ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2017, **9** (1), 63–71
doi:10.15330/cmp.9.1.63-71

http://www.journals.pu.if.ua/index.php/cmp
Карпатськi матем. публ. 2017, Т.9, №1, С.63–71

KULYAVETC' L.V.[1], MULYAVA O.M.[2]

ON THE GROWTH OF A KLASSS OF DIRICHLET SERIES ABSOLUTELY CONVERGENT IN HALF-PLANE

In terms of generalized orders it is investigated a relation between the growth of a Dirichlet series $F(s) = \sum\limits_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ with the abscissa of asolute convergence $A \in (-\infty, +\infty)$ and the growth of Dirichlet series $F_j(s) = \sum\limits_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}$, $1 \leq j \leq 2$, with the same abscissa of absolute convergence if the coefficients a_n are connected with the coefficients $a_{n,j}$ by correlation

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right) = (1+o(1))\prod_{j=1}^{m}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}, \quad n \to \infty,$$

where $\omega_j > 0$, $1 \leq j \leq m$, $\sum\limits_{j=1}^{m} \omega_j = 1$.

Key words and phrases: Dirichlet series, generalized order.

[1] Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
[2] National University of Food Technologies, 68 Volodymyrska str., 01601, Kyiv, Ukraine
E-mail: lubov.kulyavets@gmail.com (Kulyavetc' L.V.), info@nuft.edu.ua (Mulyava O.M.)

INTRODUCTION

For an entire function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ let $\varrho[f]$ be its order and $\sigma[f]$ be its type. Using Hadamard's formulas for the finding of these characteristics, E.G. Calys [1] proved the following theorems.

Theorem A. *Suppose that entire functions $f_1(z) = \sum\limits_{n=0}^{\infty} a_{n,1} z^n$ and $f_2(z) = \sum\limits_{n=0}^{\infty} a_{n,2} z^n$ have finite orders and regular growth (in sence of the equality of order $\varrho[f]$ and lower order $\lambda[f]$) and the sequences $(|a_{n,1}/a_{n+1,1}|)$ and $(|a_{n,2}/a_{n+1,2}|)$ are nondecreasing for $n \geq n_0$. If*

$$\ln(1/|a_n|) = (1+o(1))\sqrt{\ln(1/|a_{n,1}|)\ln(1/|a_{n,2}|)}$$

as $n \to \infty$, then the function f has regular growth and $\varrho[f] = \sqrt{\varrho[f_1]\varrho[f_2]}$.

Theorem B. *Suppose that functions f_1 and f_2 from Theorem A have the same order $\varrho[f_1] = \varrho[f_2] = \varrho \in (0, +\infty)$ and the types $\sigma[f_1] = \sigma_1$, $\sigma[f_2] = \sigma_2$. Also suppose that $a_{n,1} \neq 0$ and $|a_{n,2}| \geq |a_{n,1}|/l(1/|a_{n,1}|)$ for all $n \geq n_0$, where l is slowly varying function. If*

$$|a_n| = (1+o(1))\sqrt{|a_{n,1}||a_{n,2}|}$$

as $n \to \infty$, then the function f has the order $\varrho[f] = \varrho$ and the type $\sigma[f] \leq \sqrt{\sigma_1\sigma_1}$.

УДК 517.537.72
2010 *Mathematics Subject Classification:* 30B50.

In [2] Theorems A and B are generalized on the case of entire Dirichlet series of finite generalized orders by Sheremeta, moreover instead two functions f_1 and f_2 were considered $n \geq 2$ entire Dirichlet series.

Here we will obtain analogues results for Dirichlet series absolutely convergent in a half-plane.

Let $\Lambda = (\lambda_n)$ be an increasing to $+\infty$ sequence of nonnegative numbers and $S(\Lambda, A)$ be a class of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n \exp\{sl_n\}, \quad s = \sigma + it \tag{1}$$

with a given sequence (λ_n) of exponents and an abscissa of absolutely convergence $\sigma_a = A \in (-\infty, +\infty)$ and $M(\sigma, F) = \sup\{|F(\sigma + it)| : t \in \mathbb{R}\}$ for $\sigma \in (-\infty, A)$.

By L we denote a class of positive continuous functions α on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0)$ for $x \leq x_0$ and $0 < \alpha(x) \uparrow +\infty$ as $x_0 \leq x \uparrow +\infty$. We say that $\alpha \in L^0$ if $\alpha \in L$ and $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_{si}$, if $\alpha \in L$ and $\alpha(cx) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ for each $c \in (0, +\infty)$, i. e. α is slowly increasing function. Clearly, $L_{si} \subset L^0$.

For $\alpha \in L$ and $\beta \in L$ the values

$$\varrho^A_{\alpha,\beta}[F] = \varlimsup_{\sigma\uparrow A} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/(A - \sigma))}, \quad \lambda^A_{\alpha,\beta}[F] = \varliminf_{\sigma\uparrow A} \frac{\alpha(\ln M(\sigma, F))}{\beta(1/(A - \sigma))}$$

are called [3] generalized order and lower order correspondly of Dirichlet series (1) from the class $S(\Lambda, A)$.

1 ANALOGUES OF THEOREM A.

We need the following lemmas from [3].

Lemma 1.1. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and*

$$\frac{x}{\beta^{-1}(c\alpha(x))} \uparrow +\infty, \quad \alpha\left(\frac{x}{\beta^{-1}(c\alpha(x))}\right) = (1 + o(1))\alpha(x) \tag{2}$$

as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$.

If $\alpha(\lambda_n) = o\left(\beta\left(\lambda_n / \ln n\right)\right)$ as $n \to \infty$, then

$$\varrho^A_{\alpha,\beta}[F] = k^A_{\alpha,\beta}[F] =: \varlimsup_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\lambda_n / \ln\left(|a_n| e^{A\lambda_n}\right)\right)},$$

and if, moreover, $\alpha(\lambda_{n+1}) = (1 + o(1))\alpha(\lambda_n)$ and $\dfrac{\ln|a_{n+1}| - \ln|a_n|}{\lambda_{n+1} - \lambda_n} \nearrow A$ as $n_0 \leq n \to \infty$, then

$$\lambda^A_{\alpha,\beta}[F] = \varkappa^A_{\alpha,\beta}[F] =: \varliminf_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\lambda_n / \ln\left(|a_n| e^{A\lambda_n}\right)\right)}.$$

Remark 1.1 ([3]). *In order that $\lambda^A_{\alpha,\beta}[F] \geq \varkappa^A_{\alpha,\beta}[F]$, it is sufficient that $\alpha(\lambda_{n+1}) = (1 + o(1))\alpha(\lambda_n)$ as $n \to \infty$.*

Lemma 1.2. *Let $\alpha \in L_{si}$, $\beta \in L_{si}$ and*

$$\frac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty, \quad \beta\left(\frac{x}{\alpha^{-1}(c\alpha(x))}\right) = (1+o(1))\beta(x) \tag{3}$$

as $x_0 \le x \to +\infty$ for each $c \in (0, +\infty)$.

If $\alpha(\ln n) = o(\beta(\lambda_n))$ as $n \to \infty$, then

$$\varrho^A_{\alpha,\beta}[F] = k^{A*}_{\alpha,\beta}[F] =: \varlimsup_{n\to\infty} \frac{\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)},$$

and if, moreover, $\beta(\lambda_{n+1}) = (1+o(1))\beta(\lambda_n)$ and $\dfrac{\ln|a_{n+1}| - \ln|a_n|}{\lambda_{n+1}-\lambda_n} \nearrow A$ as $n_0 \le n \to \infty$, then

$$\lambda^A_{\alpha,\beta}[F] = \varkappa^{A*}_{\alpha,\beta}[F] =: \varliminf_{n\to\infty} \frac{\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}.$$

Remark 1.2 ([3]). *In order that $\lambda^A_{\alpha,\beta}[F] \ge \varkappa^{A*}_{\alpha,\beta}[F]$, it is sufficient that $\beta(\lambda_{n+1}) = (1+o(1))\beta(\lambda_n)$ as $n \to \infty$.*

Suppose that $F_j \in S(\Lambda, A)$, $1 \le j \le m$, and

$$F_j(s) = \sum_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}. \tag{4}$$

Using Lemma 1.1, at first we prove the following analog of Theorem A.

Theorem 1. *Let functions $\alpha \in L_{si}$ and $\beta \in L_{si}$ satisfy conditions* (2), *$\alpha(\lambda_n) = o\left(\beta\left(\lambda_n/\ln n\right)\right)$ and $\alpha(\lambda_{n+1}) = (1+o(1))\alpha(\lambda_n)$ as $n \to \infty$. Suppose that all functions* (4) *have regular $\alpha\beta$-growth (i.e. $\lambda^A_{\alpha,\beta}[F_j] = \varrho^A_{\alpha,\beta}[F_j] < +\infty$) and $\dfrac{\ln|a_{n+1,j}| - \ln|a_{n,j}|}{\lambda_{n+1}-\lambda_n} \nearrow A$ as $n_0 \le n \to \infty$.*

If $\omega_j > 0$, $1 \le j \le m$, $\sum\limits_{j=1}^m \omega_j = 1$ and

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right) = (1+o(1))\prod_{j=1}^m \beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}, \quad n \to \infty, \tag{5}$$

then function (1) *has regular $\alpha\beta$-growth and $\varrho^A_{\alpha,\beta}[F] = \prod\limits_{j=1}^m (\varrho^A_{\alpha,\beta}[F_j])^{\omega_j}$.*

Proof. Since $\lambda^A_{\alpha,\beta}[F_j] = \varrho^A_{\alpha,\beta}[F_j] = \varrho_j < +\infty$, by Lemma 1.1 we have

$$\lim_{n\to\infty} \frac{\alpha(\lambda_n)}{\beta\left(\lambda_n/\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)} = \varrho_j.$$

Therefore, from (5) we obtain

$$\lim_{n\to\infty} \frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right) = \lim_{n\to\infty} \frac{1}{\alpha(\lambda_n)}\prod_{j=1}^m \beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}$$
$$= \lim_{n\to\infty}\prod_{j=1}^m \left(\frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right)^{\omega_j} = \prod_{j=1}^m \lim_{n\to\infty}\left(\frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right)^{\omega_j}$$
$$= \prod_{j=1}^m (1/\varrho_j)^{\omega_j},$$

that is,

$$\lim_{n\to\infty}\frac{\alpha(\lambda_n)}{\beta\left(\lambda_n/\ln\left(|a_n|e^{A\lambda_n}\right)\right)}=\prod_{j=1}^{m}\varrho_j^{\omega_j}.$$

Using Lemma 1.1 and the Remark 1.1, hence we get $\prod\limits_{j=1}^{m}\varrho_j^{\omega_j}\le\lambda_{\alpha,\beta}^{A}[F]\le\varrho_{\alpha,\beta}^{A}[F]=\prod\limits_{j=1}^{m}\varrho_j^{\omega_j}$, that is the function F has regular $\alpha\beta$-growth and $\varrho_{\alpha,\beta}^{A}[F]=\prod\limits_{j=1}^{m}(\varrho_{\alpha,\beta}^{A}[F])^{\omega_j}$. Theorem 1 is proved. □

From (2) it follows that the function α increases less rapidly than the function β. It is easy to verify that the functions $\alpha(x)=\ln\ln x$ and $\beta(x)=\ln x$ for $x\ge x_0$ satisfy (2) and the condition $\alpha(\lambda_n)=o\left(\beta\left(\lambda_n/\ln n\right)\right)$ holds as $n\to\infty$, provided $\varlimsup\limits_{n\to\infty}(\ln\ln n)/\ln\lambda_n<1$. Therefore, Theorem 1 implies the following statement.

Corollary 1.1. *Let* $\varlimsup\limits_{n\to\infty}(\ln\ln n)/\ln\lambda_n<1$, $\ln\ln\lambda_{n+1}=(1+o(1))\ln\ln\lambda_n$ *as* $n\to\infty$. *Suppose that* $\lim\limits_{\sigma\uparrow A}\dfrac{\ln\ln\ln M(\sigma,F_j))}{\ln(1/(A-\sigma))}=\varrho_j$ *and* $\dfrac{\ln|a_{n+1,j}|-\ln|a_{n,j}|}{\lambda_{n+1}-\lambda_n}\nearrow A$ *as* $n_0\le n\to\infty$ *for all* $1\le j\le m$. *If*

$$\ln\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)=(1+o(1))\prod_{j=1}^{m}\ln^{\omega_j}\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right),\quad\sum_{j=1}^{m}\omega_j=1,$$

as $n\to\infty$ *then* $\lim\limits_{\sigma\uparrow A}\dfrac{\ln\ln\ln M(\sigma,F))}{\ln(1/(A-\sigma))}=\prod\limits_{j=1}^{m}\varrho_j^{\omega_j}$.

For the proof of the following theorem we will use Lemma 1.2.

Theorem 2. *Let the functions* $\alpha\in L_{si}$ *and* $\beta\in L_{si}$ *satisfy the condition* (3), $\alpha(\ln n)=o(\beta(\lambda_n))$ *and* $\beta(\lambda_{n+1})=(1+o(1))\beta(\lambda_n)$ *as* $n\to\infty$. *Suppose that all functions* (4) *have regular* $\alpha\beta$-*growth and* $\dfrac{\ln|a_{n+1,j}|-\ln|a_{n,j}|}{\lambda_{n+1}-\lambda_n}\nearrow A$ *as* $n_0\le n\to\infty$.

If $\omega_j>0$, $1\le j\le m$, $\sum\limits_{j=1}^{m}\omega_j=1$ *and*

$$\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)=(1+o(1))\prod_{j=1}^{m}\alpha^{\omega_j}\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right),\quad n\to\infty,\tag{6}$$

then function (1) *has regular* $\alpha\beta$-*growth and* $\varrho_{\alpha,\beta}^{A}[F]=\prod\limits_{j=1}^{m}(\varrho_{\alpha,\beta}^{A}[F_j])^{\omega_j}$.

Proof. Since $\lambda_{\alpha,\beta}^{A}[F_j]=\varrho_{\alpha,\beta}^{A}[F_j]=\varrho_j<+\infty$, by Lemma 1.2 we have

$$\lim_{n\to\infty}\frac{\alpha\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}=\varrho_j.$$

Therefore, from (6), as in the proof of Theorem 1,

$$\lim_{n\to\infty}\frac{\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}=\prod_{j=1}^{m}\lim_{n\to\infty}\left(\frac{\alpha\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)}\right)^{\omega_j}=\prod_{j=1}^{m}\varrho_j^{\omega_j},$$

whence, as above, we obtain the regular $\alpha\beta$-growth of the function f and the equality $\varrho_{\alpha,\beta}^{A}[F]=\prod\limits_{j=1}^{m}(\varrho_{\alpha,\beta}^{A}[F])^{\omega_j}$. Theorem 2 is proved. □

From (3) it follows that the function β increases less rapidly than the function β. It is easy to verify that the functions $\alpha(x) = \ln x$ and $\beta(x) = \ln\ln x$ for $x \geq x_0$ satisfy (3). Therefore, Theorem 2 implies the following statement.

Corollary 1.2. *Let* $\ln\ln n = o(\ln\ln\lambda_n))$ *and* $\ln\ln\lambda_{n+1} = (1+o(1))\ln\ln\lambda_n$ *as* $n \to \infty$. *Suppose that* $\lim\limits_{\sigma\uparrow A} \dfrac{\ln\ln M(\sigma, F_j))}{\ln\ln(1/(A-\sigma))} = \varrho_j$ *and* $\dfrac{\ln|a_{n+1,j}| - \ln|a_{n,j}|}{\lambda_{n+1}-\lambda_n} \nearrow A$ *as* $n_0 \leq n \to \infty$ *for all* $1 \leq j \leq m$. *If*

$$\ln\ln\left(|a_n|e^{A\lambda_n}\right) = (1+o(1))\prod_{j=1}^{m}\ln^{\omega_j}\ln\left(|a_{n,j}|e^{A\lambda_n}\right), \quad \sum_{j=1}^{m}\omega_j = 1,$$

as $n \to \infty$ *then* $\lim\limits_{\sigma\uparrow A}\dfrac{\ln\ln M(\sigma,F))}{\ln\ln(1/(A-\sigma))} = \prod\limits_{j=1}^{m}\varrho_j^{\omega_j}$.

2 ANALOGUES OF THEOREM B.

Suppose, as above, that $\alpha \in L_{si}$ and $\beta \in L_{si}$. In order to get the analogues of Theorem B, except the generalized order $\varrho^A_{\alpha,\beta}[F] \in (0,+\infty)$, it is needed to enter a (generalized) type. A definition of the type depends on what from the functions α or β grows slower.

Suppose at first that the function β increases less rapidly than the function α and define a type by the formula

$$T^{A*}_{\alpha,\beta}[F] = \varlimsup_{\sigma\uparrow A}\frac{\ln M(\sigma,F)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(1/(A-\sigma)))}.$$

Since $T^{A*}_{\alpha,\beta}[F] = \varrho^A_{\alpha_1,\beta_1}[F]$, where $\alpha_1(x) = x \notin L_{si}$ and $\beta_1(x) = \alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(x))$ for $x \geq x_0$, we can apply none from the lemmas indicated above. However the following statement is true [3].

Lemma 2.1. *Let* $\alpha_1(x) = x$ *for* $x \geq x_0$, $\beta_1 \in L_{si}$ *and*

$$\frac{x}{\beta_1(x)} \uparrow +\infty, \quad \beta_1\left(\frac{x}{\beta_1(x)}\right) = (1+o(1))\beta_1(x), \quad x_0 \leq x \to +\infty.$$

If $\ln n = o(\beta_1(\lambda_n))$ *as* $n \to \infty$ *then* $\varrho^A_{\alpha_1,\beta_1}[F] = \varlimsup\limits_{n\to\infty}\dfrac{\ln\left(|a_n|e^{A\lambda_n}\right)}{\beta_1(\lambda_n)}$.

Since $\beta_1(x) = \alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(x))$ for $x \geq x_0$ then Lemma 2.1 implies the following statement.

Lemma 2.2. *Let* $\alpha \in L_{si}$ *and* $\beta \in L_{si}$ *be such that* $\alpha^{-1}(c\beta(x)) \in L_{si}$ *for each* $c \in (0,+\infty)$ *and*

$$\frac{x}{\alpha^{-1}(c\beta(x))} \uparrow +\infty, \quad \alpha^{-1}\left(c\beta\left(\frac{x}{\alpha^{-1}(c\beta(x)}\right)\right) = (1+o(1))\alpha^{-1}(c\beta(x)) \tag{7}$$

as $x_0 \leq x \to +\infty$ *for each* $c \in (0,+\infty)$. *If* $\ln n = o(\alpha^{-1}(c\beta(\lambda_n))$ *as* $n \to \infty$ *for each* $c \in (0,+\infty)$, *then*

$$T^{A*}_{\alpha,\beta}[F] = \varlimsup_{n\to\infty}\frac{\ln\left(|a_n|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}.$$

The following theorem generalizes Theorem B.

Theorem 3. *Let $\beta \in L_{si}$, $\alpha(e^x) \in L^0$, $\alpha^{-1}(c\beta(x)) \in L_{si}$, conditions* (7) *hold and* $\ln n = o(\alpha^{-1}(c\beta(\lambda_n)))$ *as $n \to \infty$ for each $c \in (0, +\infty)$. Suppose that all Dirichlet series* (4) *have the same generalised order $\varrho^A_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$ and the types $T^{A*}_{\alpha,\beta}[F_j] \in (0, +\infty)$. Suppose also that $a_{n,1} \neq 0$ for all $n \geq n_0$ and for all $2 \leq j \leq m$*

$$\ln\ln\left(|a_{n,j}|e^{A\lambda_n}\right) \geq (1+o(1))\ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right), \quad n \to \infty. \tag{8}$$

If $\omega_j > 0$, $1 \leq j \leq m$, $\sum\limits_{j=1}^{m} \omega_j = 1$ and

$$\ln\left(|a_n|e^{A\lambda_n}\right) = (1+o(1))\prod_{j=1}^{m}\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)^{\omega_j}, \quad n \to \infty, \tag{9}$$

then Dirichlet series (1) *has the generalized order $\varrho^A_{\alpha,\beta}[F] = \varrho$ and the type*

$$T^{A*}_{\alpha,\beta}[F] \leq \prod_{j=1}^{m} T^{A*}_{\alpha,\beta}[F_j]^{\omega_j}.$$

Proof. Since $\alpha(e^x) \in L^0$, then for each $c \in (0, +\infty)$ we have

$$\alpha(cx) = \alpha(e^{\ln x + \ln c}) = \alpha(e^{(1+o(1))\ln x}) = (1+o(1))\alpha(e^{\ln x}) = (1+o(1))\alpha(x)$$

as $x \to +\infty$, that is $\alpha \in L_{si}$. Hence it follows that $\alpha^{-1}((1-\eta)x) = o(\alpha^{-1}(x))$ as $x \to +\infty$ for each $\eta \in (0, 1)$, because if $\alpha^{-1}((1-\eta)x_k) \geq h\alpha^{-1}(x_k))$ for some number $h > 0$ and an increasing to $+\infty$ sequence (x_k) then $(1-\eta)x_k \geq \alpha(h\alpha^{-1}(x_k)) = (1+o(1))x_k$ as $k \to \infty$, that is impossible.

Therefore, conditions (7) imply the conditions (3). Indeed, if for some $c \in (0, +\infty)$, $\eta \in (0, 1)$ and an increasing to $+\infty$ sequence (x_k) the inequality

$$\beta\left(x_k/\alpha^{-1}(c\beta(x_k))\right) \leq (1-\eta)\beta(x_k)$$

is true then $\alpha^{-1}\left(c\beta\left(x_k/\alpha^{-1}(c\beta(x_k))\right)\right) \leq \alpha^{-1}\left(c(1-\eta)\beta(x_k)\right) = o(\alpha^{-1}\left(c\beta(x_k)\right))$ as $k \to \infty$, that is impossible in view of (7).

Finally, from the condition $\ln n = o(\alpha^{-1}(c\beta(\lambda_n))$ as $n \to \infty$ for each $c \in (0, +\infty)$ we have $\ln n \leq \alpha^{-1}(c\beta(\lambda_n))$ for $n \geq n_0$ and each $c \in (0, +\infty)$, that is $\alpha(\ln n) \leq c\beta(\lambda_n)$ and in view of the arbitrariness of c we obtain $\alpha(\ln n) = o(\beta(\lambda_n))$ as $n \to \infty$.

Thus, from the conditions on the functions α and β and the sequence (λ_n) in Theorem 3 the conditions of Lemma 1.2 follows.

Since all Dirichlet series (4) have the same generalized order $\varrho^A_{\alpha,\beta}[F_j] = \varrho \in (0, +\infty)$, then by Lemma 1.2 for every $\varrho_1 > \varrho$ and all $n \geq n_0(\varrho_1)$ we have $\ln\left(|a_{n,j}|e^{A\lambda_n}\right) \leq \alpha^{-1}(\varrho_1\beta(\lambda_n))$. Therefore, from (9) we obtain

$$\begin{aligned}
\varrho^A_{\alpha,\beta}[F] &= \varlimsup_{n\to\infty} \frac{\alpha\left(\ln\left(|a_n|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)} = \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)}\alpha\left(\prod_{j=1}^{m}\left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)^{\omega_j}\right) \\
&= \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\sum_{j=1}^{m}\omega_j \ln\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right\}\right) \\
&\leq \varlimsup_{n\to\infty} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\sum_{j=1}^{m}\omega_j \ln \alpha^{-1}(\varrho_1\beta(\lambda_n))\right\}\right) = \varrho_1,
\end{aligned}$$

that is in view of the arbitrariness of ϱ_1 we obtain the inequality $\varrho^A_{\alpha,\beta}[F] \le \varrho$.

On the other hand, in view of the conditions (8) and $\alpha(e^x) \in L^0$ we have

$$\begin{aligned}
\varrho^A_{\alpha,\beta}[F] &= \overline{\lim_{n\to\infty}} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\sum_{j=1}^m \omega_j \ln\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right\}\right)\\
&= \overline{\lim_{n\to\infty}} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\omega_1 \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right) + \sum_{j=2}^m \omega_j \ln\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right\}\right)\\
&\ge \overline{\lim_{n\to\infty}} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{\omega_1 \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right) + \sum_{j=2}^m \omega_j (1+o(1)) \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right\}\right)\\
&= \overline{\lim_{n\to\infty}} \frac{1}{\beta(\lambda_n)}\alpha\left(\exp\left\{(1+o(1))\sum_{j=1}^m \omega_j \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right\}\right)\\
&= \overline{\lim_{n\to\infty}} \frac{(1+o(1))}{\beta(\lambda_n)}\alpha\left(\exp\left\{\sum_{j=1}^m \omega_j \ln\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right\}\right)\\
&= \overline{\lim_{n\to\infty}} \frac{\alpha\left(\ln\left(|a_{n,1}|e^{A\lambda_n}\right)\right)}{\beta(\lambda_n)} = \varrho.
\end{aligned}$$

Thus, $\varrho^A_{\alpha,\beta}[F] = \varrho$ and for $T^{A*}_{\alpha,\beta}[F]$ by Lemma 2.2 from (9) we obtain

$$\begin{aligned}
T^{A*}_{\alpha,\beta}[F] &= \overline{\lim_{n\to\infty}} \frac{\ln\left(|a_n|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))} = \overline{\lim_{n\to\infty}} \frac{1}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}\prod_{j=1}^m \left(\ln\left(|a_{n,j}|e^{A\lambda_n}\right)\right)^{\omega_j}\\
&= \overline{\lim_{n\to\infty}} \prod_{j=1}^m \left(\frac{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}\right)^{\omega_j} \le \prod_{j=1}^m \overline{\lim_{n\to\infty}} \left(\frac{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}{\alpha^{-1}(\varrho^A_{\alpha,\beta}[F]\beta(\lambda_n))}\right)^{\omega_j} = \prod_{j=1}^m T^{A*}_{\alpha,\beta}[F_j]^{\omega_j}.
\end{aligned}$$

The proof of Theorem 3 is complete. □

It is easy to verify that the functions $\alpha(x) = \ln x$ and $\beta(x) = \ln\ln x$ for $x \ge x_0$ satisfy the conditions of Theorem 3. Therefore, the following statement is true.

Corollary 2.1. *Let Diriclet series* (4) *be such that for all* $1 \le j \le m$

$$\lim_{\sigma\uparrow A} \frac{\ln\ln M(\sigma, F_j)}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \lim_{\sigma\uparrow A} \frac{\ln M(\sigma, F_j))}{\ln^{\varrho}(1/(A-\sigma))} = T_j,$$

and $\ln n = O(\ln\ln\lambda_n)$ *as* $n\to\infty$. *Then the conditions* (8) *and* (9) *imply*

$$\lim_{\sigma\uparrow A} \frac{\ln\ln M(\sigma, F)}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \lim_{\sigma\uparrow A} \frac{\ln M(\sigma, F)}{\ln^{\varrho}(1/(A-\sigma))} \le \prod_{j=1}^m T_j^{\omega_j}.$$

Since $\varrho^A_{\alpha,\beta}[F] = \overline{\lim_{\sigma\uparrow A}} \dfrac{\ln\exp\{\alpha(\ln M(\sigma,F)\}}{\ln\exp\{\beta(1/(A-\sigma)\}}$, we define the type also by the formula

$$T^A_{\alpha,\beta}[F] = \overline{\lim_{\sigma\uparrow A}} \frac{\exp\{\alpha(\ln M(\sigma,F)\}}{\exp\{\varrho^A_{\alpha,\beta}[F]\beta(1/(A-\sigma)\}},$$

and for the finding by the coefficients we use Lemma 1.1. We obtain the following statement.

Lemma 2.3. *Suppose that the function $e^{\alpha(x)}$ and $e^{\beta(x)}$ belongs to L_{si} and*

$$\frac{x}{\beta^{-1}(\ln c+\alpha(x))}\uparrow+\infty,\quad \exp\left\{\alpha\left(\frac{x}{\beta^{-1}(\ln c+\alpha(x))}\right)\right\}=(1+o(1))e^{\alpha(x)} \tag{10}$$

as $x\to+\infty$ for each $c\in(0,+\infty)$. If $\exp\{\alpha(\lambda_n)\}=o\left(\exp\left\{\beta\left(\lambda_n/\ln n\right)\right\}\right)$ as $n\to\infty$ then

$$T^A_{\alpha,\beta}[F]=\varlimsup_{n\to\infty}\frac{\exp\{\alpha(\lambda_n)\}}{\exp\left\{\varrho^A_{\alpha,\beta}[F]\beta\left(\dfrac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\right\}}.$$

Theorem 4. *Let the function $e^{\alpha(x)}$ and $e^{\beta(x)}$ belongs to L_{si}, the conditions* (2) *and* (10) *hold and $\alpha(\lambda_n)=o\left(\beta\left(\lambda_n/\ln n\right)\right)$ as $n\to\infty$. Suppose that all Dirichlet series* (4) *have the same generalized order $\varrho^A_{\alpha,\beta}[F_j]=\varrho\in(0,+\infty)$ and the types $T^A_{\alpha,\beta}[F_j]\in(0,+\infty)$. Suppose also that $a_{n,1}\neq 0$ for all $n\geq n_0$ and for all $2\leq j\leq m$*

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\leq(1+o(1))\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,1}|e^{A\lambda_n}\right)}\right),\quad n\to\infty. \tag{11}$$

If $\omega_j>0$, $1\leq j\leq m$, $\sum\limits_{j=1}^m\omega_j=1$ and

$$\exp\left\{\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\right\}=(1+o(1))\prod_{j=1}^m\exp\left\{\omega_j\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right\} \tag{12}$$

as $n\to\infty$ then Dirichlet series (1) *has the generalized order $\varrho^A_{\alpha,\beta}[F]=\varrho$ and type*

$$T^A_{\alpha,\beta}[F]\leq\prod_{j=1}^m T^A_{\alpha,\beta}[F_j]^{\omega_j}.$$

Proof. From (12) we have

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)=\sum_{j=1}^m\omega_j\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)+o(1) \tag{13}$$

as $n\to\infty$. Therefore, by Lemma 1.1

$$\frac{1}{\varrho^A_{\alpha,\beta}[F]}=\varliminf_{n\to\infty}\frac{1}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\geq\sum_{j=1}^m\varliminf_{n\to\infty}\frac{\omega_j}{\alpha(\omega_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)=\frac{1}{\varrho}.$$

On the other hand, in view of (11) from (13) we obtain

$$\frac{1}{\varrho^A_{\alpha,\beta}[F]}\leq\varliminf_{n\to\infty}\sum_{j=1}^m\frac{\omega_j}{\alpha(\lambda_n)}\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,1}|e^{A\lambda_n}\right)}\right)=\frac{1}{\varrho},$$

that is $\varrho^A_{\alpha,\beta}[F]=\varrho$. From (12) and Lemma 2.3 also it follows that

$$\begin{aligned}\frac{1}{T^A_{\alpha,\beta}[F]}&=\varliminf_{n\to\infty}\frac{1}{\exp\{\alpha(\lambda_n)\}}\exp\left\{\varrho\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right)\right\}\\&=\varliminf_{n\to\infty}\frac{1}{\exp\{\alpha(\lambda_n)\}}\prod_{j=1}^m\exp\left\{\varrho\omega_j\beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right\}\\&\geq\prod_{j=1}^m\varliminf_{n\to\infty}\left(\frac{\exp\left\{\varrho\beta\left(\dfrac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)\right\}}{\exp\{\alpha(\lambda_n)\}}\right)^{\omega_j}=\prod_{j=1}^m\left(\frac{1}{T^A_{\alpha,\beta}[F_j]}\right)^{\omega_j}.\end{aligned}$$

Theorem 4 is proved. $\square$

It is easy to verify that the functions $\alpha(x) = \ln\ln x$ and $\beta(x) = \ln\ln x$ for $x \ge x_0$ satisfy the conditions (2) and (10). The condition $\alpha(\lambda_n) = o\left(\beta\left(\lambda_n / \ln n\right)\right)$ as $n \to \infty$ holds, provided $\varlimsup\limits_{n\to\infty} (\ln\ln n)/\ln\lambda_n < 1$. Therefore, Theorem 4 implies the following statement.

Corollary 2.2. *Let* $\varlimsup\limits_{n\to\infty} (\ln\ln n)/\ln\lambda_n < 1$ *and for all* $1 \le j \le m$

$$\varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln\ln M(\sigma, F_j))}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln M(\sigma, F_j))}{\ln^{\varrho}(1/(A-\sigma))} = T_j \in (0, +\infty).$$

Suppose that $a_{n,1} \ne 0$ *for all* $n \ge n_0$ *and for all* $2 \le j \le m$

$$\ln\ln \frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)} \le (1+o(1)) \ln\ln \frac{\lambda_n}{\ln\left(|a_{n,1}|e^{A\lambda_n}\right)}, \quad n \to \infty.$$

If $\omega_j > 0$, $1 \le j \le m$, $\sum\limits_{j=1}^{m} \omega_j = 1$ *and*

$$\ln \frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)} = (1+o(1)) \prod_{j=1}^{m} \left(\ln \frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}$$

as $n \to \infty$ *then*

$$\varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln\ln M(\sigma, F)}{\ln\ln(1/(A-\sigma))} = \varrho, \quad \varlimsup_{\sigma\uparrow A} \frac{\ln\ln\ln M(\sigma, F)}{\ln^{\varrho}(1/(A-\sigma))} \le \prod_{j=1}^{m} T_j^{\omega_j}.$$

REFERENCES

[1] Calys E.G. *A note on the order and type of integral functions*. Riv. Mat. Univer. Parma 1964, **5** (2), 133–137.

[2] Kulyavec' L.V., Mulyava O.M. *On the growth of a class of entire Dirichlet series*. Carpathian Math. Publ. 2014, **6** (2), 300–309. doi:10.15330/cmp.6.2.300-309 (in Ukrainian)

[3] Gal' Yu.M. On the growth of analytic functions, represented by Dirichlet series absolutely convergent in half-plane. Drohobych, 1980. (in Russian)

Received 17.04.2015

Revised 01.06.2017

Кулявець Л.В., Мулява О.М. *Про зростання одного класу абсолютно збіжних у півплощині рядів Діріхле* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 63–71.

У термінах узагальнених порядків досліджено зв'язок між зростанням ряду Діріхле $F(s) = \sum\limits_{n=1}^{\infty} a_n \exp\{s\lambda_n\}$ з абсцисою абсолютної збіжності $A \in (-\infty, +\infty)$ і зростанням рядів Діріхле $F_j(s) = \sum\limits_{n=1}^{\infty} a_{n,j} \exp\{s\lambda_n\}$, $1 \le j \le 2$, з такою ж абсцисою абсолютної збіжності, якщо, наприклад, коефіцієнти a_n повязані з коефіцієнтами $a_{n,j}$ співвідношенням

$$\beta\left(\frac{\lambda_n}{\ln\left(|a_n|e^{A\lambda_n}\right)}\right) = (1+o(1)) \prod_{j=1}^{m} \beta\left(\frac{\lambda_n}{\ln\left(|a_{n,j}|e^{A\lambda_n}\right)}\right)^{\omega_j}, \quad n \to \infty,$$

де $\omega_j > 0$, $1 \le j \le m$, $\sum\limits_{j=1}^{m} \omega_j = 1$.

Ключові слова і фрази: ряд Діріхле, узагальнений порядок.