ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2016, 8 (2), 272–278
doi:10.15330/cmp.8.2.272-278

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2016, Т.8, №2, С.272–278

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A WORPITZKY BOUNDARY THEOREM FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM

For a branched continued fraction of a special form we propose the limit value set for the Worpitzky-like theorem when the element set of the branched continued fraction is replaced by its boundary.

Key words and phrases: element set, value set, branched continued fraction of special form.

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INTRODUCTION

A lot of convergence criteria for continued fractions are characterized by convergence domains. Such domains are indicated in the complex plane, that if elements a_k, b_k of a continued fraction belong to these domains then the continued fraction

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\vdots}}} = \mathop{\mathrm{D}}_{k=1}^{\infty} \frac{a_k}{b_k}$$

converges. At first convergence domains for continued fractions we can find in papers of Worpitzky (1865), Pringsheim (1899) and Van Vleck (1901) [8].

Despite of the fact that a well known convergence theorem for continued fractions was proposed by J. Worpitzky in 1865, its new proofs, generalizations and applications are actual even at present [3, 6, 8].

H. Waadeland [10] formulated the Worpitzky theorem in a slightly more general form than classical one [8], using conditions on the coefficients of the continued fraction proposed by F. Paydon and H. Wall [9].

Theorem 1. *Let $\rho \in (0, 1/2]$ be any positive number, and let all elements of a continued fraction*

$$\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\vdots}}} = \mathop{\mathrm{D}}_{i=1}^{\infty} \frac{a_i}{1}, \tag{1}$$

$a_i,\ i = 1, 2, \dots$, be complex numbers, bounded by

$$|a_i| \le \rho(1-\rho), \quad i = 1, 2, \dots. \tag{2}$$

Then the continued fraction (1) converges and its values are contained in the disk $|w| \le \rho$.

УДК 517.524
2010 *Mathematics Subject Classification:* 11A55, 11J70, 30B70, 40A15.

For the continued fraction (1) Haakon Waadeland raised the question: What happens to the set of values of the continued fraction (1) when the condition (2) in the Worpitzky theorem would be replaced by $|a_i| = \rho(1-\rho),\ i = 1, 2, \ldots$? Answering on his question H.Waadeland proved [10], that the set of all possible values of the continued fraction (1) is the annulus

$$\rho \cdot \frac{1-\rho}{1+\rho} \leq |w| \leq \rho.$$

In the classical case of the theorem ($\rho = 1/2$), i.e. $|a_i| = 1/4, \quad i = 1, 2, \ldots$, the annulus is $1/6 \leq |w| \leq 1/2$.

The same question one can put for multidimensional generalizations of the continued fraction, such as for example,

a branched continued fraction (BCF) [3]

$$1 + \sum_{i_1=1}^{N} \cfrac{a_{i_1} z_{i_1}}{1 + \sum\limits_{i_2=1}^{N} \cfrac{a_{i_1 i_2} z_{i_2}}{1 + \sum\limits_{i_3=1}^{N} \cfrac{a_{i_1 i_2 i_3} z_{i_3}}{1 + \ddots}}} = 1 + \mathop{D}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)} z_{i_k}}{1}, \tag{3}$$

where $a_{i_1 i_2 \ldots i_k}$ be complex numbers, z_{i_k} be complex variables, $i(k) = i_1 i_2 \ldots i_k$ be multiindex;

a branched continued fraction with independent variables [1]

$$\cfrac{a_{00}}{1 + \sum\limits_{i_1=1}^{N} \cfrac{a_{i_1} z_{i_1}}{1 + \sum\limits_{i_2=1}^{i_1} \cfrac{a_{i_1 i_2} z_{i_2}}{1 + \sum\limits_{i_3=1}^{i_2} \cfrac{a_{i_1 i_2 i_3} z_{i_3}}{1 + \ddots}}}} = \cfrac{a_{00}}{1 + \mathop{D}\limits_{k=1}^{\infty} \sum\limits_{i_k=1}^{i_{k-1}} \cfrac{a_{i(k)} z_{i_k}}{1}}, \tag{4}$$

where $a_{i_1 i_2 \ldots i_k}$ be complex numbers, z_{i_k} be complex variables, $i(k) = i_1 i_2 \ldots i_k$ be multiindex $1 \leq i_k \leq i_{k-1},\ k = 1, 2, \ldots,\ i_0 = N$;

or a two-dimensional continued fraction (TDCF) [6]

$$\mathop{D}_{i=0}^{\infty} \frac{a_{i,i} z_1 z_2}{\Phi_i}, \quad \Phi_i = 1 + \mathop{D}_{j=1}^{\infty} \frac{a_{i+j,i} z_1}{1} + \mathop{D}_{j=1}^{\infty} \frac{a_{i,i+j} z_2}{1}, \tag{5}$$

where $a_{i,j}, i = 0, 1, \ldots, j = 1, 2, \ldots,$ be complex numbers, z_1, z_2 be complex variables.

It was found this question for the branched continued fraction (3) with $z_1 = z_2 = \ldots = z_N = 1$ is answered by the following theorem [11].

Theorem 2. *Let $\rho \in (0, 1/2]$ and $N \geq 2$ be an integer. In the family of branched continued fractions*

$$1 + \sum_{i_1=1}^{N} \cfrac{a_{i_1}}{1 + \sum\limits_{i_2=1}^{N} \cfrac{a_{i_1 i_2}}{1 + \sum\limits_{i_3=1}^{N} \cfrac{a_{i_1 i_2 i_3}}{1 + \ddots}}} = 1 + \mathop{D}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{1}, \tag{6}$$

where $a_{i_1 i_2 \ldots i_k}$ be complex numbers, $i(k) = i_1 i_2 \ldots i_k$ be multiindex, $a_{i(k)}$ satisfy the conditions $\left|a_{i(k)}\right| = \dfrac{\rho(1-\rho)}{N}$, then the set of possible branched continued fraction values is the closed disk $|w| \leq \rho$.

Thus, in this case the set of possible BCF values is unchanged when all elements of (6) are restricted to the boundary of the disk.

For TDCF (5) with $z_1 = z_2 = 1$ the answer is proposed by the following theorem [7].

Theorem 3. *Let ρ be a real number in $(0, 1/2]$, and let F_ρ be the family of two-dimensional continued fractions*

$$\overset{\infty}{\underset{i=0}{D}} \frac{a_{i,i}}{\Phi_i}, \quad \Phi_i = 1 + \overset{\infty}{\underset{j=1}{D}} \frac{a_{i+j,i}}{1} + \overset{\infty}{\underset{j=1}{D}} \frac{a_{i,i+j}}{1}, \tag{7}$$

where $a_{i,j}$, $i = 0, 1, \dots, j = 1, 2, \dots,$ be complex numbers that satisfy conditions $|a_{i,j}| = \frac{1}{2}\rho(1 - \rho)$, $i, j \geq 1$.

Then the set of all possible values f of the TDCF (7) in F_ρ is the annulus A_ρ, given by

$$R \cdot \frac{\rho(1-\rho)}{4R - \rho(1-\rho)} \leq |f| \leq R, \quad R = \frac{1}{2}(\sqrt{1 - 2\rho(1-\rho)} + \sqrt{1 - 4\rho(1-\rho)}).$$

In the case $\rho = 1/2$ the annulus is $\left(8 + \sqrt{2}\right)/124 \leq |f| \leq 1/2\sqrt{2}$.

In the present paper the answer will be done for the branched continued fraction with independent variables (4) with $z_1 = z_2 = \ldots = z_N = 1$ (named the branched continued fraction of the special form [2, 5, 4]).

1 THE WORPITZKY-LIKE THEOREMS FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM

Since the beginning we prove the Worpitsky-like theorem in a slightly more general form than it was done in [1].

Theorem 4. *Let $\rho \in (0, 1/2]$ and $N \geq 2$ be an integer. In the BCF of the special form*

$$\frac{a_{00}}{1 + \overset{\infty}{\underset{k=1}{D}} \sum\limits_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{1}}, \tag{8}$$

where $a_{i_1 i_2 \dots i_k}$ be complex numbers, $i(k) = i_1 i_2 \dots i_k$ be multiindex $1 \leq i_k \leq i_{k-1}$, $k = 1, 2, \dots,$ $i_0 = N$, $a_{i(k)}$ satisfy the conditions $\left|a_{i(k)}\right| \leq \alpha_{i_{k-1}} = \dfrac{\rho(1-\rho)}{i_{k-1}}$, $|a_{00}| \leq \rho(1 - \rho)$.

Then the BCF of the special form (8) converges, and its values are contained in the disk $|w| \leq \rho$.

Proof. It is not difficult to show that a periodic continued fraction

$$\cfrac{\rho(1-\rho)}{1 - \cfrac{\rho(1-\rho)}{1 - \cfrac{\rho(1-\rho)}{1 - \ddots}}} \tag{9}$$

is the majorant fraction for the BCF of special form (8).

It means that approximants of these fractions satisfy the relation:

$$|f_n - f_m| \leq M \cdot |g_n - g_m|,$$

where f_n, g_n are the nth approximants of the BCF of the special form (8) and continued fraction (9) respectively, M is a certain constant, m, n are natural numbers.

For the difference between the nth and mth approximants of the BCF of the special form (8) the following relation is true [1]:

$$f_n - f_m = (-1)^m \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \ldots \sum_{i_m=1}^{i_{m-1}} \frac{a_{00} \cdot \prod\limits_{k=1}^{m} a_{i(k)}}{\prod\limits_{k=0}^{m} Q_{i(k)}^{(n-1)} \prod\limits_{k=0}^{m-1} Q_{i(k)}^{(m-1)}}, \; n > m \geq 1, \tag{10}$$

where

$$Q_{i(s)}^{(s)} = 1, \quad Q_{i(k)}^{(s)} = 1 + \sum_{i_{k+1}=1}^{i(k)} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(s)}}, \; k = \overline{1, s-1}, s \geq 2,$$

$$Q^{(s)} = Q_{i(0)}^{(s)} = 1 + \sum_{i_1=1}^{N} \frac{a_{i(1)}}{Q_{i(1)}^{(s)}}, \; s \geq 1, \quad f_n = \frac{a_{00}}{Q_{i(0)}^{(n-1)}}.$$

Using the method of complete mathematical induction it is easy to prove that

$$\left| Q_{i(k)}^{(s)} \right| \geq h_{s-k}, \tag{11}$$

where h_m is the m th approximant of the continued fraction

$$1 - \cfrac{\rho(1-\rho}{1 - \cfrac{\rho(1-\rho}{1- \vdots}}$$

for all possible index sets.

Let us write the difference formula for approximants of the continued fraction (9)

$$g_n - g_m = \frac{\rho^{m+1}(1-\rho)^{m+1}}{\prod\limits_{i=0}^{m} h_{n-i-1} \prod\limits_{i=0}^{m-1} h_{m-i-1}}. \tag{12}$$

From (11) follows that all $Q_{i(k)}^{(s)} \neq 0$. Hence, taking into account (10) and (12) we have

$$|f_n - f_m| \leq \sum_{i_1=1}^{N} \sum_{i_2=1}^{i_1} \ldots \sum_{i_m=1}^{i_{m-1}} \frac{|a_{00}| \cdot \prod\limits_{k=1}^{m} \left| a_{i(k)} \right|}{\prod\limits_{k=0}^{m} \left| Q_{i(k)}^{(n-1)} \right| \prod\limits_{k=0}^{m-1} \left| Q_{i(k)}^{(m-1)} \right|}$$
$$\leq \frac{\rho^{m+1}(1-\rho)^{m+1}}{\prod\limits_{k=0}^{m} h_{n-k-1} \prod\limits_{k=0}^{m-1} h_{m-k-1}} = g_n - g_m.$$

The continued fraction (9) converges, and therefore the BCF of the special form (8) is also convergent.

Let us write the m th approximant of (8) in the form

$$z=\frac{a_{00}}{1+\sum\limits_{i_1=1}^{N}\frac{a_{i(1)}}{Q_{i(1)}^{(m-1)}}}=\frac{a_{00}}{(1+w)}.$$

From the conditions of the theorem on the fraction coefficients and inequalities (11) one can write

$$|w|=\left|\sum_{i_1=1}^{N}\frac{a_{i(1)}}{Q_{i(1)}^{(m-1)}}\right|\leq\frac{\rho(1-\rho)}{h_{m-2}}=g_{m-1}.$$

Putting $g_n=P_n/Q_n$, where P_n is the nth numerator and Q_n is the nth denominator of the approximant g_n it is easy to find by induction that

$$Q_n=\sum_{i=0}^{n}\rho^i(1-\rho)^{n-i}.$$

If Q is the value of the infinite fraction (9), and $Q_n>0,\ n=1,2,\ldots$, then we get

$$g_n-g_{n-1}=\frac{(\rho(1-\rho))^n}{Q_nQ_{n-1}}\geq 0,$$

i.e., the sequence $\{g_n\}$ grows monotonically. Hence,$|w|\leq Q$. Since $Q=\rho(1-\rho)\cdot(1-Q)^{-1}$, and taking into account that $Q=0$, if $\rho=0$, the solution of this quadratic equation with respect to Q gives $Q=\rho$.

Therefore, $|w|\leq\rho$, and $|z|\leq\rho$. □

Now we obtain the boundary version of this theorem.

Theorem 5. *Let $\rho\in(0,1/2]$ and $N\geq 2$ be an integer. In the family of branched continued fractions of the special form F_ρ*

$$\frac{a_{00}}{1+\underset{k=1}{\overset{\infty}{\mathrm{D}}}\sum\limits_{i_k=1}^{i_{k-1}}\frac{a_{i(k)}}{1}},\tag{13}$$

where $a_{i_1i_2\ldots i_k}$ be complex numbers, $i(k)=i_1i_2\ldots i_k$ be multiindex $1\leq i_k\leq i_{k-1}$, $k=1,2,\ldots,$ $i_0=N$, $a_{i(k)}$ satisfy the conditions $\left|a_{i(k)}\right|=\dfrac{\rho(1-\rho)}{i_{k-1}}$, $|a_{00}|=\rho(1-\rho)$, the set of all possible branched continued fractions of the special form values is the annulus A_ρ, given by

$$\rho\cdot\frac{1-\rho}{1+\rho}\leq|w|\leq\rho.$$

Proof. Let f_0 be a possible value of the BCF of the special form (13). Then all values f with $|f|=|f_0|$ are possible BCF of the special form values in F_ρ. Hence the set of values of such fraction must be a disk or an annulus, in both cases centered at the origin. From the Worpitzky-like theorem (Theorem 4) follows that this disk or annulus must be contained in the disk $|f|\leq\rho$.

We shall first prove that the set of all values must be contained in A_ρ. Any BCF of the special form in F_ρ can be written in the form

$$f=\frac{\rho(1-\rho)e^{i\theta}}{1+\omega},\ \theta\in[0,2\pi),\ \omega=\sum_{i_1=1}^{N}\frac{a_{i(1)}}{1+\underset{k=1}{\overset{\infty}{D}}\sum\limits_{i_{k+1}=1}^{i_k}\frac{a_{i(k+1)}}{1}}.$$

Since $\dfrac{a_{i(1)} \cdot N}{1 + \overset{\infty}{\underset{k=1}{D}} \sum\limits_{i_{k+1}=1}^{i_k} \dfrac{a_{i(k+1)}}{1}} \in F_\rho$ we have, using the previous Theorem 4

$$\left| a_{i(1)} \cdot N \cdot \left(1 + \overset{\infty}{\underset{k=1}{D}} \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{1} \right)^{-1} \right| \leq \rho.$$

It means that

$$\left| a_{i(1)} \cdot \left(1 + \overset{\infty}{\underset{k=1}{D}} \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{1} \right)^{-1} \right| \leq \frac{\rho}{N},$$

and $|\omega| \leq \rho$. Since $|\omega| \leq \rho$ it follows that for any value f of a BCF of the special form in F_ρ we have $|f| \geq \rho \cdot \dfrac{1-\rho}{1+\rho}$.

That is sharp, follows from the fact that

$$\rho = \cfrac{\rho(1-\rho)}{1 - \cfrac{\rho(1-\rho)}{1 - \vdots}},$$

and that the right-hand side is in F_ρ.

We next prove that A_ρ is contained in the set of values of BCFs of the special form in F_ρ with independent variables $|\omega| \leq \rho$.

By the mapping $\xi = 1/1 + \omega$ the circle $\omega = \rho$ is mapped onto the circle

$$\left| \xi - \frac{1}{1-\rho^2} \right| = \frac{\rho}{1-\rho^2}.$$

Then, by $\xi \to \rho(1-\rho)e^{i\theta}\xi$, for all $\theta \in [0, 2\pi)$ we get all points in the annulus A_ρ.

Hence, A_ρ is contained in the set of BCF with independent variables values for F_ρ. $\square$

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Received 19.12.2015

Revised 25.03.2016

Кучмінська Х.Й. *Межова теорема Ворпіцького для гіллястих ланцюгових дробів спеціального вигляду* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 272–278.

Для гіллястого ланцюгового дробу спеціального вигляду запропоновано межову множину значень у теоремі типу Ворпіцького, коли множина елементів гіллястого ланцюгового дробу замінена її межею.

Ключові слова і фрази: множина елементів, множина значень, гіллястий ланцюговий дріб спеціального вигляду.