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ON THE GROWTH OF A COMPOSITION OF ENTIRE FUNCTIONS

Let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$ and f and g be arbitrary entire functions of positive lower order and finite order.

In order to

$$\lim_{r\to+\infty}\frac{\ln\ln M_{f(g)}(r)}{\ln\ln M_f(\exp\{\gamma(r)\})}=+\infty,\quad M_f(r)=\max\{|f(z)|:\,|z|=r\},$$

it is necessary and sufficient $(\ln\gamma(r))/(\ln r)\to 0$ as $r\to+\infty$. This statement is an answer to the question posed by A.P. Singh and M.S. Baloria in 1991.

Also in order to

$$\lim_{r\to+\infty}\frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})}=0,\quad F(z)=f(g(z)),$$

it is necessary and sufficient $(\ln\gamma(r))/(\ln r)\to\infty$ as $r\to+\infty$.

Key words and phrases: entire function, composition of functions, generalized order.

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INTRODUCTION

For an entire function $f\not\equiv$ const we put $M_f(r)=\max\{|f(z)|:\,|z|=r\}$. The quantities

$$\varrho[f]=\varlimsup_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln r},\quad \lambda[f]=\varliminf_{r\to+\infty}\frac{\ln\ln M_f(r)}{\ln r} \tag{1}$$

are called [7, p. 61] the order and the lower order of f accordingly.

G.D. Song and C.C. Yang [6] have proved that if f and g are transcendental entire functions, $0<\lambda[f]\le\varrho[f]<+\infty$ and $F(z)=f(g(z))$ then

$$\lim_{r\to+\infty}\frac{\ln\ln M_F(r)}{\ln\ln M_f(r)}=+\infty.$$

A.P. Singh and M.S. Baloria [3] posed a question: how to find $R=R(r)$ such that

$$\varliminf_{r\to+\infty}\frac{\ln\ln M_F(r)}{\ln\ln M_f(R)}<+\infty\,?$$

They have proved the following theorems.

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Theorem A. *Let f and g be entire functions of positive lower order and of finite order, and $F(z) = f(g(z))$. Then* $\lim\limits_{r\to+\infty} \dfrac{\ln\ln M_F(r)}{\ln\ln M_f(r^A)} = +\infty$ *for every positive constant A.*

Theorem B. *Let f and g be entire functions of finite order with $\varrho[g] < \varrho[f]$ and $F(z) = f(g(z))$. Then* $\lim\limits_{r\to+\infty} \dfrac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{r^{\varrho[f]}\})} = 0.$

The aim of proposed article is research of the above mentioned problem from [4].

1 Main results

Next theorem gives an answer to the question of A.P. Singh and M.S. Baloria.

Theorem 1. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[f] \le \varrho[f] < +\infty$ and $\lambda[g] > 0$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = +\infty, \quad F(z) = f(g(z)), \tag{2}$$

it is necessary and sufficient

$$\lim_{r\to+\infty} \frac{\ln\, \gamma(r)}{\ln\, r} = 0. \tag{3}$$

Proof. G. Polya [2] has proved that if f and g are entire functions, $|g(0)| = 0$ and $F(z) = f(g(z))$ then there exists a constant $c \in (0, 1)$ independent of f and g such that for all $r > 0$

$$M_F(r) \ge M_f\left(cM_g\left(\frac{r}{2}\right)\right) \text{ and} \tag{4}$$

$$M_F(r) \le M_f(M_g(r)). \tag{5}$$

J. Clunie [1] defines more precisely inequality (4). He proved that

$$M_F(r) \ge M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right). \tag{6}$$

We assume that the function γ satisfies (3), that is $\ln\, \gamma(r) = o(\ln\, r)$ as $r \to +\infty$. If the lower orders $\lambda[f]$ and $\lambda[g]$ are positive then for $\lambda \in (0, \min\{\lambda[f], \lambda[g]\})$ and all $r \ge r_0(\lambda)$ the inequalities $\ln\, \ln\, M_f(r) \ge \lambda \ln\, r$ and $\ln\, \ln\, M_g(r) \ge \lambda \ln\, r$ are true. Therefore, in view of (6)

$$\begin{aligned} \ln\, \ln\, M_F(r) &\ge \ln\, \ln\, M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \ge \lambda \ln \left(\frac{1}{8}M_g\left(\frac{r}{2}\right) - |g(0)|\right) \\ &= \lambda(1+o(1)) \ln\, M_g\left(\frac{r}{2}\right) \ge (1+o(1))\lambda 2^{-l} r^{\lambda}, \quad r \to +\infty. \end{aligned} \tag{7}$$

On the other hand, if $\varrho[f] < +\infty$ then $\ln\ln\, M_f(\exp\{\gamma(r)\}) \le \varrho\gamma(r)$ for $\varrho > \varrho[f]$ and all $r \ge r_0(\varrho)$. Therefore, in view of (7)

$$\frac{\ln\ln\, M_F(r)}{\ln\ln\, M_f(\exp\{\gamma(r)\})} \ge (1+o(1)) \frac{\lambda}{2^{\lambda}(\varrho[f]+\varepsilon)} \frac{r^{\lambda}}{\gamma(r)} \to +\infty, \quad r \to +\infty, \tag{8}$$

because $\lambda \ln\, r - \ln\, \gamma(r) = (1+o(1))\lambda \ln\, r \to +\infty$ as $r \to +\infty$. The sufficiency of (3) is proved.

To prove the necessity of (3) we assume that (3) does not hold. Then $\ln\gamma(r_n) \geq \delta \ln r_n$ for some $\delta > 0$ and an increasing to $+\infty$ sequence (r_n). We choose $f(z) = e^z$ and $g(z) = E_\varrho(z)$ with $\varrho < \delta$, where E_ϱ is the Mittag-Leffler function. Then $M_f(r) = e^r$ and [7, p. 115]

$$M_{E_\varrho}(r) = E_\varrho(r) = (1 + o(1))\varrho e^{r^\varrho}, \quad r \to +\infty. \tag{9}$$

Therefore,

$$\ln\,\ln\, M_F(r) = \ln\, M_g(r) = r^\varrho + \ln\,\varrho + o(1), \quad r \to +\infty. \tag{10}$$

Thus,

$$\begin{aligned} \varliminf_{r\to+\infty} \frac{\ln\ln\, M_F(r)}{\ln\ln\, M_f(\exp\{\gamma(r)\})} &\leq \varliminf_{n\to+\infty} \frac{\ln\ln\, M_F(r_n)}{\ln\ln\, M_f(\exp\{\gamma(r_n)\})} \\ &= \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\gamma(r_n)} \leq \varliminf_{n\to+\infty} \frac{r_n^\varrho}{r_n^\delta} = 0, \end{aligned} \tag{11}$$

that is, if (3) does not hold then there exist entire functions f and g with $\lambda[f] = \varrho[f] = 1$ and $\lambda[g] = \varrho[g] = \varrho \in (0, +\infty)$, for which (2) is false. Theorem 1 is proved. □

The following theorem complements Theorem 1.

Theorem 2. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[g] \leq \varrho[g] < +\infty$ and $\lambda[f] > 0$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln\, M_F(r)}{\ln\ln\, M_g(\exp\{\gamma(r)\})} = +\infty, \quad F(z) = f(g(z)),$$

it is necessary and sufficient that (3) *holds.*

Proof. As in the proof of Theorem 1 we obtain (7) and for the function g we have $\ln\ln\, M_g(\exp\{\gamma(r)\}) \leq \varrho \ln\,\gamma(r)$ for every $\varrho > \varrho[g]$ and all $r \geq r_0(\varrho)$. Therefore, estimate (8) is true with $\varrho[g]$ instead $\varrho[f]$ and the sufficiency of (3) is proved.

If there exists a sequence (r_n) such that $\ln\gamma(r_n) \geq \delta \ln r_n$, $\delta > 0$, then again we choose f and g as in the proof of Theorem 1. Then (9) holds and

$$\ln\ln\, M_g(\exp\{\gamma(r)\}) = \ln\ln\left((1 + o(1))\varrho e^{\varrho\gamma(r)}\right) = \varrho\gamma(r) + o(1), \quad r \to +\infty.$$

In view of (9) as above we have

$$\varliminf_{r\to+\infty} \frac{\ln\ln\, M_F(r)}{\ln\ln\, M_g(\exp\{\gamma(r)\})} \leq \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\varrho\gamma(r_n)} \leq \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\varrho r_n^\delta} = 0.$$

Theorem 2 is proved. □

For the functions $f(z) = e^z$, $g(z) = E_\varrho(z)$ and $F(z) = f(g(z))$ chose the proof of Theorems 1 and 2 the following equalities are true

$$\varliminf_{r\to+\infty} \frac{\ln\ln\, M_F(r)}{\ln\ln\, M_f(\exp\{\gamma(r)\})} = \varliminf_{r\to+\infty} \frac{\ln\ln\, M_F(r)}{\ln\ln\, M_g(\exp\{\gamma(r)\})} = 0.$$

The following question arises: what is condition on γ providing existence of the limit

$$\lim_{r\to+\infty} \frac{\ln\ln\, M_F(r)}{\ln\ln\, M_f(\exp\{\gamma(r)\})} \left(\lim_{r\to+\infty} \frac{\ln\ln\, M_F(r)}{\ln\ln\, M_g(\exp\{\gamma(r)\})} \right) = 0.$$

The following theorem gives an answer to this question.

Theorem 3. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[f] \leq \varrho[f] < +\infty$ and $\varrho[g] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)), \tag{12}$$

it is necessary and sufficient that

$$\lim_{r\to+\infty} \frac{\ln \gamma(r)}{\ln r} = +\infty. \tag{13}$$

Proof. We assume that the function γ satisfies (13), that is $\ln r = o(\ln \gamma(r))$ as $r \to +\infty$. If the orders $\varrho[f]$ and $\varrho[g]$ are finite then $\ln \ln M_f(r) \leq \varrho \ln r$ and $\ln \ln M_g(r) \leq \varrho \ln r$ for $\varrho > \max\{\varrho[f], \varrho[g]\}$ and all $r \geq r_0(\varrho)$. Therefore, in view of (5)

$$\ln \ln M_F(r) \leq \ln \ln M_f(M_g(r)) \leq \varrho \ln M_g(r) \leq \varrho r^{\varrho}, \quad r \geq r_0(\varrho).$$

On the other hand, for $\lambda < \lambda[f]$ and all $r \geq r_0(\lambda)$ $\ln \ln M_f(e^{\gamma(r)}) \geq l\gamma(r)$. Therefore,

$$\frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} \leq \frac{\varrho r^{\varrho}}{\lambda\gamma(r)} \to 0, \quad r \to +\infty,$$

because $\varrho \ln r - \ln \gamma(r) = (1 + o(1)) \ln \gamma(r) \to -\infty$ as $r \to +\infty$. The sufficiency of (13) is proved.

Now we assume that (13) does not hold, that is for some $\delta < +\infty$ and an increasing to $+\infty$ sequence (r_n) the inequality $\ln \gamma(r_n) \leq \delta \ln r_n$ is true. We choose $f(z) = e^z$ and $g(z) = E_{\varrho}(z)$ with $\varrho > \delta$. Then in view of (10)

$$\begin{aligned} \overline{\lim_{r\to+\infty}} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} &\geq \overline{\lim_{n\to+\infty}} \frac{\ln\ln M_F(r_n)}{\ln\ln M_f(\exp\{\gamma(r_n)\})} \\ &= \overline{\lim_{n\to+\infty}} \frac{r_n^{\varrho}}{\gamma(r_n)} \geq \underline{\lim_{n\to+\infty}} \frac{r_n^{\varrho}}{r_n^{\delta}} = +\infty, \end{aligned} \tag{14}$$

that is equality (12) does not hold. Theorem 3 is proved. □

The following theorem is proved similarly.

Theorem 4. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f and g be arbitrary entire functions with $0 < \lambda[g] \leq \varrho[g] < +\infty$ and $\varrho[f] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_g(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)),$$

it is necessary and sufficient that (13) *holds.*

Remark 1.1. *From the proofs of Theorems 1 and 3 one can see that equality* (3) *is true provided, γ is an arbitrary slowly increasing function, and* (12) *holds if γ increase rapidly than power functions.*

Remark 1.2. *If we choose f and g as in the proofs of Theorem 1 and 2 and $\gamma(r) = ar^{\varrho}$, then there exists the limit*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\alpha(r)\})} = \lim_{r\to+\infty} \frac{r^{\varrho}}{\alpha(r)} = \frac{1}{a},$$

that is for each $K \in (0, +\infty)$ there exist entire functions of a finite order and a positive lower order and a positive continuous on $[0, +\infty)$ function γ such that

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = K.$$

2 OTHER RESULTS

In [5] the following analogue of Theorem A is proved.

Theorem C. *Let f, g, h be entire functions of positive lower order and of finite order and $F(z) = f(g(z))$, $\Phi(z) = f(h(z))$. If $\varrho[h] < \lambda[g]$ then for every $A \in (0, \lambda[g]/\varrho[h])$*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_H(r^A)} = +\infty.$$

We will complement this theorem by two next statements.

Proposition 2.1. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f, g and h be arbitrary entire functions with $0 < \lambda[f] \le \varrho[f] < +\infty$, $\lambda[g] > 0$ and $\varrho[h] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_\Phi(e^{\gamma(r)})} = +\infty, \quad F(z) = f(g(z)),\ \Phi(z) = f(h(z)), \tag{15}$$

it is necessary and sufficient that

$$\lim_{r\to+\infty} \frac{\gamma(r)}{\ln r} = 0. \tag{16}$$

Proof. In view of (5) for arbitrary $\varrho > \max\{\varrho[f], \varrho[h]\}$ and all $r \ge r_0(\varrho)$ we have

$$\ln\ln M_\Phi(e^{\gamma(r)}) \le \varrho \ln M_h(e^{\gamma(r)}) \le \varrho e^{\varrho\gamma(r)}.$$

Therefore, in view of (7) $\dfrac{\ln\ln M_F(r)}{\ln\ln M_\Phi(e^{\gamma(r)})} \ge (1+o(1))\dfrac{l2^{-\lambda}}{\varrho}\dfrac{r^\lambda}{e^{\varrho\gamma(r)}} \to +\infty, \quad r \to +\infty$, because by the condition (16) $\dfrac{r^l}{e^{\varrho\gamma(r)}} = \exp\{\lambda \ln r - \varrho\gamma(r)\} \to +\infty$ as $r \to +\infty$. The sufficiency of (16) is proved.

Now we assume that (16) does not hold, that is for some $\delta < +\infty$ and an increasing to $+\infty$ sequence (r_n) the inequality $\gamma(r_n) \ge \delta \ln r_n$ is true. We choose $f(z) = h(z) = e^z$ and $g(z) = E_\varrho(z)$ with $\varrho < \delta$. Then $\ln\ln M_\Phi(r) = r$ and in view of (10)

$$\begin{aligned} \varliminf_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_\Phi(\exp\{\gamma(r)\})} &\le \varliminf_{n\to+\infty} \frac{\ln\ln M_F(r_n)}{\ln\ln M_\Phi(\exp\{\gamma(r_n)\})} \\ &= \varliminf_{n\to+\infty} \frac{r_n^\varrho}{\exp\{\gamma(r)\}} \le \varliminf_{n\to+\infty} \frac{r_n^\varrho}{r_n^\delta} = 0, \end{aligned} \tag{17}$$

that is there exist entire functions f, g and h for which (13) is false. Proposition 1 is proved. □

Proposition 2.2. *Let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$. Let f, g and h be arbitrary entire functions with $0 < l[f] \le \varrho[f] < +\infty$, $\varrho[g] < +\infty$ and $\lambda[h] > 0$. In order to*

$$\lim_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_\Phi(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)),\ \Phi(z) = f(h(z)), \tag{18}$$

it is necessary and sufficient that

$$\lim_{r\to+\infty} \frac{\gamma(r)}{\ln r} = +\infty. \tag{19}$$

Proof. We assume that the function γ satisfies (19), that is $\ln r = o(\gamma(r))$ as $r \to +\infty$. If the orders $\varrho[f]$ and $\varrho[g]$ are finite then for $\varrho > \max\{\varrho[f], \varrho[g]\}$ and all $r \geq r_0(\varrho)$ in view of (5) we have $\ln \ln M_F(r) \leq \varrho r^{\varrho}$ for $r \geq r_0(\varrho)$. On the other hand, using (6) for $0 < \lambda < \min\{\lambda[f], \lambda[f]\}$ and $r \geq r_0(\lambda)$ we obtain

$$\ln \ln M_{\Phi}(e^{\gamma(r)}) \geq \ln \ln M_f \left(\frac{1}{8} M_g \left(\frac{r}{2}\right) - |g(0)|\right) \geq (1 + o(1))\lambda 2^{-\lambda} e^{\lambda\gamma(r)}, \quad r \to +\infty.$$

Therefore, $\dfrac{\ln\ln M_F(r)}{\ln\ln M_{\Phi}(\exp\{\gamma(r)\})} \leq \dfrac{(1+o(1))\lambda}{\varrho 2^{\lambda}} e^{\varrho \ln r - \lambda\gamma(r)} \to 0, \quad r \to +\infty$. The sufficiency of (19) is proved.

Now we assume that (19) does not hold, that is for some $\delta < +\infty$ and an increasing to $+\infty$ sequence (r_n) the inequality $\gamma(r_n) \leq \delta \ln r_n$ is true. We choose $f(z) = h(z) = e^z$ and $g(z) = E_{\varrho}(z)$ with $\varrho > \delta$. Then in view of (10)

$$\begin{aligned} \varlimsup_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_{\Phi}(\exp\{\gamma(r)\})} &\geq \varlimsup_{n\to+\infty} \frac{\ln\ln M_F(r_n)}{\ln\ln M_{\Phi}(\exp\{\gamma(r_n)\})} \\ &= \varlimsup_{n\to+\infty} \frac{r_n^{\varrho}}{\exp\{\gamma(r_n)\}} \geq \varliminf_{n\to+\infty} \frac{r_n^{\varrho}}{r_n^{\delta}} = +\infty, \end{aligned} \tag{20}$$

that is (18) does not hold. Proposition 2 is proved. □

Finally, we will prove a result on the growth of a composition of entire functions in the terms of generalized orders. By L we denote a class of all positive continuous on $(-\infty, +\infty)$ functions α such that $\alpha(x) = \alpha(x_0)$ for $-\infty < x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$.

For $\alpha \in L$ and $\beta \in L$ the generalized order $\varrho_{\alpha\beta}[f]$ and a lower generalized order $\lambda_{\alpha\beta}[f]$ of an entire function f are defined [3] by the formulas

$$\varrho_{\alpha,\beta}[f] = \varlimsup_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)}, \quad l_{\alpha,\beta}[f] = \varliminf_{r\to+\infty} \frac{\alpha(\ln M_f(r))}{\beta(\ln r)}.$$

Proposition 2.3. *Let $\alpha \in L$, $\beta \in L$, $\beta(x + O(1)) = (1 + o(1)\beta(x)$ as $x \to +\infty$ and f, g be entire functions with $0 < \lambda_{\alpha,\beta}[f] \leq \varrho_{\alpha,\beta}[f] < +\infty$ and $0 < l_{\alpha,\beta}[g] \leq \varrho_{\alpha,\beta}[g] < +\infty$. In order to*

$$\lim_{r\to+\infty} \frac{\alpha(\ln M_F(r))}{\alpha(\ln M_f(r))} = +\infty, \quad F(z) = f(g(z)), \tag{21}$$

it is necessary and sufficient that

$$\lim_{x\to+\infty} \frac{\beta(x)}{\alpha(x)} = +\infty. \tag{22}$$

Proof. If (22) holds then from (6) and the definition of the lower generalized order it follows that for each $0 < \lambda < \lambda_1 < \min\{\lambda_{\alpha,\beta}[f], \lambda_{\alpha,\beta}[g]\}$ and $r \geq r_0(\lambda)$

$$\begin{aligned} \alpha(\ln M_F(r)) &\geq \alpha\left(\ln M_f \left(\frac{1}{8} M_g \left(\frac{r}{2}\right) - |g(0)|\right)\right) \geq \lambda_1 \beta\left(\ln M_g \left(\frac{r}{2}\right) + O(1)\right) \\ &= \lambda_1(1+o(1))\beta\left(\ln M_g \left(\frac{r}{2}\right)\right) = \lambda_1(1+o(1))\beta\left(\alpha^{-1}\left(\alpha\left(\ln M_g\left(\frac{r}{2}\right)\right)\right)\right) \\ &\geq \lambda_1(1+o(1))\beta(\alpha^{-1}(\lambda_1(1+o(1))\beta(\ln r))) \geq \lambda\beta(\alpha^{-1}(\lambda\beta(\ln r))). \end{aligned}$$

On the other hand, for $\varrho > \varrho_{\alpha,\beta}[f]$ and all $r \geq r_0(\varrho)$ we have $\alpha(\ln M_f(r)) \leq \varrho\beta(\ln r)$. Therefore,

$$\varliminf_{r\to+\infty} \frac{\alpha(\ln M_F(r))}{\alpha(\ln M_f(r))} \geq \varliminf_{r\to+\infty} \frac{\lambda\beta(\alpha^{-1}(\lambda\beta(\ln r)))}{\varrho\beta(\ln r)} = \frac{l^2}{\varrho} \varliminf_{x\to+\infty} \frac{\beta(x)}{\alpha(x)} = +\infty,$$

that is (21) is true. If (22) does not hold, that is $\varliminf\limits_{x\to+\infty} \beta(x)/\alpha(x) < +\infty$ then in view of (5) for $\lambda < \lambda_{\alpha,\beta}[f]$, $\varrho > \max\{\varrho_{\alpha,\beta}[f],\, \varrho_{\alpha,\beta}[f]\}$ and all r enough large

$$\varliminf_{r\to+\infty} \frac{\alpha(\ln M_F(r))}{\alpha(\ln M_f(r))} \leq \varliminf_{r\to+\infty} \frac{\varrho\beta(\ln M_g(r))}{\lambda\beta(\ln r)} = \varliminf_{r\to+\infty} \frac{\varrho\beta(\alpha^{-1}(\alpha(\ln M_g(r))))}{\lambda\beta(\ln r)}$$
$$\leq \varliminf_{r\to+\infty} \frac{\varrho\beta(\alpha^{-1}(\varrho\beta(\ln r)))}{l\beta(\ln r)} = \frac{\varrho^2}{l} \varliminf_{x\to+\infty} \frac{\beta(x)}{\alpha(x)} < +\infty,$$

that is (21) is false. Proposition 3 is proved. □

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Шеремета М.М. *Про зростання композицій цілих функцій* // Карпатські матем. публ. — 2017. — Т.9, №2. — C. 181–187.

Нехай γ — додатна, неперервна на $[0, +\infty)$ і зростаюча до $+\infty$ функція, а f і g — довільні цілі функції додатного нижнього порядку і скінченного порядку.

Для того, щоб

$$\varlimsup_{r\to+\infty} \frac{\ln\ln M_{f(g)}(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = +\infty, \quad M_f(r) = \max\{|f(z)| : |z| = r\},$$

необхідно і досить, щоб $(\ln \gamma(r))/(\ln r) \to 0$ при $r \to +\infty$. Це твердження є відповіддю на питання, поставлене А. Сінхом і М. Балоріа у 1991 р.

Також для того, щоб

$$\varliminf_{r\to+\infty} \frac{\ln\ln M_F(r)}{\ln\ln M_f(\exp\{\gamma(r)\})} = 0, \quad F(z) = f(g(z)),$$

необхідно і достатньо, щоб $(\ln \gamma(r))/(\ln r) \to \infty$ при $r \to +\infty$.

Ключові слова і фрази: ціла функція, композиція функцій, узагальнений порядок.