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ON GENERALIZED COMPLEX SPACE FORMS SATISFYING CERTAIN CURVATURE CONDITIONS

We study Ricci soliton (g, V, λ) of generalized complex space forms when the Riemannian, Bochner and W_2 curvature tensors satisfy certain curvature conditions like semi-symmetric, Einstein semi-symmetric, Ricci pseudo symmetric and Ricci generalized pseudo symmetric. In this study it is shown that shrinking, steady and expansion of the generalized complex space forms depend on the solenoidal property of vector V. Also we prove that generalized complex space form with conservative Bochner curvature tensor is constant scalar curvature.

Key words and phrases: generalized complex space forms, Ricci soliton, Einstein manifold, Einstein semi-symmetric, pseudo symmetric.

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1 INTRODUCTION

A Kähler manifold with constant holomorphic sectional curvature is a complex space form and it has a specific form of its curvature tensor. More generally an almost Hermition manifold M is called a generalized complex space form $M(f_1, f_2)$ if its Riemannian curvature tensor R satisfies,

$$\begin{aligned} R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,JZ)JY \\ - g(Y,JZ)JX + 2g(X,JY)JZ\}, \end{aligned} \tag{1}$$

for all $X, Y, Z \in TM$, where f_1 and f_2 are smooth functions on M [21]. In [21], an important obstruction for such a space was presented by Tricerri and Vanhecke: if M is connected, dim ≥ 6 and f_2 is not identically zero, then M is a complex-space-form (in particular, f_1 and f_2 must be constant). Olszak [16] proved the existence of generalized complex space form. The authors Alegre and Carriazo studied structures on generalized Sasakian space forms [1]. The authors De [7], Kim [12], Atceken [13], Nagaraja [14], et. al., have contributed to the study of Sasakian space forms in which they put different symmetric conditions on projctive curvature tensor etc.

A Riemannian manifold (M, g) is called locally symmetric if its curvature tensor R is parallel [5], i.e. $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. As a proper generalization of locally symmetric manifold the notion of semi-symmetric manifold was defined by

$$(R(X,Y) \cdot R)(U, V, W) = 0, \quad X, Y, U, V, W \in \chi(M)$$

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and it is studied by many authors [15,17]. A complete intrinsic classification of these was given by Szabo [20].

For a $(0,k)$-tensor field T on M, $k \geq 1$, and a symmetric $(0,2)$ tensor fields g and S on M, we define the $(0,k+2)$ tensor fields $R \cdot T$, $Q(A,T)$ and $Q(B,T)$ by

$$
\begin{aligned}
&(R \cdot T)(X_1, \ldots, X_k, X, Y) = \\
&\qquad - T(R(X,Y)X_1, X_2, \ldots, X_k - \cdots - T(X_1, X_2, \ldots, X_{k-1}, R(X,Y)X_k), \\
&Q(g,T)(X_1, \ldots, X_k, X, Y) = \\
&\qquad - T((X \wedge_g Y)X_1, X_2, \ldots, X_k - \ldots - T(X_1, X_2, \ldots, X_{k-1}, (X \wedge_g Y)X_k), \\
&Q(S,T)(X_1, \ldots, X_k, X, Y) = \\
&\qquad - T((X \wedge_S Y)X_1, X_2, \ldots, X_k - \cdots - T(X_1, X_2, \ldots, X_{k-1}, (X \wedge_S Y)X_k),
\end{aligned}
$$

where $(X \wedge_g Y)$ and $(X \wedge_S Y)$ are the endomorphism given by

$$(X \wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y, \quad (X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y.$$

A Riemannian manifold is said to be pseudo symmetric (in the sense of Deszcz [6,9]) if

$$R \cdot R = L_R Q(g,R)$$

holds on the set $U_R = \{x \in M \mid R - \frac{r}{n(n-1)}G \neq 0 \text{ at } x\}$, where G is the $(0,4)$–tensor defined by $G(X_1, X_2, X_3, X_4)$=$g((X_1 \wedge X_2)X_3, X_4)$ and L_R is some function on U_R.

A Riemannian manifold is said to be Ricci generalized pseudo symmetric (in the sense of Deszcz [6,9]) if

$$R \cdot R = L_R Q(S,R)$$

holds on the set $U_R = \{x \in M : Q(S,R) \neq 0 \text{ at } x\}$, and L_R is some function on U_R. A Riemannian manifold is said to be Bochner Ricci generalized pseudo symmetric if

$$R \cdot B = L_B Q(S,B)$$

holds on the set $U_B = \{x \in M : B \neq 0 \text{ at } x\}$, and L_B is some function on U_B and B is the Bochner curvature tensor. If $L_B = 0$ on U_B, then a Bochner Ricci generalized pseudo symmetric manifold is Bochner semisymmetric. But L_B need not be zero, in general and hence there exists Bochner Ricci generalized pseudo symmetric manifolds which are not Bochner semisymmetric manifolds. Thus the class of Bochner Ricci generalized pseudo symmetric manifolds is a natural extension of the class of Bochner semisymmetric manifolds.

Also we need the notion of Ricci solitons. It is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M,g). A Ricci soliton is a triple (g,V,λ) with g a Riemannian metric such that

$$L_V g + 2S + 2\lambda g = 0, \tag{2}$$

where V is the potential vector field, λ a real scalar, S is Ricci tensor of M and L_V denotes the Lie derivative operator along V. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively [10].

In the context of generalized complex space forms, the authors Bharathi and Bagewadi [3], Bagewadi and Praveena [2, 19] extended the study to W_2 curvature, H-projective, Bochner and pseudoprojective curvature tensors. Motivated by these ideas, in this paper, we extend the study of Ricci soliton in which curvature tensor on generalized complex space forms satisfy several semi-symmetric and pseudo-symmetric conditions. The paper is organized as follows. In the section 2 we give definitions, notions and basic results for generalized complex space forms. In sections 3 and 4 we study Bochner semi-symmetric and Einstein semi-symmetric on generalized complex space forms. In sections 5 and 6 we find the characterizations of generalized complex space forms satisfying the pseudo-symmetric conditions like $R \cdot B = L_B Q(S, B)$. and $B \cdot W_2 = L_1 Q(g, W_2)$. Finally we obtain generalized complex space form with conservative Bochner curvature tensor is of constant scalar curvature.

2 PRELIMINARIES

Let M be a complex n-dimensional Kähler manifold, with a complex structure J and a positive–definite metric g which satisfies the following conditions [4]

$$J^2 = -I, \quad g(JX, JY) = g(X, Y) \quad \text{and} \quad \nabla J = 0,$$

where ∇ means covariant derivation according to the Levi-civita connection. The scalar curvature $r = \Sigma S(e_i, e_i)$, therefore

$$(\nabla_X S)(e_i, e_i) = \nabla_X r = dr(X).$$

Let Q be the Ricci operator defined by $g(QX, Y) = S(X, Y)$. Then

$$(\nabla_Z S)(X, Y) = g((\nabla_Z Q)(X), Y).$$

Taking $Y = Z = e_i$ and taking summation over i in the above equation we get

$$(\nabla_{e_i} S)(X, e_i) = g((\nabla_{e_i} Q)(X), e_i),$$
$$(divQ)(X) = tr(Z \to (\nabla_Z Q)(X)) = \sum g((\nabla_{e_i} Q)(X), e_i).$$

But it is known [8, 18] that $(divQ)(X) = \frac{1}{2}dr(X)$. Hence $(\nabla_{e_i} S)(X, e_i) = \frac{1}{2}dr(X)$ and $(\nabla_{e_i} S)(JX, e_i) = \frac{1}{2}dr(JX)$. It is known [11] that in a Kähler manifold the Ricci tensor S satisfies

$$(divR)(X, Y)Z = (\nabla_Z S)(X, Y) - (\nabla_X S)(Z, Y) = (\nabla_{JY} S)(JX, Z). \tag{3}$$

Using equation (1) we have

$$S(X, Y) = \{(n-1)f_1 + 3f_2\}g(X, Y), \tag{4}$$
$$QX = [(n-1)f_1 + 3f_2]X, \tag{5}$$
$$r = n[(n-1)f_1 + 3f_2], \tag{6}$$

where S is the Ricci tensor, Q is the Ricci operator and r is scalar curvature of the space form $M(f_1, f_2)$.

Given a complex n-dimensional Kähler manifold M, the Bochner curvature tensor and W_2 curvature tensor are given by [11]

$$\begin{aligned}B(X,Y,Z,U) &= R(X,Y,Z,U) - \frac{1}{2n+4}[g(Y,Z)S(X,U) - S(X,Z)g(Y,U)\\ &\quad + g(JY,Z)S(JX,U) - S(JX,Z)g(JY,U) + S(Y,Z)g(X,U)\\ &\quad - g(X,Z)S(Y,U) + S(JY,Z)g(JX,U) - g(JX,Z)S(JY,U)\\ &\quad - 2S(Y,JX)g(JZ,U) - 2S(JZ,U)g(JX,Y)]\\ &\quad + \frac{r}{(2n+2)(2n+4)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U) + g(JY,Z)g(JX,U)\\ &\quad - g(JX,Z)g(JY,U) - 2g(JX,Y)g(JZ,U)],\end{aligned} \tag{7}$$

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX]. \tag{8}$$

Definition 1. *The Einstein Tensor denoted by E is defined by*

$$E(X,Y) = S(X,Y) - \frac{r}{n}g(X,Y), \tag{9}$$

where S is a Ricci tensor and r is the scalar curvature.

Definition 2 ([9,20]). *A n-dimensional generalized complex space form is said to be:*

1) *Bochner-Semi-symmetric if it satisfies*

$$(R(X,Y)\cdot B)(U,V,W) = 0 \text{ for all } X,Y \in \chi(M);$$

2) *Einstein-Semi-symmetric if it satisfies*

$$(R(X,Y)\cdot E)(U,V,W) = 0 \text{ for all } X,Y \in \chi(M).$$

3 BOCHNER SEMI-SYMETRIC GENERALIZED COMPLEX SPACE FORMS

Let generalized complex space form $M(f_1, f_2)$ be Bochner semi-symmetric and by definition it satisfies the equation $R \cdot B = 0$, i.e. for any tangent vectors X, Y, U, Z and W, this implies

$$(R(X,Y)\cdot B)(U,Z,W) = 0.$$

Therefore

$$R(X,Y)B(U,Z)W - B(R(X,Y)U,Z)W - B(U,R(X,Y)Z)W - B(U,Z)R(X,Y)W = 0.$$

Taking inner product with T we have,

$$\begin{aligned}g(R(X,Y)B(U,Z)W,T) - g(B(R(X,Y)U,Z)W,T) - g(B(U,R(X,Y)Z)W,T)\\ - g(B(U,Z)R(X,Y)W,T) = 0.\end{aligned} \tag{10}$$

Using equations (1) and (7) in (10) and putting $X = Z = e_i$, further again putting $Y = T = e_i$ to the simplified equation, where e_i is an an orthonarmal basis of the tangent space at each point of the manifold and taking summation over $i, 1 \le i \le n$, we get

$$f_2\left\{\frac{2n-8}{2n+4}S(U,W) - \frac{5n+2}{(2n+4)(2n+2)}rg(U,W)\right\} = 0.$$

If $f_2 \neq 0$, then

$$\left\{\frac{2n-8}{2n+4}S(U,W) - \frac{5n+2}{(2n+4)(2n+2)}rg(U,W)\right\} = 0.$$

This implies,

$$S(U,W) = \frac{5n+2}{(2n-8)(2n+2)}rg(U,W). \tag{11}$$

That is $M(f_1, f_2)$ is an Einstein manifold. Hence we can state the following result.

Theorem 1. *A generalized complex space form $M(f_1, f_2)$ is an Einstein manifold provided by $f_2 \neq 0$ if Bochner curvature tensor satisfies $R \cdot B = 0$.*

Using equation (11) in (2), we get

$$(L_V g)(U,W) + 2\left[\frac{5n+2}{(2n-8)(2n+2)}\right] rg(U,W) + 2\lambda g(U,W) = 0, \tag{12}$$

setting $U = W = e_i$ in (12) and then taking summation over $i, 1 \leq i \leq n$, we obtain

$$divV + \frac{5n+2}{(2n-8)(2n+2)}rn + \lambda n = 0. \tag{13}$$

If V is solenoidal then $divV = 0$. Therefore the equation (13) can be reduced to

$$\lambda = -\frac{5n+2}{(2n-8)(2n+2)}r.$$

Thus, we can state the following.

Corollary 1. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying Bochner semi-symmetric. If V is solenoidal then it is shrinking, steady and expanding accordingly scalar curvature is positive, zero and negative respectively.*

4 EINSTEIN SEMI-SYMMETRIC GENERALIZED COMPLEX SPACE FORM

Let R and E satisfy the equation $R \cdot E = 0$ in $M(f_1, f_2)$. Then this equation leads to

$$(R(X,Y) \cdot E(U,W)) = 0,$$

where X, Y, U and W are any tangent vectors. The above equation can be expressed as

$$E(R(X,Y)U,W) + E(U,R(X,Y)W) = 0. \tag{14}$$

In view of (9) equation (14) becomes

$$S(R(X,Y)U,W) - \frac{r}{2}g(R(X,Y)U,W) + S(U,R(X,Y)W) - \frac{r}{2}g(U,R(X,Y)W) = 0. \tag{15}$$

Using equation (1) in (15) and by replacing $X = U = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over $i, 1 \leq i \leq n$, we get

$$f_1[-nS(Y,W) + rg(Y,W)] = 0.$$

If $f_1 \neq 0$, then

$$S(Y,W) = \frac{r}{n}g(Y,W). \tag{16}$$

Then we can state the following.

Theorem 2. *If generalized complex space form is Einstein semi-symmetric then it is an Einstein manifold provided $f_1 \neq 0$.*

Using equation (16) in (2), we get

$$(L_V g)(Y,W) + 2\frac{r}{n}g(Y,W) + 2\lambda g(Y,W) = 0. \tag{17}$$

Let $\{e_i : i = 1,2,...,n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Then setting $Y = W = e_i$ in (17) and then taking summation over $i, 1 \leq i \leq n$, we obtain

$$(L_V g)(e_i,e_i) + 2\frac{r}{n}g(e_i,e_i) + 2\lambda g(e_i,e_i) = 0.$$

This implies

$$divV + r + \lambda n = 0. \tag{18}$$

If V is solenoidal then $divV = 0$. Therefore the equation (18) can be reduced to

$$\lambda = -\frac{r}{n}.$$

Thus we can state the following.

Corollary 2. *Let (g,V,λ) be a Ricci soliton in a generalized complex space form satisfying Einstein semi-symmetric condition. Then V is solenoidal if and only if it is shrinking, steady and expanding accordingly scalar curvature is positive, zero and negative respectively.*

5 BOCHNER RICCI-GENERALIZED PSEUDO-SYMMETRIC GENERALIZED COMPLEX SPACE FORMS

Let us consider the Ricci-generalized Bochner pseudosymmetric generalized complex space form $M(f_1,f_2)$. Then we have

$$(R(X,Y)\cdot B)(U,Z,W) = L_B((X\Lambda_S Y \cdot B)(U,Z,W).$$

This implies

$$\begin{aligned}&R(X,Y)B(U,Z)W - B(R(X,Y)U,Z)W - B(U,R(X,Y)Z)W - B(U,Z)R(X,Y)W\\ &= L_B[(X\Lambda_S Y)B(U,Z)W - B((X\Lambda_S Y)U,Z)W - B(U,(X\Lambda_S Y)Z)W - B(U,W)(X\Lambda_S Y)W].\end{aligned}$$

Taking inner product with T we have,

$$\begin{aligned}&g(R(X,Y)B(U,Z)W,T) - g(B(R(X,Y)U,Z)W,T) - g(B(U,R(X,Y)Z)W,T)\\ &\quad - g(B(U,Z)R(X,Y)W,T) = L_B[g((X\Lambda_S Y)B(U,Z)W,T) - g(B((X\Lambda_S Y)U,Z)W,T)\\ &\quad - g(B(U,(X\Lambda_S Y)Z)W,T) - g(B(U,Z)(X\Lambda_S Y)W,T)].\end{aligned} \tag{19}$$

Using equations (7), (4) and (5) in (19) and substituting $X = Z = e_i$, further again substituting $Y = T = e_i$ in the resulting equation, where $\{e_i\}, i, 1 \leq i \leq n$, is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, we get

$$\begin{aligned}&f_2\left\{\frac{2n-8}{2n+4}S(U,W) - \frac{5n+2}{(2n+4)(2n+2)}rg(U,W)\right\}\\ &\quad = L_B\left[\frac{4((n-1)f_1+3f_2-1)-n(r+1)}{2n+4}S(U,W) + \frac{r(n+2)-(n+4)}{(2n+2)(2n+4)}rg(U,W)\right].\end{aligned}$$

This implies that

$$\left[\frac{f_2(2n-8) - L_B(4((n-1)f_1 + 3f_2 - 1) - n(r+1))}{2n+4}\right] S(U,W)$$
$$- \left[\frac{f_2(5n+2) + L_B(r(n+2) - (n+4))}{(2n+4)(2n+2)}\right] rg(U,W) = 0.$$

The above equation implies

$$[\alpha S(U,W) - \beta rg(U,W)] = 0,$$

where $\alpha = \left[\frac{f_2(2n-8) - L_B(4((n-1)f_1+3f_2-1)-n(r+1))}{2n+4}\right]$ and $\beta = \left[\frac{f_2(5n+2)+L_B(r(n+2)-(n+4))}{(2n+4)(2n+2)}\right]$. This implies

$$S(U,W) = \frac{\beta r}{\alpha} g(U,W). \tag{20}$$

Theorem 3. *A Bochner Ricci-generalized pseudo-symmetric generalized complex space form is an Einstein manifold.*

Using equation (20) in (2), we get

$$(L_V g)(U,W) + 2\frac{\beta r}{\alpha} g(U,W) + 2\lambda g(U,W) = 0. \tag{21}$$

Contraction of (21) over U and W gives

$$(L_V g)(e_i, e_i) + 2\frac{\beta r}{\alpha} g(e_i, e_i) + 2\lambda g(e_i, e_i) = 0.$$

This implies

$$divV + \frac{\beta r}{\alpha} n + \lambda n = 0. \tag{22}$$

If V is solenoidal then $divV = 0$. Therefore the equation (22) can be reduced to

$$\lambda = -\frac{\beta r}{\alpha}.$$

Thus we can state the following.

Corollary 3. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying Bochner Ricci-Generalized pseudo-symmetric generalized complex space forms. Then V is solenoidal if and only if it is shrinking or steady or expanding depending upon the sign of scalar curvature.*

6 GENERALIZED COMPLEX SPACE FORM SATISFYING $B \cdot W_2 = L_1 Q(g, W_2)$

We assume that $B \cdot W_2 = L_1 Q(g, W_2)$ hold on $M(f_1, f_2)$, then we have

$$(B(X,Y) \cdot W_2)(U,V,Z) = L_1[((X \wedge Y) \cdot W_2)(U,V)Z].$$

This implies,

$$B(X,Y)W_2(U,V)Z - W_2(B(X,Y)U,V)Z - W_2(U,B(X,Y)V)Z - W_2(U,V)B(X,Y)Z$$
$$= L_1[(X\Lambda_g Y)W_2(U,V)Z - W_2((X\Lambda_g Y)U,V)Z - W_2(U,(X\Lambda_g Y)V)Z - W_2(U,V)(X\Lambda_g Y)Z.$$

Taking inner product with T we have,

$$\begin{aligned} &g(B(X,Y)W_2(U,V)Z,T) - g(W_2(B(X,Y)U,V)Z,T) - g(W_2(U,B(X,Y)V)Z,T) \\ &- g(W_2(U,V)B(X,Y)Z,T) = L_B[g((X\Lambda_g Y)W_2(U,V)Z,T) - g(W_2((X\Lambda_g Y)U,V)Z,T) \\ &- g(W_2(U,(X\Lambda_g Y)V)Z,T - g(W_2(U,V)(X\Lambda_g Y)Z,T)]. \end{aligned} \tag{23}$$

Applying equations (1), (7) and (8) in (23) and putting $X = V = e_i$, further again putting $Y = T = e_i$ in the resulting equation and taking summation over $i, 1 \le i \le n$, we get

$$\frac{\gamma}{(n-1)}S(U,Z) + \frac{\delta}{(n-1)}rg(U,Z) = L_1[\frac{1}{n-1}[nS(U,Z) - rg(U,Z)]] \tag{24}$$

where

$$\gamma = \frac{(2n+2)[(6n^3-8n^2-39n-22)f_2 - 2(n^3+4n^2+7n-18)(n+1)f_1] + rn(2n+4)}{(2n+2)(2n+4)},$$
$$\delta = \frac{-f_1(2n+2)(4n+2) + 6f_2(2n+2)(n+1) + r}{(2n+2)(2n+4)}.$$

Equation (24) implies

$$[\gamma S(U,W) + \delta rg(U,W)] = L_1[nS(U,Z) - rg(U,Z)].$$

The above equation implies

$$S(U,W) = Arg(U,W), \tag{25}$$

where $A = \frac{(L_1+\delta)}{L_1 n - \gamma}$. Thus we can state.

Theorem 4. *A n-dimensional generalized complex space form satisfying $B \cdot W_2 = L_1 Q(g, W_2)$ is an Einstein manifold.*

Using equation (25) in (2), we get

$$(L_V g)(U,W) + 2Arg(U,W) + 2\lambda g(U,W) = 0. \tag{26}$$

Taking $U = W = e_i$ and summing over $i = 1,2,...,n$ in (26) we obtain

$$(L_V g)(e_i,e_i) + 2Arg(e_i,e_i) + 2\lambda g(e_i,e_i) = 0.$$

This implies

$$divV + Arn + \lambda n = 0. \tag{27}$$

If V is solenoidal then $divV = 0$. Therefore the equation (27) can be reduced to

$$\lambda = -Ar. \tag{28}$$

Thus we can state the following.

Corollary 4. *Let (g, V, λ) be a Ricci soliton in a generalized complex space form satisfying $B \cdot W_2 = L_1 Q(g, W_2)$. Then V is solenoidal if and only if it is shrinking or steady or expanding depending upon the sign of scalar curvature.*

7 GENERALIZED COMPLEX SPACE FORM WITH $divB = 0$

Assume that the Bochner curvature tensor of a generalized complex space form is conservative that is $divB = 0$. Using equations (4) and (5) in (7), then we obtain

$$\begin{aligned} B(X,Y,Z) = R(X,Y,Z) - 2\frac{[(n-1)f_1+3f_2]}{2n+4}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ] \\ + \frac{r}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ], \end{aligned} \tag{29}$$

Differentiating (29) covariantly, contracting and our assumption yields.

$$\begin{aligned} 0 = (divR)(X,Y)Z - 2\frac{d[(n-1)f_1+3f_2]}{2n+4}[g(Y,Z)X \\ -g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ] \\ + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ], \end{aligned} \tag{30}$$

Using equation (3) in (30) we obtain

$$\begin{aligned} 0 = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - 2\frac{d[(n-1)f_1+3f_2]}{2n+4}[g(Y,Z)X - g(X,Z)Y \\ + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ] \\ + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ]. \end{aligned} \tag{31}$$

Taking $[(n-1)f_1 + 3f_2] = constant = k_1 \neq 0$ in equation (31) we obtain

$$\begin{aligned} 0 = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y \\ + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ]. \end{aligned} \tag{32}$$

Again using equation (3) in (32) we get

$$\begin{aligned} 0 = (\nabla_{JZ} S)(JY,X) + \frac{dr}{(2n+2)(2n+4)}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ - g(JX,Z)JY - 2g(JX,Y)JZ]. \end{aligned}$$

Replace Z by JZ in the above equation we get

$$\begin{aligned} (\nabla_Z S)(JY,X) = \frac{dr}{(2n+2)(2n+4)}[g(Y,JZ)X - g(X,JZ)Y + g(Y,Z)JX \\ - g(X,Z)JY + 2g(JX,Y)Z]. \end{aligned} \tag{33}$$

Contraction of (33) over Y and Z after simplification we get $dr(JX) = 0$. If $dr(JX) = 0$ then $dr(X) = 0$ so r is constant. Using $r = constant$ in (32) we get

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z).$$

We can state the following.

Theorem 5. *A n-dimensional generalized complex space form with conservative Bochner curvature tensor is constant scalar curvature provided* $[(n-1)f_1 + 3f_2] = k_1(constant)$.

Theorem 6 ([8])**.** *Let M be a Kaehler manifold of dimension* $n \geq 4$. *Then div R=0 and div C=0 are equivalent.*

Using above Theorem we can state the following.

Theorem 7. *Let M be a generalized complex space form of dimension* $n \geq 4$. *Then div R=0, div C=0 and div B=0 are equivalent provided* $[(n-1)f_1 + 3f_2] = k_1(constant)$.

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Правіна М.М., Баґеваді Ц.С. *Про узагальнені форми в комплексному просторі, які задовільняють певні умови кривини* // Карпатські матем. публ. — 2016. — Т.8, №2. — C. 284–294.

Ми вивчаємо солітон Річчі (g, V, λ) на узагальнених формах в комплексному просторі при умовах, що тензори з кривиною Рімана, Бохнера і W_2 задовільняють певні умови кривини, а саме напівсиметричності, Ейнштейнової напівсиметричності, псевдосиметричності Річч та узагальненої псевдосиметричності Річчі. У роботі показано, що стиснення, випрямлення і розширення узагальнених форм в сомплексному просторі залежить від соленоїдальних властивостей вектора V. Також доведено, що узагальнена форма у комплексному просторі з звичайним тензором кривизни Бохнера має сталу скалярну кривизну.

Ключові слова і фрази: узагальнені форми у комплексному просторі, многовид Ейнштейна, напівсиметричність Ейнштейна, псевдосиметричність.