THE NONLOCAL PROBLEM FOR THE $2n$ DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATOR COEFFICIENTS AND THE INVOLUTION

We study a problem with periodic boundary conditions for a $2n$-order differential equation whose coefficients are non-self-adjoint operators. It is established that the operator of the problem has two invariant subspaces generated by the involution operator and two subsystems of the system of eigenfunctions which are Riesz bases in each of the subspaces. For a differential-operator equation of even order, we study a problem with non-self-adjoint boundary conditions which are perturbations of periodic conditions. We study cases when the perturbed conditions are Birkhoff regular but not strongly Birkhoff regular or nonregular. We found the eigenvalues and elements of the system $V$ of root functions of the operator which is complete and contains an infinite number of associated functions. Some sufficient conditions for which this system $V$ is a Riesz basis are obtained. Some conditions for the existence and uniqueness of the solution of the problem with homogeneous boundary conditions are obtained.

Key words and phrases: operator of involution, differential-operator equation, eigenfunctions, Riesz basis.

1 INTRODUCTION

The theory of differential equations with an unbounded operator coefficient was initiated by Hill and Yosida where the first theorems on the existence of the Cauchy problem solution for a linear homogeneous differential equation with respect to a function with values in a Banach space were obtained. Among works on this subject should be noted works of Kato T., Krein S.G., Mizohata S., Phillips R.S.


During recent years the number of publications with the use of an involution operator in various sections of the theory of ordinary differential equations (see [2, 8–10, 12, 13, 15, 16]), partial differential equations (see [1, 7, 11, 14, 17, 18]) and differential equations with operator coefficients (see [3–6]) increased significantly.

In our article we will use the following notations. Let $H$ be a separable Hilbert space and $A : D(A) \subset H \rightarrow H$ be the closed unbounded linear operator with the discrete spectrum

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\[ \sigma(A) \equiv \{ z_k \in \mathbb{R}, \ z_k = \alpha(k)^2, \ \alpha, \gamma > 0, \ k = 1,2,\ldots \}. \] We denote by \( V(A) \equiv \{ v_k \in H : k = 1,2,\ldots \} \) the system of the eigenfunctions of \( A \) which forms a Riesz basis in \( H \), by \( W(A) \equiv \{ w_m \in H : m = 1,2,\ldots \} \) the biorthogonal system of the functions in the sense of equalities \((v_k, w_m; H) = 0, k \neq m, (v_k, w_m; H) = 1, k, m = 1,2,\ldots; H(A) \equiv \{ h \in H : A^h \in H \}, s \geq 0). \]

Let \( H_1 \equiv L_2((0,1), H) \) and \( D_x : H_1 \to H_1 \) is a strong derivative in the space \( H_1; \)

\[ \left\| \frac{u(x + \triangle x) - u(x)}{\triangle x} - D_x u; H_1 \right\| \to 0, \triangle x \to 0. \] Denote by \( H_2 \equiv \{ u \in H_1 : D_x^2 u \in H_1, A^{2n} u \in H_1 \}; \) by \( [H] \) the algebra of the bounded linear operators \( B : H \to H \). Denote by \( H_0 \equiv L_2(0,1); \)

let \( I \) be the operator of the involution in the space \( H_0, I y(x) \equiv y(1 - x), \) and let \( E \) be the identity transformation in \( H_0, p_j \equiv \frac{1}{2}(E + (-1)^j I) \) are the orthoprojectors in the space \( H_0, H_{0,j} \equiv \{ y \in H_0 : y \equiv p_j y \}, j = 0,1. \) Let us denote by \( W_{2n}(0,1) \equiv \{ y \in H_0 : y^{(m)} \in C[0,1], m = 0,2n - 1, y^{(2n)} \in H_0 \}, \) by \( W^* \) the space of the continuous linear functionals on the space \( W_{2n}(0,1) \) and by \( W^*_j \equiv \{ l \in W^* : ly = 0, y \in H_{2,1 - j} \cap W_{2n}(0,1) \}; j = 0,1. \)

We consider the following boundary problem

\[ \begin{aligned}
Lw & \equiv (-1)^n D_x^{2n} w(x) + A^{2n} w(x) \\
& + \sum_{s=1}^n a_s \left( D_x^{2s-1} w(x) + D_x^{2s-1} w(1 - x) \right) = f(x), \quad x \in (0,1), \\
\ell_j w & \equiv D_x^{2j-1} w(0) - D_x^{2j-1} w(1) + \ell_j^1 w = \varphi_j, \\
\ell_{n+j} w & \equiv D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = \varphi_{n+j},
\end{aligned} \]

where

\[ \ell_j^1 w \equiv \sum_{r=0}^{m_j} \left( b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1) \right), \quad j = 1,2,\ldots, n. \]

The function \( w \) is called the solution of the problem (1)–(4) if

\[ \|Lw - f; H_1\| = 0, \quad \|I_j w - \varphi_j; H \left( A^2 \right)\| = 0, \]

\[ \beta_{n+j} = 2n - 2j + \frac{3}{2}, \quad \beta_j = 2n - \max(m_j, 2j - 1) - \frac{1}{2}, \]

\[ a_{j, b_{j,r,s}} \in \mathbb{R}, \quad r = 0,1,\ldots, \quad m_j \leq 2n - 1, \quad s = 0,1, \quad j = 1,2,\ldots, n. \]

The paper is arranged as follows. In Section 2 we investigate the properties of the operator of problem with periodic conditions for the equation \((-1)^n y^{(2n)} = \lambda y \). In Section 3 we study the spectral properties of the operator of a problem with boundary conditions that are periodic perturbations. In Sections 4 we construct a commutative group of operators that map the root functions of the operators of perturbed boundary-value problems. In Section 5 using these operators, systems of root functions of boundary-value problem operators are constructed and conditions for the completeness and basis property of these systems are established. In Section 6 some analogous results are obtained for the operators of boundary problems generated by differential equations with an involution.

## 2 A Spectral Problem with Periodic Boundary Conditions for a Differential-Operator Equation

Consider the partial case of the problem (1)–(4) with \( a_j = 0, \ b_{j,r,s} = 0, \ r = 0,1,\ldots, m_j, \)

\[ s = 0,1, \ j = 1,2,\ldots, n, \] namely

\[ \begin{aligned}
(-1)^n D_x^{2n} u(x) + A^{2n} u(x) & = f(x), \quad x \in (0,1), \\
\ell_{0,j} u & \equiv D_x^{2j-1} u(0) - D_x^{2j-1} u(1) = 0, \\
\ell_{0,n+j} u & \equiv D_x^{2j-2} u(0) - D_x^{2j-2} u(1) = 0, \quad j = 1,2,\ldots, n.
\end{aligned} \]
Let $L_0$ be the operator of the problem (5)–(7),
$$L_0 u \equiv (-1)^n D_x^{2n} u + A^{2n} u, \quad u \in D(L_0), \quad D(L_0) \equiv \{ u \in H_2 : l_0 j u = 0, \ j = 1, 2, \ldots, 2n \}.$$

Consider the spectral problem

$$(-1)^n D_x^{2n} u(x) + A^{2n} u(x) = \lambda u(x), \quad \lambda \in \mathbb{C}, \quad (8)$$
$$\ell_{0,j} u \equiv u^{(2j-1)}(0) - u^{(2j-1)}(1) = 0, \quad j = 1, 2, \ldots, 2n. \quad (9)$$

We find the solution of the spectral problem (8), (9) as the product $u(x) = y(x) v_k$, $v_k \in V(A)$, $k = 1, 2, \ldots$.

To determine the unknown function $y \in W^{2n}(0, 1)$ we obtain the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \quad (10)$$
$$\ell_{0,j} y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j = 1, 2, \ldots, n, \quad (11)$$
$$\ell_{0,n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \ldots, n. \quad (12)$$

Let $L_{0,k}$ be the operator of the problem (10)–(12),
$$L_{0,k} y \equiv (-1)^n y^{(2n)} + z_k^{2n} y, \quad y \in D(L_{0,k}), \quad D(L_{0,k}) \equiv \{ y \in W^{2n}(0, 1) : l_{0,j} y = 0, j = 1, 2, \ldots, 2n \}.$$

The roots $\rho_j$ of the characteristic equation $(-1)^n \rho^{2n} = \lambda - z_k^{2n}$ of the differential equation

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad (13)$$

are defined by the relations

$$\rho_j = \omega_j \rho, \quad \omega_1 = i, \quad \omega_j = i \exp \left( \frac{\pi(j - 1)}{2n} \right), \quad j = 2, 3, \ldots, n.$$

The fundamental system of the solutions of the differential equation (13) is defined by the formulas

$$Y_j(x, \rho) \equiv \frac{1}{2} \left( \exp \omega_j \rho x + \exp \omega_j \rho(1 - x) \right) \in H_{2,0}, \quad j = 1, 2, \ldots, n, \quad (14)$$
$$Y_{n+j}(x, \rho) \equiv \frac{1}{2} \left( \exp \omega_j \rho x - \exp \omega_j \rho(1 - x) \right) \in H_{2,1}, \quad j = 1, 2, \ldots, n. \quad (15)$$

Substituting the general solution

$$y(x, \rho) = \sum_{s=1}^{2n} C_s Y_s(x, \rho)$$

of the differential equation (13) into the boundary conditions (11), (12) we obtain the equation for determination the eigenvalues of the operator $L_{0,k}$

$$\Delta(\rho) = \det(l_{0,j} Y_j)^{2n}_{r_{j}=1} = 0. \quad (16)$$

By substituting the functions (14), (15) in the boundary conditions (11), (12), we obtain $l_{0,r} Y_{n+j} = 0$, $l_{0,n+r} Y_j = 0$, $j, r = 1, 2, \ldots, n$. Therefore $\Delta(\rho) = \Delta_0(\rho) \Delta_1(\rho) = 0$, where $\Delta_s(\rho) = \det(l_{s,n+r} Y_{n+j})_{s,j=1}^{2n}$. Therefore the solutions of the equation (16) are $\rho_q = 2q\pi i$, $q = 0, 1, 2, \ldots$, which are numbered in ascending order and lie on the half-line $1m\rho = 0$, $\text{Re} \rho \geq 0$.

The operator $L_{0,k}$ is self-adjoint (see [15]). Therefore the solutions of the equation (16) are $\rho_q = 2q\pi i$, $q = 0, 1, 2, \ldots$, which are numbered in ascending order and lie on the half-line $1m\rho = 0$, $\text{Re} \rho \geq 0$.

Thus, the operator $L_{0,k}$ has eigenvalues $\lambda_{q,k} = (\rho_q)^{2n} + z_k^{2n}$, $q = 0, 1, \ldots$. We obtain the following result.
Lemma 2.1. The self-adjoint operator $L_{0,k}$ has a point spectrum

$$
\sigma (L_{0,k}) = \{ \lambda_q \in \mathbb{R} : \lambda_q = (2\pi q)^2 + z_k^{2n}, q = 0, 1, \ldots \}
$$

and a system of eigenfunctions

$$
V (L_{0,k}) \equiv \{ v_q (x) \in L_2 (0, 1) : v_0 (x) = 1, v_{2q} (x) \equiv \sqrt{2} \cos 2\pi qx, v_{2q-1} (x) \equiv \sqrt{2} \sin 2\pi qx, q = 1, 2, \ldots \},
$$

which is an orthonormal basis of the space $H_0$.

Remark 2.1. The systems

$$
V_0 (L_{0,k}) \equiv \{ v_{2q} (x) : q = 0, 1, \ldots \}, V_1 (L_{0,k}) \equiv \{ v_{2q-1} (x) : q = 1, 2, \ldots \}
$$

form an orthonormal basis in spaces $H_{0,0}$ and $H_{0,1}$, respectively.

Therefore, the operator $L_0$ has the following eigenfunctions in the space $H_1$

$$
V (L_0) \equiv \{ v_{q,k} (x, L_0) \in H_1 : v_{q,k} (x, L_0) \equiv v_q (x)v_k, q = 0, \infty, k = 1, \infty \}.
$$

A system of functions $\{ h_s \}_{s=1}^{\infty} \subset H$ is called a Riesz basis in a Hilbert space $H$, if $\{ h_s \}_{s=1}^{\infty}$ is complete in the space $H$, and for any orthonormal basis $\{ \epsilon_s \}_{s=1}^{\infty} \subset H$ there exists an isomorphism $B : H \rightarrow H$, $B \epsilon_s = h_s, s = 1, 2, \ldots$.

The product of a system $V (A)$ and an orthonormal system $V (L_{0,k})$ is the Riesz basis (see [9]) in the space $H_1$. Thus, the following theorem is true.

Theorem 1. The operator $L_0$ has a discrete spectrum

$$
\sigma (L_0) \equiv \{ \lambda_{q,k} \in \mathbb{R} : \lambda_{q,k} = \rho_q^{2n} + z_k^{2n}, k = 1, \infty, q = 0, \infty \},
$$

and the system of the eigenfunctions $V (L_0)$ forms the Riesz basis in the space $H_1$.

Let us consider the functions

$$
y_r (x, \rho_q) \equiv \frac{1}{2} (1 + e^{2\rho_q x} - 1) \left( e^{2\rho_q x} + e^{2\rho_q (1-x)} \right),
$$

$$
y_1 (x, \rho_q) \equiv \frac{1}{2} (1 - 2x) \sin \rho_q x, \quad r = 2, 3, \ldots, n, \quad q = 1, 2, \ldots, \quad (17)
$$

and determine the square matrix

$$
B_{0,p} (x, \rho_q) \equiv (\beta_{p,s}^{0})_{p,s=1}^{n}
$$

of the order $n$ according to the following: the row with number $p$ is determined by the elements of the system (17) $\beta_{p,s}^{0} (x, \rho_q) = y_s (x, \rho_q)$ and the other lines by the formulas $\beta_{j,s}^{0} (x, \rho_q) \equiv (\omega_s)^{2j-1}$ with $j \neq p, j, s = 1, 2, \ldots, n$.

We denote the determinant of the matrix $B_{0,p} (x, \rho_q)$ by $y_{1,p} (x, \rho_q)$.

Substituting the determinant into conditions (11), (12), we obtain

$$
l_{0,r} y_{1,p} = 0, \quad r \neq p, \quad l_{0,p} y_{1,p} = (\rho_q)^{2p-1} h(i) \mathcal{W}^n, \quad (18)
$$
where the Vandermonde determinant $W^n$ is constructed by the numbers
\[ 1, (\omega_2)^2, \ldots, (\omega_n)^2, h(i) = (-i)^{n-1}i, i = \sqrt{-1}, r = 1, \ldots, 2n, m = 1, 2, \ldots, n. \]

Consider the functions $y_{2,p}(x, \rho_q) \equiv (h(i)W^n)^{-1}y_{1,p}(x, \rho_q)$, $p = 1, 2, \ldots, n$.
From the relation (18) we obtain
\[ l_{0,r}y_{2,p} = 0, r \neq p, l_{0,p}y_{2,p} = (\rho_q)^{2p-1}, p, r = 1, 2, \ldots, n. \] (19)

Similarly, let us consider the system of functions
\begin{align*}
y_{n+r}(x, \rho_q) & \equiv \frac{1}{2} (1 - e^{i\omega q x})^{-1} \left(e^{i\rho q x} - e^{i\rho q(1-x)}\right), \\
y_{n+1}(x, \rho_q) & \equiv \frac{1}{2} (1 - 2x) \cos \rho q x, \quad r = 2, 3, \ldots, n, q = 1, 2, \ldots,
\end{align*}
and a square matrix
\[ B_{1,r}(x, \rho_q) \equiv (\beta_{1,s}^{1})_{p,s=1} \]
of the order $n$ which rows are determine by following: the row with number $r$ is determined by the elements of the system (20) $\beta_{1,s}^{1} (x, \rho_q) \equiv y_{n+s}(x, \rho_q)$ and the other lines by the equalities
\[ \beta_{j,s}^{1} (x, \rho_q) \equiv (\omega_q)^{2j-2}, j \neq r, r, s = 1, 3, \ldots, n. \]

We denote the determinant of the matrix by $y_{1,n+r}(x, \rho_q)$.
Substituting it into conditions (11), (12), we get
\[ l_{0,j}y_{1,n+r} = 0, j \neq n + r, l_{0,n+r}y_{1,n+r} = W^n (\rho_q)^{2r-2}. \] (21)

Let us define the functions $y_{2,n+r}(x, \rho_q) \equiv (W^n)^{-1}y_{1,n+r}(x, \rho_q)$, $r = 1, 2, \ldots, n$.
Taking the relation (21) for the functions $y_{2,n+r}(x, \rho_q)$ into account, we obtain
\[ l_{0,j}y_{2,n+r} = 0, j \neq n + r, l_{0,n+r}y_{2,n+r} = (\rho_q)^{2r-2}, j = 1, \ldots, 2n, r = 1, 2, \ldots, n. \]

**Remark 2.2.** There exist positive numbers $K_0, K_1$ such that
\[ K_1 \leq \|y_{2,q}(x, \rho_q); H_0\| \leq K_2 < \infty, \quad j = 1, 2, \ldots, 2n, q = 1, 2, \ldots. \] (22)

Here $K_s$, $s \in \mathbb{N}$, are positive constants.

### 3 Nonlocal boundary value problem

For the differential-operator equation (5) and an arbitrary fixed $p \in \{1, 2, \ldots, n\}$ and $b \in \mathbb{R}$ we consider the boundary value problem
\begin{align*}
\ell_{1,j} u & \equiv D^{2j-1}_x u(0) - D^{2j-1}_x u(1) = 0, \quad j \neq p, j = 1, 2, \ldots, n, \quad (23) \\
\ell_{1,p} u & \equiv D^{2p-1}_x u(0) - D^{2p-1}_x u(1) + l_p^2 u = 0, \quad (24) \\
\ell_{1,n+j} u & \equiv D^{2j-2}_x u(0) - D^{2j-2}_x u(1) = 0, \quad j = 1, 2, \ldots, n, \quad (25) \\
\end{align*}
with
\[ l_p^2 u \equiv b(D^{2p-1}_x u(0) + D^{2p-1}_x u(1)) = 0, \quad b \in \mathbb{R}. \] (26)
We will use following notations. Let $L_1$ be the operator of the problem (5), (23)–(26) and $L_1u \equiv (-1)^nD_+^{2n}u(x) + A^2u(x)$, $u \in D(L_1)$, $D(L_1) \equiv \{u \in H_2 : l_1u = 0, r = \overline{1,2n}\}$.

We find the solution of the spectral problem (8), (23)–(26) as the product $u(x) = y(x)v_k$, $v_k \in V(A), k = 1, 2, \ldots$.

To determine the unknown function $y \in W^{2n}(0,1)$ we consider the spectral problem

\[
(-1)^n y^{(2n)}(x) + z_k^{2n}y(x) = \lambda y(x), \quad \lambda \in \mathbb{C},
\]

\[
l_{1,j}y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j \neq p, j = 1, 2, \ldots, n,
\]

\[
l_{1,p}y \equiv y^{(2p-1)}(0) - y^{(2p-1)}(1) + l_k^2y = 0,
\]

\[
l_{1,n+j}y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \ldots, n,
\]

with

\[
l_k^2y \equiv b \left( y^{(2p-1)}(0) + y^{(2p-1)}(1) \right).
\]

Let $L_{1,k} \equiv L_{1,k,p}$ be the operator of the problem (27)–(31) and

$L_{1,k}y \equiv (-1)^n y^{(2n)}(x) + z_k^{2n}y(x), \quad y \in D(L_{1,k}), D(L_{1,k}) \equiv \{y \in W^{2n}(0,1) : l_{1,j}y = 0, j = \overline{1,2n}\}$.

Let $V(L_{1,k})$ be the system of root functions of the operator $L_{1,k}$, let $R(L_{1,k}) \equiv E + S(L_{1,k})$ be the operator which maps the system $V(L_{0,k})$ into the system $V(L_{1,k}).$

**Theorem 2.** For any $b \in \mathbb{R}$, $p \in \{1, 2, \ldots, n\}$, the operator $L_{1,k}$ has the point spectrum $\sigma(L_{0,k})$ and the system of root functions $V(L_{1,k})$ forms a Riesz basis in $H_0$.

**Proof.** We will show that eigenvalues of the operators $L_{0,k}$ and $L_{1,k}$ coincide.

We substitute the fundamental system (14), (15) for the solutions of the differential equation (27) into the boundary conditions (28)–(31). Using $l_k^2y_n+(x, \rho) = 0$, $j, p = \overline{1,n}$, we obtain the same equations for determination the spectrum

\[
\det(l_{1,j}y_1(x, \rho))_{j,r=1}^{2n} = \det(l_{1,j}y_r(x, \rho))_{j,r=1}^{n} \det(l_{1,n+j}y_{n+r}(x, \rho))_{j,r=1}^{n}.
\]

Let us define elements of the system $V(L_{1,k}).$

It is easy to see that $v_{2q}(x) \in D(L_{1,k}), L_{1,k}v_{2q}(x) = \lambda_q v_{2q}(x), q = 0, 1, 2, \ldots$. Hence

\[
v_{2q}(x, L_{1,k}) \equiv v_{2q}(x), \quad q = 0, 1, \ldots.
\]

We define the root functions of the operator $L_{1,k}$ as

\[
v_{2q-1}(x, L_{1,k}) \equiv v_{2q-1}(x) + c_{b,p}y_{2,p}(x, \rho_q).
\]

Substituting (33) into the boundary condition (29) and taking the equality (19) into account, we obtain $c_{b,p} = -\sqrt{2}b$,

\[
v_{2q-1}(x, L_{1,k}) \equiv v_{2q-1}(x) - \sqrt{2}b y_{2,p}(x, \rho_q), \quad q = 1, 2, \ldots.
\]

Thus, the operator $L_{1,k}$ has a system of root functions (32)–(34) in the sense of equalities

\[
L_{1,k}v_{2q-1}(x, L_{1,k}) = \lambda_q v_{2q-1}(x, L_{1,k}) + \xi_{b,q} v_{2q}(x, L_{1,k}),
\]

\[
\xi_{b,q} = -4\sqrt{2b}n (\rho_q) \overline{2n-1}, L_{1,k}v_{2q}(x, L_{1,k}) = \lambda_q v_{2q}(x, L_{1,k}), \quad q = 1, 2, \ldots.
\]

Given the regularity according to Birkhoff (see [15]) of boundary conditions (28)–(31) we obtain that the system $V(L_{1,k})$ is complete and minimal in the space $H_0$. $\square$
Let $W^{n-1}$ be the Vandermonde determinant constructed by elements $\omega^2, \omega^2, \ldots, \omega^2$; let $R_1 \equiv E + S_1$ be the operator which maps the system $V(L_0)$ into the system $V_1$ and elements of this system are

$$
v_{2q-1,1}(x) \equiv (1 - \sqrt{2}^{-1}W^n(h(i)W^{n-1})^{-1}b(1-2x))v_{2q}(x),
$$

$$
v_{0,1}(x) \equiv v_{0}(x), \quad v_{2q,1}(x) \equiv v_{2q}(x), \quad q = 1, 2, \ldots.
$$

**Lemma 3.1.** The system $V_1$ forms a Riesz basis in $H_0$.

**Proof.** For an arbitrary function $\varphi \in H_0$ we have

$$
\varphi = \varphi_0 v_0(x) + \sum_{q=0}^{\infty} \left( \varphi_{2q} v_{2q}(x) + \varphi_{2q-1} v_{2q-1}(x) \right) \in H_0,
$$

$$
\|\varphi; H_0\|^2 = |\varphi_0|^2 + \sum_{q=1}^{\infty} \left( |\varphi_{2q}|^2 + |\varphi_{2q-1}|^2 \right) < \infty,
$$

consider the function

$$
\varphi_1 = R_1 \varphi = \varphi_0 v_{0,1}(x) + \sum_{q=1}^{\infty} \left( \varphi_{2q} v_{2q,1}(x) + \varphi_{2q-1} v_{2q-1,1}(x) \right),
$$

$$
\|R_1 \varphi; H_0\|^2 \leq K_3 \|\varphi; H_0\|^2, \quad K_3 = 2 \left( 1 + |W^n(W^{n-1})^{-1}b|^2 \right).
$$

Therefore $\|R_1; H_0\|^2 \leq K_3 < \infty$, $R_1 \equiv E + S_1 \in [H_0]$, $R_1^{-1} \equiv E - S_1 \in [H_0]$. Taking into account the Bari Theorem (see [9]), we obtain the following statement: the system $V_1$ forms the Riesz basis in $H_0$.

Therefore, the operator $L_1$ has the following system of root functions in the space $H_1$

$$
V(L_1) \equiv \left\{ v_{q,k}(x, L_1) \equiv v_{q}(x, L_{1,k}) : q = 0, \infty, k = 1, \infty \right\}.
$$

**Remark 3.1.** The operator $L_1$ has a system of root functions in the means of equalities

$$
L_1 v_{2q-1,k}(x, L_1) = \lambda_{q,k} v_{2q-1,k}(x, L_1) + 2b_n \left( p_k \right)^{2n-1}, \quad q, k = 1, 2, \ldots,
$$

$$
L_1 v_{2q,k}(x, L_1) = \lambda_{q,k} v_{2q,k}(x, L_1), \quad q = 0, 1, \ldots, k = 1, 2, \ldots.
$$

**Theorem 3.** For any fixed numbers $p \in \{1, 2, \ldots, n\}$, $b \in \mathbb{R}$, the system $V(L_1)$ is the Riesz basis of the space $H_1$.

**Proof.** Let $R(L_{1,k}) \equiv E + S(L_{1,k}) : V(L_{0,k}) \rightarrow V(L_{1,k})$, let $p_k$ be a projection in $H$, $p_k y \equiv (y, w_k(A); H) v_k$, $R(L_1) \equiv \sum_{k=1}^{\infty} R(L_{1,k}) p_k$.

From the definition of the operator $R(L_1) = E + S(L_1)$ it follows that $R^{-1}(L_1) = E - S(L_1)$. Therefore the system $V(L_1)$ is complete and minimal in the space $H_1$. Taking into account the representations of the elements of the system $V(L_{1,k})$ and Theorem 2, we obtain $\|R(L_1); [H_1]\| \leq K_4 \|R(L_{1,k}); [H_0]\| < \infty$.

Taking into account the Bari Theorem (see [9]), we obtain the following statement: the system $V_1$ forms the Riesz basis in $H_1$. 

\[\square\]
4 Transformation operators

For any $k \in \mathbb{N}$, $p \in \{1, 2, \ldots, n\}$, we define the operator $B_p : H_0 \to H_0$ as the operator whose eigenvalues coincide with eigenvalues of the operator $L_{0,k}$, and the root functions are defined by

$$v_{2s}(x, B_p) \equiv v_{2s}(x), \ v_{2q-1}(x, B_p) \equiv v_{2q-1}(x) + c_q(B_p)y_{2,p}(x, \rho_q),$$

(36)

where $c_q(B_p) \in \mathbb{R}$, $s = 0, 1, \ldots, q = 1, 2, \ldots$.

The operator which maps the system $V(L_{0,k})$ into the system $V(B_p)$ of the root functions of the operator $B_p$ is denoted by $R(B_p) \equiv E + S(B_p)$, where $S(B_p) : H_{0,0} \to H_{0,1}$, $S(B_p) : H_{0,1} \to 0$.

We denote by $G_{0,p}(L_{0,k}) \equiv \{ R(B_p) \}$ such that the root functions of the operator $B_p$ are defined by the equalities (36), and $G_{0,p,c}(L_{0,k}) \equiv G_{0,p}(L_{0,k}) \cap [H_0]$.

**Remark 4.1.** Using that $S(B_p) : H_{0,0} \to H_{0,1}$, $S(B_p) : H_{0,1} \to 0$ we obtain $S^2(B_p) = 0$, $R^{-1}(B_p) \equiv E - S(B_p)$.

Consequently, the operator $R(B_p)$ has a dense domain in the space $H_0$ and the system of root functions is complete and minimal in $H_0$.

Similarly, using the root functions of an adjoint operator $L_{1,k}^*$, we define the functions

$$w_0 (x, B_p) \equiv v_0 (x) + c_0 (1 - 2x),
$$

$$w_{2q} (x, B_p) \equiv v_{2q}(x) + c_q(B_p) y_{2,2n-1}(x, \rho_q), \ w_{2q-1} (x, B_p) \equiv v_{2q-1}(x), \ q = 1, 2, \ldots,$$

and the set of operators $G_{1,p}(L_{0,k}) \equiv \{ R(B_p) = E + S(B_p^*), R(B_p) \in G_{0,p}(L_{0,k}) \}$.

**Theorem 4.** For any $b \in \mathbb{R}$, $p \in \{1, 2, \ldots, n\}$, the operator $B_p$ has the point spectrum $\sigma(L_{0,k})$ and the system of root functions $V(L_{1,k})$ forms the Riesz basis in $H_0$ if and only if the sequence $c_q(B_p)$ is bounded, i.e., $|c_q(B_p)| \leq K_5 < \infty, \ q = 1, 2, \ldots$.

**Proof.** The necessity. Let the system $V(B_p)$ be the Riesz basis in $H_0$, i.e., $R(B_p) \in [H_0]$, then $S(B_p) = E - K_p(B) \in [H_0]$. From the definition of the operator $B_p$ we have

$$S(B_p) v_{2q-1}(x) = c_q(B_p) y_{2,p}(x, \rho_q), \ q = 1, 2, \ldots.$$

Therefore, taking the estimate (22) into account, we obtain

$$|c_q(B_p)| \leq \|S(B_p)[H_0]| |y_{2,p}(x, \rho_q); H_0|^{-1} \leq K_6 < \infty,$$

$$K_6 = K_5^{-1} \|S(B_p)[H_0]|, \ q = 1, 2, \ldots.$$

The sufficiency. The completeness of the system $V(B_p)$ in the space $H_0$ follows from Remark 4.1. Let $\varphi \in H_0$, $\varphi = \varphi_0 + \varphi_1$, $\varphi_s \in H_{0,s}$, $s = 0, 1$. Then we have

$$\varphi = \varphi_0 v_0 (x) + \sum_{q=1}^{\infty} (\varphi_{2q} v_{2q}(x) + \varphi_{2q-1} v_{2q-1}(x)) \in H_0,$$

$$\|\varphi; H_0\|^2 = \|\varphi_0\|^2 + \sum_{q=1}^{\infty} (|\varphi_{2q}|^2 + |\varphi_{2q-1}|^2) < \infty,$$

$$R(B_p) \varphi = \varphi_0 v_0 (x, B_p) + \sum_{q=1}^{\infty} (\varphi_{2q} v_{2q}(x, B_p) + \varphi_{2q-1} v_{2q-1}(x, B_p)) \in H_0,$$

$$R(B_p) \varphi = \varphi_0 v_0 (x, B_p) + \sum_{q=1}^{\infty} (\varphi_{2q} v_{2q}(x) + \varphi_{2q-1} c_q(B_p) (v_{2q-1}(x, L_{1,k}) - v_{2q-1}(x)))$$,

$$\|R(B_p) \varphi; H_0\|^2 \leq K_7 \|\varphi; H_0\|^2, \ K_7 = 3 \left(1 + K_6^2 + K_5^2 \|R(L_{1,k}); [H_0]|^2\right).$$
Therefore, \( \|R(B_p); [H_0]\|^2 \leq K_7 < \infty \).

Consider equalities \( R(B_p) = E + S(B_p), \ R^{-1}(B_p) = E - S(B_p) \). We have

\[
R^{-1}(B_p) = 2E - R(B_p).
\]

Therefore, \( \|R^{-1}(B_p); [H_0]\|^2 \leq K_8 < \infty, \ K_8 = 8 + 2K_7 \). Taking into account the Bari Theorem (see [9]), we obtain that the system \( V \) is a set of operators \( R \equiv \bigcap_{j \in \mathbb{N}} R_{j} \equiv (E + S_1) (E + S_2) = E + S_1 + S_2, \ R_1, R_2 \in Q_0(I) \).

In particular, \( (E + S)(E - S) = E - S^2 = E, \ R = E + S \in Q_0(I) \).

Therefore, for each operator \( R = E + S \in Q_0(I) \) there exists a unique inverse operator \( R^{-1} = E - S \).

According to the definition of the operator \( B_p \) and the set \( G_{0,p}(L_{0,k}) \) we have the inclusions

\[
G_{0,p}(L_{0,k}) \subset Q_0(I), \ G_{c,0,p}(L_{0,k}) \subset Q_{0,c}(I), \ p \in \{1, 2, \ldots, n\}.
\]

Thus, the set \( Q_0(I) \) is an Abelian group which contains the Abelian subgroups \( Q_{c,0}(I), \ G_{0,p}(L_{0,k}), \ G_{c,0,p}(L_{0,k}), \ p \in \{1, 2, \ldots, n\} \). Therefore, for all operators \( R_j = E + S_j \in Q_0(I), \ j = 1, 2, \ldots, d, \ d \in \mathbb{N}, \) the following equality

\[
\prod_{j=1}^{d} R_j = \prod_{j=1}^{d} (E + S_j) = E + \sum_{j=1}^{d} S_j, \ d \in \mathbb{N},
\]

holds.

5 Nonlocal boundary value problems for a differential-operator equation

5.1. For the differential-operator equation (5) and arbitrary fixed indices \( b_{p,r,s} \in \mathbb{R}, p \in \{1, 2, \ldots, n\}, r = 0, 1, \ldots, k_r, s = 0, 1, j = 1, 2, \ldots, n \), we consider the boundary problem generated by nonlocal conditions

\[
\ell_{2,j}w \equiv D_x^{2j-1}w(0) - D_x^{2j-1}w(1) = 0, \ j \neq p,
\]

\[
\ell_{2,p}w \equiv D_x^{2p-1}w(0) - D_x^{2p-1}w(1) + l_p^1w = 0,
\]

\[
\ell_{2,n+i}w \equiv D_x^{2j-2}w(0) - D_x^{2j-2}w(1) = 0, \ j = 1, 2, \ldots, n,
\]

where

\[
\ell_p^1w \equiv \sum_{r=0}^{m_p} (b_{p,r,0}D_x^r w(0) + b_{p,r,1}D_x^r w(1)).
\]

Assumption \( P_1: b_{p,r,0} = (-1)^{r+1} b_{p,r,1}, \ r = 0, 1, \ldots, m_p, \ j = 1, 2, \ldots, n. \)

Assumption \( P_2: m_p \leq 2p - 1, \ p = 1, 2, \ldots, n. \)

Remark 5.1. Assumption \( P_1 \) implies that \( l_p^1 \in W_r^1, \ p = 1, 2, \ldots, n. \)
Let \( L_2 \equiv L_{2,p} \) be the operator of the problem \((5), (38)–(41)\) and
\[
L_2u \equiv (-1)^n D_x^{2n} u(x) + A^{2n} u(x), \quad u \in D(L_2), \quad D(L_2) \equiv \{ u \in H_2 : l_{2,j}u = 0, \ j = 1, 2, \ldots, 2n \}.
\]
The solution of the spectral problem \((5), (38)–(41)\) is defined as the product \( w(x) = y(x)v_k, \) \( v_k \in V(A), \ k = 1, 2, \ldots. \)

To determine the unknown function \( y \in W^{2n}(0, 1) \) we consider the spectral problem
\[
(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \quad (42)
\]
\[
\ell_{2,j}y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j \neq p, \quad (43)
\]
\[
\ell_{2,p}y \equiv y^{(2p-1)}(0) - y^{(2p-1)}(1) + l_1^p y = 0, \quad (44)
\]
\[
\ell_{2,n+j}y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \ldots, n. \quad (45)
\]

Let \( L_{2,k} \equiv L_{2,k,p} \) be the operator of the problem \((42)–(45)\) and
\[
L_{2,k}y \equiv (-1)^n y^{(2n)}(x) + z_k^{2n} y(x), \quad y \in D(L_{2,k}), \quad D(L_{2,k}) \equiv \{ y \in W^{2n}(0, 1) : l_{2,j}y = 0, \ j = 1, 2, \ldots, 2n \}.
\]

**Theorem 5.** Suppose that the Assumption \( P_1 \) holds. Then for arbitrary numbers \( b_{p,r,s} \in \mathbb{R}, \ s = 0, 1, \ r = 0, 1, \ldots, m_p, \ p \in \{ 1, 2, \ldots, n \}, \) the following statements hold

1) the eigenvalues of the operators \( L_{0,k} \) and \( L_{2,k} \) coincide;

2) the system \( V(L_{2,k}) \) is complete and minimal in the space \( H_0; \)

3) if in addition the Assumption \( P_2 \) holds, then the system \( V(L_{2,k}) \) is the Riesz basis of the space \( H_0. \)

**Proof.** The proof of part 1 of the theorem can be made in the same way as in Theorem 2.

Let us define the elements of the system \( V(L_{2,k}) \). A direct substitution gives that the function \( v_{2q}(x), \ q = 0, 1, \ldots, \) satisfies the conditions \((43)–(45)\). Therefore, the root function of the operator \( L_{2,k} \) with respect to the eigenvalue \( \lambda_{q,k} \) is defined by
\[
\begin{align*}
v_{2q}(x, L_{2,k}) &= v_{2q}(x, L_{0,k}), \quad q = 0, 1, \ldots, \\
v_{2q-1}(x, L_{2,k}) &= v_{2q-1}(x) + c_{q,p}y_{2q}(x, \rho_q), \\
c_{q,p} &= -l_1^p (v_{2q-1}(x))(l_{2,p}y_{2p}(x, \rho_q))^{-1}, \quad q = 1, 2, \ldots,
\end{align*}
\]
Consequently \( L_{2,k} \in Q_0(I). \) If the Assumption \( P_2 \) holds, then from the inequality \( |l_1^p v_{2q-1}| \leq K_9 (\rho_q)^{2p-2} \) we obtain the inequality
\[
|l_1^p (v_{2q-1}(x)(l_{2,n+p} y_{2p}(x, \rho_{0,q}))^{-1}| \leq K_{10} < \infty, \quad (46)
\]
Taking Theorem 4 into account, we obtain the third statement of the theorem. \( \square \)

Therefore, the operator \( L_2 \) has the following system of root functions in the space \( H_1 \)
\[
V(L_2) \equiv \left\{ v_{q,k}(x, L_2) : q = 0, 1, \ldots, k = 1, 2, \ldots \right\}
\]

**Remark 5.2.** The operator \( L_2 \) has a system of root functions in the means of equalities
\[
\begin{align*}
L_2 v_{2q-1,k}(x, L_2) &= \lambda_{q,k} v_{2q-1,k}(x, L_2) + \xi_{q,p} v_{2q,k}(x, L_2), \quad (47) \\
\xi_{q,p} &= -4\sqrt{2} (\rho_q)^{2n-1} c_{q,p}, \quad q, k = 1, 2, \ldots, \\
L_2 v_{2q,k}(x, L_2) &= \lambda_{q,k} v_{2q,k}(x, L_2), \quad q = 0, 1, \ldots, k = 1, 2, \ldots, \quad (49)
\end{align*}
\]
Theorem 6. Suppose that the Assumption $P_1$ holds. Then, for arbitrary numbers $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, $p \in \{1, 2, \ldots, n\}$, the following statements hold

1) the eigenvalues of the operators $L_0$ and $L_2$ coincide;
2) the system $V(L_2)$ is complete and minimal in the space $H_1$;
3) if in addition the Assumption $P_2$ holds, then the system $V(L_2)$ forms the Riesz basis of the space $H_1$.

Proof. Taking Theorem 5 into account, it is possible to determine the elements of a system $W(L_{2,h})$ which is biorthogonal to the system $V(L_{2,k})$ in the space $H_0$.

Therefore, there exists $W(L_2) \equiv \{w_{q,k}(x, L_2)w_k : q = 0, 1, \ldots, k = 1, 2, \ldots \}$ which is the biorthogonal system of functions to the system $V(L_2)$ in the space $H_1$.

Thus the second statement of the theorem is proved.

Suppose that the Assumption $P_2$ holds. Taking the inequalities (46) into account, we obtain the estimate

$$\|R(L_2); [H_1]\| \leq K_{11} < \infty.$$  

From the Bari Theorem (see [9]) we obtain the statement: the system $V_1$ forms the Riesz basis in $H_1$. □

5.2. Consider the spectral problem

$$(-1)^n D_x^{2n}w(x) + A^{2n}w(x) = \lambda w(x), \quad \ell_j w \equiv D_x^{2j-1}w(0) - D_x^{2j-1}w(1) + l_j^1 w = 0, \quad \ell_{n+j} w \equiv D_x^{2j-2}w(0) - D_x^{2j-2}w(1) = 0,$$

where

$$\ell_j^1 w \equiv \sum_{r=0}^{w_j}(b_{j,r}D_x^r w(0) + b_{j,r,1} D_x^r w(1)), \quad j = 1, 2, \ldots, n. \quad (53)$$

Let $L_3$ be the operator of the problem (50)–(53) and

$$L_3u \equiv (-1)^n D_x^{2n}u + A^{2n}u, \quad u \in D(L_3), \quad D(L_3) \equiv \{u \in H_2 : l_j u = 0, j = 1, 2, \ldots, 2n\}.$$  

We find the solution of the spectral problem (50)–(53) as the product $w(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \ldots$.

To determine the unknown function $y \in W^{2n}(0,1)$ we obtain the spectral problem

$$(-1)^n y^{(2n)} + z_k^{2n}y = \lambda y, \quad \lambda \in \mathbb{C}, \quad \ell_j y \equiv y^{(2j-1)}(0) + y^{(2j-1)}(1) + l_j^1 y = 0, \quad \ell_{n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \ldots, n. \quad (56)$$

Let $L_{3,k}$ be the operator of the problem (54)–(56);

$$L_{3,k}y(x) \equiv (-1)^n y^{(2n)}(x) + z_k^{2n};$$  

$$y \in D(L_{3,k}); \quad D(L_{3,k}) \equiv \{y \in W^{2n}(0,1) : l_j y = 0, j = 1, 2, \ldots, 2n\};$$

let $V(L_{3,k})$ be the system of root functions of the operator $L_{3,k}$.

We can prove that

$$v_{2q} (x) \in D(L_{3,k}), \quad L_{3,k}v_{2q} (x) = \lambda_{q,k}v_{2q} (x), \quad q = 0, 1, \ldots.$$
Therefore,

\[ v_{2q} (x, L_{3,k}) \equiv v_{2q} (x), \quad q = 0, 1, \ldots \]

The root functions of the operator \(L_{3,k}\) are determined by the equalities

\[ v_{2q-1} (x, L_{3,k}) \equiv v_{2q-1} (x) + \sum_{p=1}^{n} c_{1,q,p} y_{2,p} (x, \rho_q), \quad q = 1, 2, \ldots \]

Substituting the expression (58) into the boundary conditions (55), (56), we obtain

\[ c_{1,q,p} = -\sqrt{2} \sum_{r=0}^{m_{pq}} (-1)^{r-2p+1} b_{p,r,0} (\rho_q)^{2+r-2p}, \quad p = 1, 2, \ldots, n, \quad q = 1, 2, \ldots \]

Thus, the operator \(L_{3,k}\) has the system of root functions (57)–(59) in the means of equalities

\[ L_{3,k} v_{2q-1} (x, L_{3,k}) = \lambda_{q,k} v_{2q-1} (x, L_{3,k}) + \sum_{p=1}^{n} c_{p,q} v_{2q} (x, L_{3,k}), \]

\[ \xi_q = 2 \sqrt{2} n (\rho_q)^{2n-1} \sum_{p=1}^{n} c_{p,q}, \quad q = 1, 2, \ldots, \]

\[ L_{3,k} v_{2q} (x, L_{3,k}) = \lambda_{q,k} v_{2q} (x, L_{3,k}), \quad q = 0, 1, \ldots \]

Let \(R (L_{3,k})\) be the operator which acts as \(V (L_{0,k}) \to V (L_{3,k})\). From the formulas (37), (58), we obtain the relation

\[ R(L_{3,k}) = \prod_{p=1}^{n} R(L_{2p,k,p}) = E + \sum_{p=1}^{n} S(L_{2p,k,p}). \]

Therefore, we have the inclusion \(R(L_{3,k}) \in G_0 (L_{0,k}) \subset Q_0 (I)\). Thus, we obtain the following statement.

**Lemma 5.1.** Suppose that the Assumption \(P_1\) holds. Then, for the any fixed \(b_{p,r,0} \in \mathbb{R}, r = 0, 1, \ldots, m_{pq}, p = 1, 2, \ldots, n\), the system \(V(L_{3,k})\) is complete and minimal in the space \(H_0\).

Consider the system \(V_2\) of functions

\[ v_{0,2} (x) \equiv v_0 (x), \quad v_{2q,2} (x) \equiv v_{2q} (x), \quad q = 1, 2, \ldots, \]

\[ v_{2q-1,2} (x) \equiv (1 + \tau_2) (1 - 2x) v_{2q-1} (x), \]

\[ \tau_2 \equiv W^n(W^{n-1})^{-1} c_b, \quad c_b \equiv \sum_{p=1}^{n} b_{p,2p-1,0}. \]

Let \(R_2 = E + S_2\) be the operator which acts as \(V (L_{0,k}) \to V_2\).

Using that \(S_2 : H_{0,1} \to 0, \quad S_2 : H_{0,0} \to H_0\), we obtain that \(R_2 \in Q_0(I)\).

**Lemma 5.2.** Suppose that the Assumptions \(P_1, P_2\) hold. Then the system \(V_2\) forms the Riesz basis in the space \(H_0\).

The proof is carried out analogously in Lemma 3.1.

**Remark 5.3.** Suppose that the Assumptions \(P_1, P_2\) hold. Then the following relations hold

\[ v_{2q-1} (x, L_{3,k}) = v_{2q-1,2} (x) + \sum_{j=2}^{n} c_{1,j,q} y_j (x, \rho_q) + (\rho_q)^{-1} c_q^2 (1 - 2x) v_{2q-1} (x), \]

where

\[ |c_{1,j,q}| \leq K_{12}, \quad |c_q^2| \leq K_{12} < \infty, \quad q = 0, 1, \ldots. \]

Therefore, the systems \(V(L_{3,k})\) and \(V_2\) are squarely close in the space \(H_0\).
Lemma 5.3. Suppose that the Assumptions $P_1$, $P_2$ hold. Then, for any fixed $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, $p = 1, 2, \ldots, n$, the system $V(L_{3,k})$ forms the Riesz basis in the space $H_0$.

The statement follows from Lemma 5.1, Lemma 5.2, Remark 5.4 and the Bari Theorem (see [9]).

Theorem 7. Suppose that the Assumption $P_1$ holds. Then, for any $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, $p = 1, 2, \ldots, n$, the following assertions are true

1) the eigenvalues of the operators $L_{0,k}$ and $L_{3,k}$ coincide;

2) the system $V(L_{3,k})$ is complete and minimal in the space $H_0$;

3) if in addition the Assumption $P_2$ holds, then the system $V(L_{3,k})$ forms the Riesz basis of the space $H_0$.

Proof. The proof of part 1 of the theorem follows from Theorem 3, the second statement follows from Lemma 5.1 and the third statement follows from Lemma 5.3.

Let

$$V(L_3) \equiv \left\{ v_{q,k}(x, L_3) \equiv v_q(x, L_{3,k})v_k \in W_1 : q = 0, \infty, \ k = 1, \infty \right\}$$

forms the system of the root functions of the operator $L_3$. Let

$$W(L_{3,k}) \equiv \left\{ w_{q,k}(x, L_3) \in H_1 : q = 0, \infty \right\}$$

be the biorthogonal system of functions to the system $V(L_{3,k})$ in the space $H_0$. Let

$$W(L_3) \equiv \left\{ w_{q,k}(x, L_3) \equiv w_q(x, L_{3,k})w_k \in H_1 : q = 0, \infty, \ k = 1, \infty \right\},$$

be the biorthogonal system of functions to the system $V(L_3)$ in the space $W_1$ and $R(L_3)$ be the operator which acts as $V(L_0) \to V(L_3)$.

Theorem 8. Suppose that the Assumption $P_1$ holds. Then, for all numbers $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \ldots, m_p$, $p = 1, 2, \ldots, n$, the following assertions are true

1) the eigenvalues of the operators $L_0$ and $L_3$ coincide;

2) the system $V(L_3)$ is complete and minimal in the space $H_1$;

3) if in addition the Assumption $P_2$ holds, then the system $V(L_3)$ forms the Riesz basis of the space $H_1$.

Proof. The proof of part 1 and 2 of the theorem follows from Theorem 7. Taking the relations (60), (61) into account we obtain the equality

$$R(L_3) \equiv \prod_{j=1}^{n} R(L_{2,j}) \equiv \prod_{j=1}^{n} (E + S(L_{2,j})) = E + \sum_{j=1}^{n} S(L_{2,j}). \tag{62}$$

Let Assumptions $P_1$ and $P_2$ hold. Then from the equality (62) and the assertion 3 of Theorem 7 we obtain

$$R(L_3) \in [H_1], \ |R(L_3)|^{-1} = E - S(L_3) \in [H_1].$$

Therefore, the system $V(L_3)$ forms the Riesz basis of the space $H_1$.

Remark 5.4. The operator $L_3$ has the system of root functions in the means of the equalities

$$L_3v_{q,2k-1,k}(x, L_3) = \lambda_{q,k}v_{q,2k-1,k}(x, L_3) + \xi_{q,k}v_{q,2k}(x, L_3),$$

$$\xi_{q,k} = 2\sqrt{2}r_2n (\rho_q)^{2n-1} \sum_{p=1}^{n} c_{p,q}, \quad q = 1, 2, \ldots,$$

$$L_3v_{q,k}(x, L_3) = \lambda_{q,k}v_{q,2k-1,k}(x, L_3), \quad q = 0, 1, \ldots.$$
We consider the system of functions
\[ V_3 \equiv \left\{ v_{q,k,3}(x) \in H_1 : v_{q,k,3}(x) = v_{q,1}(x)v_k, \quad q = 0, \infty, \quad k = 1, \infty \right\}. \]

**Remark 5.5.** The systems \( V(L_3) \) and \( V_3 \) are squarely close in the space \( H_1 \).

### 6 The Spectral Boundary Value Problem for a Differential-Operator Equation with Involution

Consider the spectral problem
\[ (-1)^n D_x^{2n} u + A_2^n u + \sum_{s=1}^n a_s \left( D_x^{2s-1} u(x) + D_x^{2s-1} u(1 - x) \right) = \lambda u, \quad (63) \]
\[ \ell_j u \equiv D_x^{2j-1} u(0) - D_x^{2j-1} u(1) + \ell_j^1 u = 0, \quad (64) \]
\[ \ell_{n+j} u \equiv D_x^{2j-2} u(0) - D_x^{2j-2} u(1) = 0, \quad (65) \]

with
\[ \ell_j^1 u \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r u(0) + b_{j,r,1} D_x^r u(1)), \quad j = 1, 2, \ldots, n. \quad (66) \]

Let \( L \) be the operator of the problem (63)–(66) and
\[ Lu \equiv (-1)^n D_x^{2n} u + A_2^n u + \sum_{s=1}^n a_s \left( D_x^{2s-1} u(x) + D_x^{2s-1} u(1 - x) \right); \]
\[ u \in D(L), \quad D(L) \equiv \{ u \in H_2 : \ell_j u = 0, \quad j = 1, 2, \ldots, 2n \}. \]

We can prove that
\[ L v_{2q-1,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3) + \xi_{q,k}^1 v_{2q,k}(x, L_3), \quad q = 1, 2, \ldots, \]
\[ \xi_{q,k}^1 = c_{q,k}^0 + 2\sqrt{2} r_2 \sum_{j=1}^{m_j} (-1)^{j-1} a_j (\rho_q)^{2j-1} (-4j + 2), \quad q = 1, 2, \ldots, \]
\[ L v_{2q,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3), \quad q = 0, 1, \ldots. \]

Consequently, \( V(L) \equiv V(L_3) \) and the following theorem is true.

**Theorem 9.** Suppose that the Assumption \( P_1 \) holds. Then for the any numbers \( b_{p,r,q}, a_j \in \mathbb{R}, \quad r = 0, 1, \ldots, m_p, \quad j = 1, 2, \ldots, n \) we have 1) the eigenvalues of the operators \( L_0 \) and \( L \) coincide;

2) the system \( V(L) \) of the root functions of the operator \( L \) is complete and minimal in the space \( H_1 \);

3) if in addition the Assumption \( P_2 \) holds, then the system \( V(L) \) forms the Riesz basis in the space \( H_1 \).

Let
\[ f = \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} f_{q,k} v_{q,k}(x, L), \quad f_{q,k} = (f, w_{q,k}(x, L); H_1). \]

**Remark 6.1.** From the definition of the Riesz basis of Hilbert space and the third statement of Theorem 9 for any \( f \in H_1 \) we obtain the relation
\[ K_{13} \| f; H_1 \|^2 \leq \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} |f_{q,k}|^2 \leq K_{14} \| f; H_1 \|^2. \quad (67) \]
Theorem 10. Suppose that the Assumption $P_1$ holds. Then for arbitrary numbers $b_{p,r,\theta}, a_j \in \mathbb{R}$, $r = 0, 1, \ldots, m_p, j, p \in \{1, 2, \ldots, n\}$, and function $f \in H_1$ there exists a unique solution of the problem (68)–(71).

Proof. The solution of the problem (68)–(71) can be determined by the relation

$$w = \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} w_{q,k} v_{q,k}(x, L).$$

Substituting the relations (67), (72) into the equation (68) we obtain

$$(w_{2q-1,k} = \lambda_{q,k}^{-1} f_{2q-1,k}, \quad w_{2q,k} = \lambda_{q,k}^{-1} f_{2q,k} - \lambda_{q,k}^{-2} x_{1} f_{2q-1,k}, \quad q, k = 1, 2, \ldots.)$$

Therefore,

$$|w_{2q-1,k}|^2 \leq K_{15} |f_{2q-1,k}|^2,$$

$$|w_{2q,k}|^2 \leq K_{16} (|f_{2q-1,k}|^2 + |f_{2q,k}|^2), \quad q, k = 1, 2, \ldots.$$
Consider the function $h_3 = \sum_{s=1}^{n} a_s(D_x^{2s-1}w(x) - D_x^{2s-1}w(1-x))$

\[ h_3 = 2 \sum_{k=1}^{\infty} \sum_{q=1}^{n} \sum_{j=1}^{n} a_j (-1)^{j-1} f_q^{2j-1}(\lambda_q^{1}, f_{2q,k} - \lambda_q^{1} f_{2q,k} f_{2q-1,k}) u_{2q-1,k}(x). \]  

(77)

Taking the assumption $f \in H_1$ and the equalities (77) into account we obtain that

\[ \|h_3; H_1\| \leq K_20 \|f; H_1\|. \]

From the definition of the space $H_2$, inequalities (75)–(77) and Cauchy’s inequality we get

\[ \|w; H_2\| \leq K_21 \|f; H_1\| < \infty, \quad K_21 = 3(\max(K_{17}^2, K_{18}^2, K_{19}^2))^{\frac{1}{2}}. \]

Thus $w \in H_2$. □

REFERENCES


Вивчається нелокальна крайова задача для диференціально-опера-торного рівняння парного порядку, який містить оператор інволюції. Досліджується задача з періодичними краєвими умовами для диференціального рівняння, коефіцієнти якого є несамоспряменими опере-торами. Встановлено, що оператор задачі має два інваріантні підпростори, породжені опера-тором інволюції та дві підсистеми системи власних функцій, які є базисами Рісса в кожно-му з підпросторів. Для диференціально-опера-торного рівняння парного порядку вивчається задача з несамоспряменими крайовими умовами, які є збуреннями періодичних умов. Ви-вчені випадки, коли збурені умови є регулярними, але не сильно регулярними за Біркгофом та нерегулярними за Біркгофом. Визначено власні значення і елементи системи кореневих функцій $V$ оператора задачі, яка є повною та містить нескінченнє число приєднаних функцій. Отримано достатні умови, при яких система $V$ є базисом Рісса. Визначено умови існування та єдності розв’язків задачі з однорідними крайовими умовами, який побудовано у вигляді ряду за системою $V$.

Ключові слова і фрази: оператор інволюції, диференціально-опера-торне рівняння, власні функції, базис Рісса.