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DIVISOR PROBLEM IN SPECIAL SETS OF GAUSSIAN INTEGERS

Let A_1 and A_2 be fixed sets of gaussian integers. We denote by $\tau_{A_1,A_2}(\omega)$ the number of representations of ω in form $\omega = \alpha\beta$, where $\alpha \in A_1, \beta \in A_2$. We construct the asymptotical formula for summatory function $\tau_{A_1,A_2}(\omega)$ in case, when ω lie in the arithmetic progression, A_1 is a fixed sector of complex plane, $A_2 = \mathbb{Z}[i]$.

Key words and phrases: Gaussian numbers, divisor problem, asymptotic formula, arithmetic progression.

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INTRODUCTION

Let A_1 and A_2 be fixed infinite sets of natural numbers. We let $\tau_{A_1,A_2}(n)$ denote the number of representations of n in form $n = m_1m_2$, where $m_1 \in A_1, m_2 \in A_2$. To investigate average order of function $\tau_{A_1,A_2}(n)$, it is usual to consider the summatory function

$$\sum_{n\leqslant x} \tau_{A_1,A_2}(n),$$

where x is a large real variable. For $A_1 = A_2 = \mathbb{N}$, this is the classical Dirichlet divisor problem about the number of lattice points (u, v) under the hyperbola $uv \leqslant x,\ u, v \geqslant 1$. Historical review results on the divisor problem can be found in the monograph of Krätzel [4]. The best estimate to-date is due to Huxley [3]

$$\sum_{n\leqslant x} \tau_{\mathbb{N},\mathbb{N}}(n) = x\log x + (2\gamma - 1) + O(x^{\frac{131}{416}} (\log x)^{\frac{26947}{8320}}).$$

In articles [5–9] the authors discussed special cases of sets of natural numbers A_1, A_2.

The similar problem was considered over the ring of the Gaussian integers $\mathbb{Z}[i]$ in the work of Varbanets and Zarzycki [9] in case, when

$$A_1 = \mathbb{Z}[i], \quad A_2 = \{\alpha \in \mathbb{Z}[i] : \alpha \equiv \alpha_0 \pmod{\gamma}\}, \quad \alpha_0, \gamma \in \mathbb{Z}[i].$$

The following asymptotic formula was obtained

$$\sum_{\substack{\omega=\alpha\beta \\ \alpha\equiv\alpha_0 \pmod{\gamma} \\ N(\alpha\beta)\leqslant x}} 1 = \frac{\pi^2 x\log x}{N(\gamma)} + c(\alpha_0,\gamma)\frac{x}{N(\gamma)} + O\left(\left(\frac{x}{N(\gamma)}\right)^{\frac{1}{2}+\varepsilon}\right) + O\left(\left(\frac{x}{N(\alpha_1)}\right)^{\theta}\right) + O(x^{\varepsilon}),$$

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where $\theta < \frac{1}{3}$, α_1 is a number of form $\alpha_0 + \beta\gamma$, $\beta \in \{0, \pm 1, \pm i\}$ with the smallest norm, the constant $c(\alpha_0, \gamma)$ is computable and depends on α_0 and γ.

In the present paper, we investigate the distribution of values of the divisor function not only in an arithmetic progression, but in narrow sectorial region also. By the $\tau_S(\omega)$ we denote the function $\tau_{A_1,A_2}(\omega)$ in case, when $A_1 = \mathbb{Z}[i]$, $A_2 = S(\varphi)$ is a fixed sector of complex plane

$$S(\varphi) := \{\alpha \in \mathbb{Z}[i] : \varphi_1 < \arg\alpha \leqslant \varphi_2,\ \varphi = \varphi_2 - \varphi_1\}.$$

The main point of this paper is to construct an asymptotic formula for sum

$$T(x, \gamma, \omega_0, S(\varphi)) = \sum_{\substack{\omega \equiv \omega_0 \pmod{\gamma}, \\ N(\omega) \leq x}} \tau_S(\omega),$$

in particular to investigate the ranges of γ and x for which this formula is nontrivial. Applying the method of Vinogradov we get the asymptotic formula in case, when the norm of a difference of progression grows.

In this paper we denote by $\mathbb{Z}[i]$ the ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}.$$

For $\alpha \in \mathbb{Z}[i]$ we put $Sp(\alpha) = \alpha + \overline{\alpha} = \operatorname{Re}\alpha$, $N(\alpha) = \alpha \cdot \overline{\alpha}$, where $\overline{\alpha}$ denotes a complex conjugate with α. $Sp(\alpha)$ and $N(\alpha)$ we name a trace and a norm (respectively) of α from $\mathbb{Z}[i]$. Moreover, $\exp(x) := e^x$, $\quad e_q(z) := e^{2\pi i \frac{z}{q}}$ for $q \in \mathbb{N}$. The Vinogradov's symbol as in $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; ε is an arbitrary small positive number that is not necessarily the same at each occurence; the constants implied by the O (or $\ll$) — notation depend at most on ε. $\zeta(s)$ is the Riemann zeta-function; $L(s, \chi_4)$ is the Dirichlet L-function with the non-principal character modulo 4. $\mathfrak{B} := \{0, \pm 1, \pm i\}$. $\overline{\varphi}(\alpha) = N(\alpha)\prod_{p|\alpha}(1 - N(p)^{-1})$ denotes the Euler function in $\mathbb{Z}[i]$.

1 Preliminaries

We begin this section with few background definitions and facts. Note that every non-zero Gaussian number has associated element in each quadrant of the complex plane. Therefore without loss of generality, we assume $0 \leqslant \varphi_1 < \varphi_2 \leqslant \frac{\pi}{2}$. Let $\chi(\varphi)$ be a characteristic function of sector S. We will follow the idea of Vinogradov [1]. We first mention some classical results.

Lemma 1 ([1]). *Suppose r is an integer, $r > 0$, $\Omega > 0$, $0 < \Delta < \frac{1}{2}\Omega$, φ_1, φ_2 are real numbers, $\Delta \leqslant \varphi_2 - \varphi_1 \leqslant \Omega - 2\Delta$. Then there exists a periodic function $f(\varphi) = f(\varphi; \varphi_1, \varphi_2)$ with period Ω such that:*

1. *$f(\varphi) = 1$ in the interval $[\varphi_1, \varphi_2]$; $0 \leqslant f(\varphi) \leqslant 1$ in the intervals $[\varphi_1 - \Delta, \varphi_1]$ and $[\varphi_2, \varphi_2 + \Delta]$;*
2. *$f(\varphi) = 0$ in the interval $[\varphi_2 + \Delta, \varphi_2 + \Omega - \Delta]$;*
3. *$f(\varphi)$ can be expanded into Fourier series of the form*

$$f(\varphi) = \sum_{m=-\infty}^{\infty} a_m \exp\Big(2\pi i \frac{m\varphi}{\Omega}\Big),$$

where $a_0 = \frac{1}{\Omega}(\varphi_2 - \varphi_1 + \Delta)$, $|a_m| \leqslant \begin{cases} \Omega^{-1}(\varphi_2 - \varphi_1 + \Delta), \\ 2(\pi|m|)^{-1}, \\ 2(\pi|m|)^{-1}(r\Omega(\pi|m|\Delta)^{-1})^r. \end{cases}$

Remark 1. *There exist numbers* θ_i, $|\theta_i| \leqslant 1$, $i = 1, 2$, *such that*

$$\chi(\varphi) = f(\varphi; \varphi_1, \varphi_2) + \theta_1 f(\varphi; \varphi_1 - \Delta, \varphi_1) + \theta_2 f(\varphi; \varphi_2, \varphi_2 + \Delta). \tag{1}$$

Let δ, $\delta_0 \in \mathbb{Q}[i]$ and $m \in \mathbb{Z}$. Let for $\operatorname{Re} s > 1$ we define the Hecke Z-function with the shift

$$Z_m(s; \delta, \delta_0) = \sum_{\substack{\omega \in \mathbb{Z}[i] \\ \omega \neq -\delta}} \frac{\exp\left(4mi \arg\left(\omega + \delta\right)\right)}{N(\omega + \delta)} \exp\left(2\pi i \operatorname{Re}(\delta_0 \omega)\right).$$

Lemma 2. $Z_m(s; \delta, \delta_0)$ *is an entire function if* $m \neq 0$ *and* $\delta_0 \notin \mathbb{Z}[i]$. *For* $m = 0$ *and* $\delta_0 \in \mathbb{Z}[i]$ *Hecke* Z*-function* $Z_0(s; \delta, \delta_0)$ *is a holomorphic function in the whole complex plane except at* $s = 1$, *where it has a simple pole with residue* π. *It satisfies the functional equation*

$$\pi^{-s}\Gamma(2|m| + s) Z_m(s; \delta, \delta_0) = \pi^{-(1-s)}\Gamma(2|m| + 1 - s) Z_m(1 - s; -\overline{\delta_0}, \delta) \exp\left(-2\pi i \operatorname{Re}(\delta_0 \delta)\right) \tag{2}$$

in all cases.

For the proof in the case $\delta = \delta_0 = 0$ see [2]. The proof in other cases similar.

Lemma 3 ([9])**.** *Let* δ *be a Gaussian rational,* $N(\delta) < 1$. *Then* $Z_0(s; \delta, 0)$ *has the following Laurent expansion*

$$Z_0(s; \delta, 0) = \frac{\pi}{s - 1} + a_0(\delta) + a_1(\delta)(s - 1) + \ldots,$$

where

$$a_0(\delta) = \begin{cases} \pi\gamma + 4L'(s, \chi_4), & if\ \delta \in \mathbb{Z}[i], \\ \pi\gamma + 4L'(s, \chi_4) + \sum\limits_{\beta \in \mathfrak{B}} (N(\delta + \beta))^{-1} + b_0(\gamma), & if\ 0 < N(\delta) < 1; \end{cases}$$

γ *is the Euler's constant,* $b_0(\gamma) = -4 + O\left(N^{\frac{1}{2}}(\delta)\right)$.

By the Stirling's formula for Gamma-function to the terms of the second order $O(t^{-2})$ we have for $|t| > 1$, $\sigma > 0$

$$\begin{aligned} &\Gamma(\sigma + it) = \sqrt{2\pi} t^{\sigma - \frac{1}{2}} \\ &\times \exp\left(i\left(t \log t - t + \frac{\pi}{2}\left(\sigma - \frac{1}{2}\right) + \left(\sigma - \sigma^2 - \frac{1}{6}\right)(2t)^{-1} + O(t^{-2})\right)\right) \exp\left(-\frac{\pi|t|}{2}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\Gamma(2|m| + 1 - s)}{\Gamma(2|m| + s)} &= \exp\left(it(2 - \log(4m^2 + t^2) + \frac{|2m| + 1}{4m^2 + t^2} + \frac{(2|m| + 1)^2}{(4m^2 + t^2)^2})\right) \\ &\times \frac{1}{2}(4m^2 + t^2)^{1 - 2\sigma} \exp\left(\sigma - \frac{1}{2} + \frac{t^2}{16}(4m^2 + 2|m| + t^2)^{-1}\right) \\ &\times \left(1 + O(m^2 + t^2)^{-\frac{1}{2}}\right). \end{aligned} \tag{3}$$

Applying the estimations for $|t| \geqslant 2$, $\sigma = 1$

$$Z_m^*(s;\delta,\delta_0) := Z_m(s;\delta,\delta_0) - \sum_{\omega\in\mathfrak{B}} \frac{e^{4mi\arg(\omega+\delta)}}{N(\omega+\delta)^s} e^{2\pi i\,\mathrm{Re}(\delta_0\omega)} \ll \log^4(t^2+m^2)$$

and functional equation (2), from (3) and Phragmen–Lindelöf theorem in the strip $-1 \leqslant \mathrm{Re}(s) \leqslant 1$ we infer

$$Z_m^*(s;\delta,\delta_0) \ll (m^2+t^2)^{\frac{1-\sigma}{2}} \left(\log(m^2+t^2)\right)^{\frac{1-\sigma}{2}}, \quad |m| \geqslant 1. \tag{4}$$

Let $\alpha, \beta, \gamma \in \mathbb{Z}[i]$. We define the Kloosterman sum for the ring of Gaussian integers

$$K(\alpha,\beta;\gamma) = \sum_{\substack{\xi,\xi' (\mathrm{mod}\,\gamma) \\ \xi\cdot\xi' \equiv 1(\gamma)}} e^{\pi i Sp\left(\frac{\alpha\xi+\beta\xi'}{\gamma}\right)}.$$

Lemma 4. *Let $\alpha, \beta, \gamma \in \mathbb{Z}[i], \gamma \neq 0$. Then the estimate*

$$K(\alpha,\beta;\gamma) \ll (N(\gamma)N((\alpha,\beta,\gamma)))^{\frac{1}{2}}\tau(\gamma)$$

holds. Moreover,

$$K(\alpha,\beta;\gamma) = \sum_{\delta|(\alpha,\beta,\gamma} N(\delta)K\left(1, \frac{\alpha\beta}{\delta^2}; \frac{\gamma}{\delta}\right). \tag{5}$$

Proof. This lemma follows from multiplicative property of $K(\alpha,\beta;\gamma)$ on γ and the Bombieri estimate of an exponential sum on the algebraic curve over the finite field. The formula (5) is a generalized Kuznetsov's identity for Kloosterman sums. □

2 THE MAIN RESULTS

Lemma 5. *Let $\gamma, \omega_0 \in \mathbb{Z}[i], N(\gamma) > 1$, $(\omega_0,\gamma) = \beta, N(\beta) < N(\gamma)$. Then for every $\varepsilon > 0$, $N(\gamma) \ll x^{\frac{2}{3}-\varepsilon}$ we have*

$$T_0(x,\gamma,\omega_0) = c_0(\gamma,\omega_0)\frac{x}{N(\gamma)}\log\frac{x}{N(\beta)} + c_1(\gamma,\omega_0)\frac{x}{N(\gamma)} + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right),$$

where $c_0(\gamma,\omega_0), c_1(\gamma,\omega_0)$ are computable constants

$$c_0(\gamma,\omega_0) = \pi^2 N(\beta)\overline{\varphi}\left(\frac{\gamma}{\beta}\right) N^{-1}(\gamma)\tau(\beta), \tag{6}$$

$$c_1(\gamma,\omega_0) = \pi^2 \sum_{\delta|\gamma}\left[2E - 1 + 2\frac{L'(1,\chi_4)}{L(1,\chi_4)} + \sum_{p|\gamma/\delta}{}^{*}\frac{\log N(p)}{N(p)-1}\right]\prod_{p|\gamma/\delta}{}^{*}(1-N(p)^{-1}). \tag{7}$$

Proof. Without loss of generality we will consider a case $(\omega_0,\gamma) = 1$. For $\mathrm{Re}\, s > 1$ we denote

$$F(s) := \sum_{\substack{\omega\in\mathbb{Z}[i] \\ \omega\equiv\omega_0(\gamma)}} \frac{\tau(\omega)}{N(\omega)^s}, \quad F^*(s) := F(s) - \sum_{\beta\in\mathfrak{B}} \frac{\tau(\omega_0+\beta\gamma)}{N(\omega_0+\beta\gamma)^s}.$$

It is clear, that

$$F(s) = N^{-2s}(\gamma) \sum_{\substack{\alpha_i \in (\mathrm{mod}\gamma) \\ \alpha_1\alpha_2 \equiv \omega_0(\gamma)}} Z_0\left(s, \frac{\alpha_1}{\gamma}, 0\right) Z_0\left(s, \frac{\alpha_2}{\gamma}, 0\right).$$

By using the Abel's Lemma about partial summation of Dirichlets' series, we have for $c = 1+\varepsilon$, $1 < T \leq x$, where $\varepsilon > 0$ is arbitrary small

$$T_0(x, \gamma, \omega_0) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F^*(s)\frac{x^s}{s} ds + O\left(\frac{x^c}{TN(\gamma)}\right). \tag{8}$$

From Lemma 4 we have the functional equation

$$F(s) = \frac{\pi^{2(2s-1)}}{N^{2s}(\gamma)} \frac{\Gamma^2(1-s)}{\Gamma^2(s)} \Psi(1-s),$$

where

$$\Psi(1-s) = \sum_{\omega} \frac{1}{N(\omega)^s} \sum_{\alpha\beta=\omega} \Phi(\alpha, \beta; \gamma), \quad \Phi(\alpha, \beta; \gamma) = \sum_{\substack{\alpha_1, \alpha_2 (\mathrm{mod}\gamma) \\ \alpha_1\alpha_2 \equiv \omega_0(\gamma)}} e^{\pi i Sp\left(\frac{\alpha\alpha_1 + \beta\alpha_2}{\gamma}\right)}.$$

We consider the function $F^*(s)$ in the strip $-\frac{1}{4} \leq \mathrm{Re}\, s \leq 1+\varepsilon$. It is obviously that $F^*(1+\varepsilon+it) \ll N(\gamma)^{-1-\varepsilon}$. On the line $\mathrm{Re}\, s = -\frac{1}{4}$ we apply the functional equation for $Z_0(s; \delta, 0)$, (3), Lemma 4 and then obtain $F^*(1+\varepsilon+it) \ll N(\gamma)^{1/2+\varepsilon}(|t|+3)^3$.

Applying the Phragmen-Lindelöf theorem in the strip $-\frac{1}{4} \leqslant \mathrm{Re}(s) \leqslant 1+\varepsilon$ we infer for $|t| \leq T$

$$F^*(-\varepsilon+it) \ll N(\gamma)^{1/5+\varepsilon} T^{12/5+\varepsilon}.$$

To deal with integral in (8) we shift the line of integration to $\mathrm{Re}\, s = -\varepsilon$. By the Theorem of residues we obtain

$$\begin{aligned} T_0(x, \gamma, \omega_0) = \underset{s=0}{res}\left(F^*(s))\frac{x^s}{s}\right) + \underset{s=1}{res}\left(F^*(s))\frac{x^s}{s}\right) + \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} F^*(s)\frac{x^s}{s} ds \\ + O\left(x^\varepsilon\right) + O\left(x^{-\varepsilon} N(\gamma)^{1/5+\varepsilon} T^{7/5+\varepsilon}\right) + O\left(\frac{x^{1+\varepsilon}}{TN(\gamma)}\right). \end{aligned} \tag{9}$$

Further, applying Lemma 2 we get

$$\begin{aligned} \underset{s=1}{res}\left(F^*(s))\frac{x^s}{s}\right) = \frac{\pi^2 x \log x}{N(\gamma)} \prod_{p|\gamma}{}^*(1 - N(p)^{-1}) \\ + \frac{\pi^2 x}{N(\gamma)} \prod_{p|\gamma}{}^*(1 - N(p)^{-1}) \left[-1 + 2\left(E + \frac{L'(1,\chi_4)}{L(1,\chi_4)} + \sum_{p|\gamma}{}^*(\frac{\log N(p)}{N(p)-1})\right)\right], \end{aligned} \tag{10}$$

where sign $\prod^*$ means that the product conducts by all the non-associated prime Gaussian numbers. Moreover, $F(0) = 0$ if $N(\gamma) > 1$.

$$\underset{s=0}{res}\left(F^*(s)\frac{x^s}{s}\right) = \underset{s=0}{res}\left(-\sum_{\beta \in \mathfrak{B}} \frac{\tau(\omega_0 + \beta\gamma)}{N(\omega_0 + \beta\gamma)^s} \frac{x^s}{s}\right) \ll N(\gamma)^\varepsilon. \tag{11}$$

Observe that by Lemma 4

$$\sum_{\alpha\beta=\omega} |\Phi(\alpha,\beta;\gamma)| = \sum_{\alpha\beta=\omega} |K(\alpha,\beta\omega_0;\gamma)| \ll N(\gamma)^{1/2} N((\omega,\gamma))^{1/2} \tau(\gamma)\tau(\omega).$$

Now by the termwise integration and applying the Stirling formula for gamma function and the method of stationary phase we get

$$\begin{aligned} \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} F^*(s)\frac{x^s}{s}ds = &\sum_{\substack{\omega \\ 0<N(\omega)\le Y}} \frac{\pi^2}{N(\omega)} \sum_{\alpha\beta=\omega} \Phi(\alpha,\beta;\gamma)\frac{y^{3/8}}{4\sqrt{2/\pi}} e\left(-\frac{1}{8}-\frac{1}{2\pi}y^{1/4}\right) \\ &\times (1+O(y^{-1/8})) + O\left(\frac{x^{1+\varepsilon}}{TN(\gamma)}\right) + O\left(x^{\varepsilon}\right) \\ &+ O\left(\sum_{\substack{\omega \\ N(\omega)>Y}} y^{-\varepsilon} T^{1+4\varepsilon} N(\gamma)^{1/2+\varepsilon} N((\omega,\gamma))^{1/2}\tau(\omega)N(\omega)^{-1}\right) \end{aligned} \tag{12}$$

where $Y \le X = \left(\frac{4}{\pi}\right)^4 \frac{T^4 N^2(\gamma)}{x}$, $y = \frac{\pi^4 x N(\omega)}{N^2(\gamma)}$. Thus, by combining (8)—(12) and taking $T = x^{1/2}N(\gamma)^{-3/4}$, $Y = x^{1/3}$ we obtain the assertion of Lemma 5. □

Theorem 1. *Let $\gamma, \omega_0 \in \mathbb{Z}[i]$, $N(\gamma) > 1$, $(\omega_0,\gamma) = \beta$, $N(\beta) < N(\gamma)$. Then for every $\varepsilon > 0$, $x \ge N^{\frac{3}{2}}(\gamma)$ and $\varphi_2 - \varphi_1 \gg \frac{N^{\frac{3}{4}}(\gamma)}{x^{\frac{1}{2}-\varepsilon}}$, the following formula holds*

$$\begin{aligned} T(x,\gamma,\omega_0,S(\varphi)) = \frac{2(\varphi_2-\varphi_1)}{\pi} &\left(c_0(\gamma,\omega_0)\frac{x}{N(\gamma)}\log\frac{x}{N(\beta)}\right. \\ &\left. + \left(c_1(\gamma,\omega_0)+A_0(\varphi)\right)\frac{x}{N(\gamma)}\right) + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right), \end{aligned}$$

where $c_0(\gamma,\omega_0), c_1(\gamma,\omega_0), A_0(\varphi)$ are computable constants, which defined in (6), (7). *The constant in symbol "O" depends only on ε.*

Proof. Let $m \ne 0$. Denote

$$c_m(\omega) = \sum_{\alpha\beta=\omega} e^{4mi\arg\alpha}.$$

For $\operatorname{Re} s > 1$ we have

$$\begin{aligned} F_m(s) &= \sum_{\substack{\omega\in\mathbb{Z}[i] \\ \omega\equiv\omega_0(\gamma)}} \frac{c_m(\omega)}{N(\omega)^s} = \frac{e^{4mi\arg\gamma}}{N^{2s}(\gamma)} \sum_{\substack{\alpha_i\in(\mathrm{mod}\gamma) \\ \alpha_1\alpha_2\equiv\omega_0(\gamma)}} Z_m\left(s,\frac{\alpha_1}{\gamma},0\right) Z_0\left(s,\frac{\alpha_2}{\gamma},0\right), \\ F_m^*(s) &= F_m(s) - \sum_{\substack{\beta\in\mathfrak{B} \\ \alpha\beta=\omega_0+\beta\gamma}} \frac{e^{4mi\arg\alpha}}{N(\omega_0+\beta\gamma)^s}. \end{aligned}$$

Thus, repeating the arguments of the proof of Lemma 5, we obtain for $m \ne 0$

$$\begin{aligned} T_m(x,\gamma,\omega_0) = \sum_{\substack{\omega\in\mathbb{Z}[i] \\ \omega\equiv\omega_0(\gamma)}} c_m(\omega) &= \frac{\pi x e^{4mi\arg\gamma}}{N^2(\gamma)} \sideset{}{'}\sum_{\alpha_1\in(\mathrm{mod}\gamma)} Z_m\left(1,\frac{\alpha_1}{\gamma},0\right) \\ &+ O\left(N(\gamma)^{1/5+\varepsilon}|m|^{6/5+\varepsilon}\right) + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{N^{\frac{1}{4}}(\gamma)}\right), \end{aligned} \tag{13}$$

where sign $'$ in summation $\sum$ denotes that α_1 runs reduced residue system modulo γ.

By the Lemma 1 and (1) we have

$$
\begin{aligned}
T(x,\gamma,\omega_0,S(\varphi)) &= \sum_{\substack{\omega\equiv\omega_0 \pmod{\gamma},\\ N(\omega)\le x}} \tau_S(\omega) = \sum_{\substack{\omega\equiv\omega_0 \pmod{\gamma},\\ N(w)\leqslant x}} \sum_{\alpha|\omega} \chi_s(\arg\alpha) \\
&= \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} (f(\arg\alpha;\varphi_1,\varphi_2) + \theta_1 f(\arg\alpha;\varphi_1-\Delta,\varphi_1) \\
&\quad +\theta_2 f(\arg\alpha;\varphi_2,\varphi_2+\Delta)) := \sum\nolimits_0 + \theta_1 \sum\nolimits_1 + \theta_2 \sum\nolimits_2 ,
\end{aligned} \tag{14}
$$

where f is the function from Lemma 1, associated, respectively, with segments $[\varphi_1,\ \varphi_2]$, $[\varphi_1-\Delta,\ \varphi_1]$, $[\varphi_2,\ \varphi_2+\Delta]$. The sums $\sum_0$, $\sum_1$, $\sum_2$ can be investigated similarly, so we consider the case $\sum_0$. We have

$$
\begin{aligned}
\sum\nolimits_0 &= \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} f(\arg\alpha;\varphi_1,\varphi_2) = \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} \sum_{m=-\infty}^{+\infty} a_m \exp(4mi\arg\alpha) \\
&= \sum_{m=-\infty}^{+\infty} a_m \sum_{\substack{\alpha\beta\equiv\omega_0 \pmod{\gamma}\\ N(\alpha\beta)\leqslant x}} \exp(4mi\arg\alpha) = a_0 T_0(x,\gamma,\omega_0) + \sum_{|m|\geqslant 1} a_m T_m(x,\gamma,\omega_0),
\end{aligned} \tag{15}
$$

where $a_0 = \frac{1}{\Omega}(\varphi_2-\varphi_1+\Delta)$, $\Omega = \frac{\pi}{2}$, the exact value of Δ will be defined later. The sum over m we split into two parts: $1 \le |m| \leqslant \Delta^{-1}$, $|m| > \Delta^{-1}$. For $|m| \leqslant \Delta^{-1}$ we use the estimation $|a_m| \le (2\pi|m|)^{-1}$, when $|m| > \Delta^{-1}$ we apply $|a_m| \le 2(\pi|m|)^{-1}(r\Omega(\pi|m|\Delta)^{-1})^r$, $r=2$. Substituting these estimates into (15), using the Lemma 5, (4) and (13) we obtain

$$
\begin{aligned}
\sum\nolimits_0 &= \frac{2}{\pi}(\varphi_2-\varphi_1+\Delta)T_0(x,\gamma,\omega_0) + \frac{\pi x}{N^2(\gamma)} \sum_{\alpha_1\in(\mathrm{mod}\,\gamma)}{}' \sum_{|m|\geqslant 1} a_m e^{4mi\arg\gamma} Z_m\left(1,\frac{\alpha_1}{\gamma},0\right) \\
&\quad + O\left(\sum_{1\le|m|\le\Delta^{-1}} m^{-1}\left(N(\gamma)^{1/5+\varepsilon}|m|^{6/5+\varepsilon} + \frac{x^{\frac12+\varepsilon}}{N^{\frac14}(\gamma)}\right)\right) \\
&\quad + O\left(\sum_{|m|>\Delta^{-1}} m^{-3}\Delta^{-2}\left(N(\gamma)^{1/5+\varepsilon}|m|^{6/5+\varepsilon} + \frac{x^{\frac12+\varepsilon}}{N^{\frac14}(\gamma)}\right)\right) \\
&= \frac{1}{2\pi}(\varphi_2-\varphi_1+\Delta)T_0(x,\gamma,\omega_0) + (\varphi_2-\varphi_1)\frac{x}{N(\gamma)}A_0(\varphi_2-\varphi_1,\Delta) \\
&\quad + O\left(N(\gamma)^{1/5+\varepsilon}\Delta^{-2-\varepsilon}\right) + O\left(\frac{x^{\frac12+\varepsilon}}{N^{\frac14}(\gamma)}\right),
\end{aligned}
$$

where $A_0(\varphi_2-\varphi_1,\Delta) = A_0(\varphi) + O(\Delta)$ limited for $\varphi_2-\varphi_1 \to 0$ and $\Delta \to 0$. Let $\Delta^{-1} = \frac{x^{\frac14}}{N^{\frac{9}{40}}(\gamma)}$.
In such case we have

$$
\begin{aligned}
\sum\nolimits_0 &= \frac{2(\varphi_2-\varphi_1)}{\pi}\left(c_0(\gamma,\omega_0)\frac{x}{N(\gamma)}\log\frac{x}{N(\beta)}\right. \\
&\quad \left. + \left(c_1(\gamma,\omega_0)+A_0(\varphi)\right)\frac{x}{N(\gamma)}\right) + O\left(\frac{x^{\frac12+\varepsilon}}{N^{\frac14}(\gamma)}\right).
\end{aligned} \tag{16}
$$

The sums $\sum_1$, $\sum_2$ have similar representations, but we write Δ instead $\varphi_2 - \varphi_1$. $A_0(\varphi)$ can be obtained using Lemma 1 for the case $r = 1$. The assertion of the Theorem 1 follows from (14), (16). The proof is completed. $\square$

In the same way the asymptotic formula for summary function of $\tau_{A_1,A_2}(\omega)$ can be proved, where $A_1 = S(\varphi)$, $A_2 = \{\alpha \in \mathbb{Z}[i] : \alpha \equiv \alpha_0 (\bmod\ \gamma)\}$.

The asymptotic formula for the $T_0(x, \gamma, \omega_0)$ can be used for investigation of number of solutions in Gaussian integers of the equation $\alpha_1\alpha_2 - \alpha_3\alpha_4 = \beta$, $N(\alpha_1\alpha_2) \leq x$.

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О.В.Савастру *Проблема дільників на спеціальних множинах цілих гаусових чисел* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 305–312.

Нехай A_1 та A_2 — це задані множини цілих гаусових чисел. Через $\tau_{A_1,A_2}(\omega)$ позначимо кількість уявлень ω у вигляді $\omega = \alpha\beta$, де $\alpha \in A_1, \beta \in A_2$. Побудована асимптотична формула для суматорної функції, яка відповідає функції $\tau_{A_1,A_2}(\omega)$, у випадку, коли ω належить арифметичній прогресії, A_1 — сектор роствору φ у комплексній площині, $A_2 = \mathbb{Z}[i]$.

Ключові слова і фрази: гаусові числа, проблема дільників, асимптотична формула, арифметична прогресія.