sigma, I) \leq \Phi(\sigma)$ for $\sigma \geq \sigma_0(\varepsilon)$, $\ln \Phi \in L^0$ and it is well known ( [4, 10]) that

$$G_1(x_k, x_{k+1}, \Phi) = \frac{1}{\varrho}\frac{x_k x_{k+1}}{x_{k+1}-x_k} \ln \frac{x_{k+1}}{x_k}$$

and

$$G_2(x_k, x_{k+1}, \Phi) = \frac{1}{e\varrho} \exp\left\{\frac{x_{k+1}\ln x_{k+1} - x_k \ln x_k}{x_{k+1}-x_k}\right\}.$$

Since $\Phi(\sigma)\Phi''(\sigma)/\Phi'(\sigma)^2 = 1$, condition (9) holds and by Theorem 1 we have

$$\lambda_R[I] \leq \varrho \varliminf_{k\to\infty} \frac{(x_{k+1}-x_k)\ln\left(\dfrac{x_k x_{k+1}}{x_{k+1}-x_k}\ln\dfrac{x_{k+1}}{x_k}\right)}{x_{k+1}\ln x_{k+1} - x_k \ln x_k} \tag{28}$$

provided $\ln f(x_k) = (1+o(1)) \ln f(x_{k+1})$ as $k \to \infty$, and

$$t_R[I] \leq eT \varliminf_{k\to\infty} \frac{\dfrac{x_k x_{k+1}}{x_{k+1}-x_k}\ln\dfrac{x_{k+1}}{x_k}}{\exp\left\{\dfrac{x_{k+1}\ln x_{k+1} - x_k\ln x_k}{x_{k+1}-x_k}\right\}} \tag{29}$$

provided $\ln f(x_k) - \ln f(x_{k+1}) = O(1)$ as $k \to \infty$.

We suppose that $\beta < 1$. Then there exist a number $\beta^* \in (\beta, 1)$ and an increasing sequence $(k_j)$ of positive integers such that $\ln x_{k_j} \leq \beta^* \ln x_{k_j+1}$, that is $x_{k_j} = o(x_{k_j+1})$ as $j \to \infty$. Therefore, from (28) we obtain

$$\begin{aligned}\lambda_R[I] &\leq \varrho \varliminf_{j\to\infty} \frac{(x_{k_j+1}-x_{k_j})\ln\left(\dfrac{x_{k_j}x_{k_j+1}}{x_{k_j+1}-x_{k_j}}\ln\dfrac{x_{k_j+1}}{x_{k_j}}\right)}{x_{k_j+1}\ln x_{k_j+1} - x_{k_j}\ln x_{k_j}} \\ &\leq \varrho \varliminf_{j\to\infty} \frac{\ln x_{k_j} + o(1) + \ln\ln x_{k_j+1}}{\ln x_{k_j+1}} \leq \varrho\beta^*,\end{aligned}$$

whence in view of the arbitrariness of $\beta^*$ and $\varepsilon$ we obtain inequality (26) follows.

Further, if $\gamma \in (0, 1)$, then $x_{k_j} = (1+o(1))\gamma x_{k_j+1}$ as $j \to \infty$ for some increasing sequence $(k_j)$ of positive integers and from (29) we obtain

$$\begin{aligned}t_R[I] &\leq eT \varliminf_{j\to\infty} \frac{x_{k_j}x_{k_j+1}\ln(x_{k_j+1}/x_{k_j})}{(x_{k_j+1}-x_{k_j})\exp\left\{\dfrac{x_{k_j+1}\ln x_{k_j+1} - x_{k_j}\ln x_{k_j}}{x_{k_j+1}-x_{k_j}}\right\}} \\ &= eT \varliminf_{j\to\infty} \frac{\gamma x_{k_j+1}\ln(1/\gamma)}{(1-\gamma)\exp\{\ln x_{k_j+1} - (\gamma\ln\gamma)/(1-\gamma)\}} = T\frac{\gamma}{1-\gamma}\ln\frac{1}{\gamma}\exp\left\{1+\frac{\gamma\ln\gamma}{1-\gamma}\right\},\end{aligned}$$

whence in view of the arbitrariness of $\varepsilon$ we get (27). Since $\dfrac{\gamma}{1-\gamma}\ln\dfrac{1}{\gamma}\exp\left\{1+\dfrac{\gamma\ln\gamma}{1-\gamma}\right\} \to 1$ as $\gamma \to 1$, then inequality (27) is obvious if $\gamma = 1$. Finally, if $\gamma = 0$, then $\ln x_{k_j} = o(\ln x_{k_j+1})$ as $j \to \infty$ for some increasing sequence $(k_j)$ of positive integers and from (29) we obtain

$$t_R[I] \leq eT \varliminf_{j\to\infty} \frac{x_{k_j}(\ln x_{k_j+1} - \ln x_{k_j})}{\exp\{\ln x_{k_j+1} + o(1)\}} = eT \varliminf_{j\to\infty} \frac{x_{k_j}}{x_{k_j+1}}\ln\frac{x_{k_j+1}}{x_{k_j}} = 0,$$

i.e. inequality (27) holds. The proof of Theorem 3 is complete. □

Now we consider the case $\sigma_\mu = 0$. The order $\varrho_0[I]$, the lower order $\lambda_0[I]$ and (if $0 < \varrho_0[I] < +\infty$) the type $T_0[I]$ and the lower type $t_0[I]$ are defined by formulas

$$\varrho_0[I] = \varlimsup_{\sigma\uparrow 0} \frac{\ln\ln I(\sigma)}{\ln(1/|\sigma|)}, \quad \lambda_0[\varphi] = \varliminf_{\sigma\uparrow 0} \frac{\ln\ln I(\sigma)}{\ln(1/|\sigma|)},$$
$$T_0[I] = \varlimsup_{\sigma\uparrow 0} |\sigma|^{\varrho_0[I]} \ln I(\sigma), \quad t_0[I] = \varliminf_{\sigma\uparrow 0} |\sigma|^{\varrho_*[I]} \ln I(\sigma).$$

We will show that in this formulas $\ln I(\sigma)$ can be replaced by $\ln \mu(\sigma, I)$ and will use for this purpose the following lemmas.

**Lemma 6** ( [4,9]). *Let $F \in V$, $\sigma_\mu = 0$ and $\ln F(x) \le h \ln f(x)$ for $x \ge x_0$. Then for every $\varepsilon > 0$ and all $\sigma \in [\sigma_0(\varepsilon), 0)$*

$$\ln I(\sigma) \le (1+h+\varepsilon) \ln \mu\left(\frac{\sigma}{1+h+\varepsilon}, I\right) + K, \quad K = K(\varepsilon) = const.$$

**Lemma 7** ( [4,9]). *Let $F \in V$, $\sigma_\mu = 0$ and $\ln F(x) = o(x\gamma(x))$ as $x \to +\infty$, where $\gamma$ is a positive continuous and decreasing to 0 function on $[0, +\infty)$ such that $x\gamma(x) \uparrow +\infty$ as $x \to +\infty$. Then for every $\varepsilon > 0$ and all $\sigma \in [\sigma_0(\varepsilon), 0)$*

$$\ln I(\sigma) \le \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + \frac{\varepsilon|\sigma|}{1+\varepsilon} \gamma^{-1}\left(\frac{|\sigma|}{\varepsilon(1+\varepsilon)^2}\right).$$

Lemmas 4, 6 and 7 imply the following statement.

**Proposition 2.** *Let $F \in V$, $\sigma_\mu = +\infty$, $f$ has regular variation in regard to $F$ and $f(x) \nearrow +\infty$ as $x \to +\infty$. If either $\ln F(x) = O(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ then*

$$\varrho_0[I] = \varlimsup_{\sigma\uparrow 0} \frac{\ln\ln \mu(\sigma, I)}{\ln(1/|\sigma|)}, \quad \lambda_0[\varphi] = \varliminf_{\sigma\uparrow 0} \frac{\ln\ln \mu(\sigma, I)}{\ln(1/|\sigma|)}, \tag{30}$$

*and if either $\ln F(x) = o(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ then*

$$T_0[I] = \varlimsup_{\sigma\uparrow 0} |\sigma|^{\varrho_0[I]} \ln \mu(\sigma, I), \quad t_0[I] = \varliminf_{\sigma\uparrow 0} |\sigma|^{\varrho_0[I]} \ln \mu(\sigma, I)). \tag{31}$$

*Proof.* If $\ln F(x)) = O(\ln f(x))$ (accordingly $\ln F(x) = o(\ln f(x))$) as $x \to +\infty$ then formulas (30) (accordingly (31)) easy follows from Lemmas 4 and 6.

If we choose function $\gamma$ such that $\gamma(x) = x^{\delta-1}$ for $x \ge x_0$, where $\delta \in (0,1)$ is an arbitrary numbers, then $\gamma$ satisfies the conditions of Lemma 7. Therefore, if $\ln F(x) = o(x^\delta)$ as $x \to +\infty$ then

$$\ln I(\sigma) \le \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + \frac{\varepsilon|\sigma|}{1+\varepsilon}\left(\frac{\varepsilon(1+\varepsilon)^2}{|\sigma|}\right)^{1-\delta}$$
$$= \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + \varepsilon^{2-\delta}(1+\varepsilon)^{1-2\delta}|\sigma|^\delta = \ln \mu\left(\frac{\sigma}{1+\varepsilon}, I\right) + o(1), \quad \sigma \uparrow 0,$$

whence the formulas (30) and (31) follow. It remained to notice that the condition $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ implies the condition $\ln F(x) = o(x^\delta)$ as $x \to +\infty$ for $\delta \in (0,1)$. Proposition 2 is proved. □

Using Theorem 2 and Proposition 2 we prove the following theorem.

**Theorem 4.** *Let $F \in V$, $\sigma_\mu = 0$ and $X = (x_k)$ be some sequence of positive numbers increasing to $+\infty$. Suppose that $f$ has regular variation in regard to $F$ and $f(x) \nearrow +\infty$ as $x \to +\infty$.*

*If either $\ln F(x) = O(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ and $\ln f(x_{k+1}) = O(\ln f(x_k))$ as $k \to \infty$ then*

$$\lambda_0[I] \le \beta\varrho_0[I], \quad \beta = \varliminf_{k\to\infty} \frac{\ln x_k}{\ln x_{k+1}}. \tag{32}$$

*If either $\ln F(x) = o(\ln f(x))$ or $\ln\ln F(x) = o(\ln x)$ as $x \to +\infty$ and $\ln f(x_{k+1}) = (1+o(1))\ln f(x_k)$ as $k \to \infty$ then*

$$t_0[I] \le T_0[I]A(\gamma), \quad \gamma = \varliminf_{k\to\infty} \frac{x_k}{x_{k+1}}, \tag{33}$$

*where*

$$A(\gamma) =: \frac{\gamma^{\varrho/(\varrho+1)}(1-\gamma^{1/(\varrho+1)})(1-\gamma^{\varrho/(\varrho+1)})^{\varrho}}{(1-\gamma)^{\varrho+1}}.$$

*Proof.* If $\varrho_0[I] < +\infty$ ($T_0[I] < +\infty$) then $\ln \mu(\sigma, I) \le \Phi(\sigma) = \dfrac{T}{|\sigma|^{\varrho}}$ for all $\sigma \in [\sigma_0(\varepsilon), 0)$, where either $\varrho = \varrho_0[I] + \varepsilon$ and $T = 1$ or $\varrho = \varrho_0[I]$ and $T = T_0[I] + \varepsilon$. Clearly, $\Phi \in \hat{L}^0$ and $\ln \Phi \in \hat{L}_{si}$. It is known [4, p. 40] that for this function

$$G_1(x_k, x_{k+1}, \Phi) = \frac{T(\varrho+1)}{(T\varrho)^{\varrho/(\varrho+1)}} \frac{x_k x_{k+1}}{x_{k+1}-x_k} \left( \frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}} \right)$$

and

$$G_2(x_k, x_{k+1}, \Phi) = T \left( \frac{(\varrho+1)(T\varrho)^{1/(\varrho+1)}}{\varrho} \frac{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}{x_{k+1}-x_k} \right)^{-\varrho}.$$

We remark that

$$\left( \frac{\Phi(\sigma)\Phi''(\sigma)}{(\Phi'(\sigma))^2} - 1 \right) \ln \Phi(\sigma) = \frac{1}{\varrho} \ln \frac{T}{|\sigma|^{\varrho}} \uparrow +\infty, \quad \sigma \uparrow 0,$$

that is (19) holds.

Therefore, if $\ln f(x_{k+1}) = O(\ln f(x_k))$ as $k \to \infty$ then by Theorem 2 in view of arbitrariness of $\varepsilon$

$$\lambda_0[I] \le \varrho_0[I] \varliminf_{k\to\infty} \frac{\ln \left( \dfrac{x_k x_{k+1}}{x_{k+1}-x_k} \left( \dfrac{1}{x_k^{1/(\varrho+1)}} - \dfrac{1}{x_{k+1}^{1/(\varrho+1)}} \right) \right)}{\ln \left( \dfrac{x_{k+1}-x_k}{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}} \right)^{\varrho}} \tag{34}$$

and if $\ln f(x_{k+1}) = (1+o(1))\ln f(x_k)$ as $k \to \infty$ then

$$t_0[I] \le T_0[I] \frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}} \varliminf_{k\to\infty} \frac{x_k x_{k+1}}{x_{k+1}-x_k} \left( \frac{1}{x_k^{1/(\varrho+1)}} - \frac{1}{x_{k+1}^{1/(\varrho+1)}} \right) \left( \frac{x_{k+1}^{\varrho/(\varrho+1)} - x_k^{\varrho/(\varrho+1)}}{x_{k+1}-x_k} \right)^{\varrho}. \tag{35}$$

We suppose that $\beta < 1$. Then there exists a number $\beta^* \in (\beta, 1)$ and an increasing sequence $(k_j)$ of positive integers such that $\ln x_{k_j} \le \beta^* \ln x_{k_j+1}$, that is $x_{k_j} = o(x_{k_j+1})$ as $j \to \infty$. Therefore, from (34) we obtain

$$\lambda_0[I] \le \varrho_0[I] \varliminf_{j\to\infty} \frac{\ln\left(\dfrac{x_{k_j}x_{k_j+1}}{x_{k_j+1}-x_{k_j}}\left(\dfrac{1}{x_{k_j}^{1/(\varrho+1)}} - \dfrac{1}{x_{k_j+1}^{1/(\varrho+1)}}\right)\right)}{\ln\left(\dfrac{x_{k_j+1}-x_{k_j}}{x_{k_j+1}^{\varrho/(\varrho+1)} - x_{k_j}^{\varrho/(\varrho+1)}}\right)^{\varrho}}$$

$$= \varrho_0[I] \varliminf_{j\to\infty} \frac{\ln x_{k_j}^{\varrho/(\varrho+1)}}{\varrho \ln x_{k_j+1}^{1/(\varrho+1)}} = \varrho_0[I] \varliminf_{j\to\infty} \frac{\ln x_{k_j}}{\ln x_{k_j+1}} \le \varrho_0[I]\beta^*,$$

i.e. in view of arbitrariness of $\beta^*$ we obtain the inequality $\lambda_0[I] \le \beta\varrho_0[I]$. For $\beta = 1$ this inequality is trivial.

Now we suppose that $\gamma \in (0, 1)$. Then there exists an increasing sequence $(k_j)$ of positive integers such that $x_{k_j} = (1 + o(1))\gamma\, x_{k_j+1}$ as $j \to \infty$. Therefore, from (35) we obtain

$$t_0[I] \le T_0[I]\frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}} \varliminf_{j\to\infty} \frac{x_{k_j}x_{k_i+1}}{x_{k_j+1}-x_{k_j}}\left(\frac{1}{x_{k_j}^{1/(\varrho+1)}} - \frac{1}{x_{k_j+1}^{1/(\varrho+1)}}\right)\left(\frac{x_{k_j+1}^{\varrho/(\varrho+1)} - x_{k_j}^{\varrho/(\varrho+1)}}{x_{k_j+1}-x_{k_j}}\right)^{\varrho}$$

$$\le T_0[I]\frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}}\frac{\gamma}{\gamma-1}\left(\frac{1}{\gamma^{1/(\varrho+1)}} - 1\right)\frac{(1-\gamma^{\varrho/(\varrho+1)})^{\varrho}}{(1-\gamma)^{\varrho}} = T_0[I]\frac{(\varrho+1)^{\varrho+1}}{\varrho^{\varrho}}A(\gamma).$$

It is easy to show that $A(\gamma) \to \frac{\varrho^{\varrho}}{(\varrho+1)^{\varrho+1}}$ as $\gamma \to 1$ that (2) is transformed in obvious inequality $t_0[\varphi] \le T_0[\varphi]$ as $\gamma \to 1$. If $\gamma = 0$ then $x_{k_j} = o(x_{k_j+1})$ as $j \to \infty$ and from (2) we obtain easy that $t_0[I] = 0$, because $A(0) = 0$. The proof of Theorem 4 is complete. □

## REFERENCES


[1] Whittaker J.M. *The lower order of integral functions*. J. London Math. Soc. 1933, **8** (1), 20–27. doi:10.1112/jlms/s1-8.1.20

[2] Sons L.R. *Regularity of growth and gaps*. J. Math. Anal. Appl. 1968, **24** (2), 296–306. doi:10.1016/0022-247X(68)90031-0

[3] Filevych P.V., Sheremeta M.M. *On a L. Sons theorem and asymptotical behaviour of Dirichlet series*. Ukr. Math. Bull. 2006, **3** (2), 187–198. (in Ukrainian)

[4] Sheremeta M.M. Asymptotical behaviour of Laplace-Stiltjes integrals. In: Mathematical Studies, 15. Lviv, VNTL Publishers, 2010.

[5] Sheremeta M.M., Fedynyak S.I. *On the derivative of a Dirichlet series*. Siberian Math. J. 1998, **39** (1), 181–197. doi:10.1007/BF02732373 (translation of Sibirsk. Mat. Zh. 1998, **39** (1), 206–223. (in Russian))

[6] Sheremeta M.M., Sumyk O.M. *Connection detween the growth of Young conjugated functions*. Mat. Stud. 1999, **11** (1), 41–47. (in Ukrainian)

[7] Sheremeta M.M. *On two classes of positive functions and belonging to them of main characteristic of entire functions*. Mat. Stud. 2003, **19** (1), 73–82. (in Ukrainian)

[8] Posiko O.S., Sheremeta M.M. *Asymptotic estimates for Laplace-Stiltjes integrals*. Ukr. Math. Bull. 2005, **2** (4), 541–549. (in Ukrainian)

[9] Posiko O.S., Skaskiv O.B., Sheremeta M.M. *Estimates of Laplace-Stiltjes integral*. Mat. Stud. 2004, **21** (2), 179–196. (in Ukrainian)

[10] Sumyk O.M., Sheremeta M.M. *Estimates from below for maximal term of Dirichlet series*. Izv. Vys. Uch. Zav. Matem. 2001, (4), 53–57. (in Russian)

*Received 06.09.2016*

Добушовський М.С., Шеремета М.М. *Аналоги теореми Уіттекера для інтегралів Лапласа-Стілтьєса* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 239–250.

Для максимуму підінтегрального виразу інтегралу Лапласа-Стілтьєса знайдено нижні оцінки на деякій послідовності. Використовуючи ці оцінки, отримано аналоги тереми Уіттекера для цілих функцій, зображених лакунарними степеневими рядами.

*Ключові слова і фрази:* інтеграл Лапласа-Стілтьєса, максимум підінтегрального виразу, теорема Уіттекера.