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WIMAN'S INEQUALITY FOR ANALYTIC FUNCTIONS IN $\mathbb{D}\times\mathbb{C}$ WITH RAPIDLY OSCILLATING COEFFICIENTS

Let $\mathcal{A}^2$ be a class of analytic functions f represented by power series of the from

$$f(z)=f(z_1,z_2)=\sum_{n+m=0}^{+\infty}a_{nm}z_1^nz_2^m$$

with the domain of convergence $\mathbb{T}=\{z\in\mathbb{C}^2\colon |z_1|<1, |z_2|<+\infty\}$ such that $\frac{\partial}{\partial z_2}f(z_1,z_2)\not\equiv 0$ in $\mathbb{T}$ and there exists $r_0=(r_1^0,r_2^0)\in[0,1)\times[0,+\infty)$ such that for all $r\in(r_1^0,1)\times(r_2^0,+\infty)$ we have $r_1\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)+\ln r_1>1$, where $\mathfrak{M}_f(r)=\sum_{n+m=0}^{+\infty}|a_{nm}|r_1^nr_2^m$. Let $\mathcal{K}(f,\theta)=\{f(z,t)=\sum_{n+m=0}^{+\infty}a_{nm}e^{2\pi it(\theta_n+\theta_m)}\colon t\in\mathbb{R}\}$ be class of analytic functions, where (θ_{nm}) is a sequence of positive integer such that its arrangement (θ_k^*) by increasing satisfies the condition

$$\theta_{k+1}^*/\theta_k^*\geq q>1, k>0.$$

For analytic functions from the class $\mathcal{K}(f,\theta)$ Wiman's inequality is improved.

Key words and phrases: Wiman's type inequality, analytic functions of several variables.

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1 INTRODUCTION

In this paper we consider some analog of the classical inequality of A.Wiman (in this regard, see [1–7]) for the class $\mathcal{A}_0^2$ of analytic functions f represented by power series of the form

$$f(z)=f(z_1,z_2)=\sum_{n+m=0}^{+\infty}a_{nm}z_1^nz_2^m,\quad z=(z_1,z_2)\in\mathbb{C}^2, \tag{1}$$

with the domain of convergence $\mathbb{T}=\mathbb{D}\times\mathbb{C}=\{z\in\mathbb{C}^2\colon |z_1|<1, |z_2|<+\infty\}$. Let $\mathcal{A}_1^2$ be the class of functions $f\in\mathcal{A}_0^2$ such that

$$\frac{\partial}{\partial z_2}f(z_1,z_2)\not\equiv 0 \tag{2}$$

in $\mathbb{T}$, $\mathcal{A}_2^2$ be the class of functions $f\in\mathcal{A}_0^2$ there exists $r_0=(r_1^0,r_2^0)\in T{:=}[0,1)\times[0,+\infty)$ such that for all $r\in(r_1^0,1)\times(r_2^0,+\infty)$ we have

$$r_1\frac{\partial}{\partial r_1}\ln\mathfrak{M}_f(r)+\ln r_1>1,\quad \mathfrak{M}_f(r):=\sum_{n+m=0}^{+\infty}|a_{nm}|r_1^nr_2^m \tag{3}$$

and $\mathcal{A}^2=\mathcal{A}_1^2\cap\mathcal{A}_2^2$.

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Proposition 1. 1. *If $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_1^2$ then for every $\delta > 0$ there exists a set $E = E_f(\delta) := E_1 \times [1, r_2^0]$, $\int_{E_1 \cap [r_1^0, 1)} d \ln r_1 < +\infty$, such that for all $r \in T \setminus E$ the inequlity*

$$\mathfrak{M}_f(r) \le \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}, \quad \mu_f(r) := \max\{|a_{nm}| r_1^n r_2^m \colon n, m \ge 0\}, \tag{4}$$

holds.
2. *If $f \in \mathcal{A}_0^2 \setminus \left(\mathcal{A}_1^2 \cup \mathcal{A}_2^2\right)$ then for all $r \in T$, $\mathfrak{M}_f(r) \le C < +\infty$.*

Proof. 1. Remark that every function $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_1^2$ is the function of the form $f(z) \equiv f_1(z_1)$ for all $z_2 \in \mathbb{C}$, i.e. is identical function of z_2 and analytic function of $z_1 \in \mathbb{D}$. Therefore the result of T.Kővari (see [8,9]) implies that inequality (4) holds for all $r \in T \setminus E$.

2. Further, $r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) = r_1 \frac{d}{dr_1} \ln \mathfrak{M}_{f_1}(r_1)$, where $\mathfrak{M}_{f_1}(r_1) = \sum_{n=0}^{+\infty} |a_{n0}| r_1^n$. Well known that the function $r_1 \frac{d}{dr_1} \ln \mathfrak{M}_{f_1}(r_1)$ is nondecreasing on $(0,1)$. Therefore, with the denial of inequality (3) we obtain that the inequality $r_1 \frac{d}{dr_1} \ln \mathfrak{M}_{f_1}(r_1) + \ln r_1 \le 1$ for all $r_1 \in (0,1)$ holds. Hence, $\mathfrak{M}_{f_1}(r_1) = O(1)$ $(r_1 \to 1-0)$. □

Remark 1. For the function $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_2^2$ similarly as in proof of 2) we obtain

$$\ln \mathfrak{M}_f(r_1, r_2) - \ln \mathfrak{M}_f(r_1^0, r_2) \le \frac{1}{2}(2 - \ln r_1 r_1^0) \ln \frac{r_1}{r_1^0} < (1 - \ln r_1^0) \ln \frac{1}{r_1^0},$$

for all $(r_1, r_2) \in (r_1^0, 1) \times (0, +\infty)$.

For $r = (r_1, r_2) \in T$ and a function $f \in \mathcal{A}^2$ we denote

$$\triangle_r = \{(t_1, t_2) \in T \colon t_1 > r_1,\ t_2 > r_2\},$$
$$M_f(r) = \max\{|f(z)| \colon |z_1| \le r_1, |z_2| \le r_2\},$$
$$\mu_f(r) = \max\{|a_{nm}| r_1^n r_2^m \colon (n,m) \in \mathbb{Z}_+^2\}.$$

We call $E \subset T$ a set of *asymptotically finite logarithmic measure on T* $(E \in \mathrm{Y})$ if there exists $R \in T$ such that

$$\nu_{\ln}(E \cap \triangle_R) := \iint\limits_{E \cap \triangle_R} \frac{dr_1 dr_2}{(1-r_1) r_2} < +\infty,$$

i.e. the set $E \cap \triangle_R$ is a set of *finite logarithmic measure on T*.

We note that for a function $f \in \mathcal{A}^2$ of the form $f(z) = f_1(z_1) \cdot f_2(z_2)$, where f_1 is analytical in $\mathbb{D}$ and f_2 is entire function of one variable, the inequality

$$M_f(r) \le \mathfrak{M}_f(r) \le \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1/2+\delta} r_2 \tag{5}$$

for every $r \in \Delta_{r^0} \setminus E$, $E = E_1 \times E_2 \subset T$,

$$\int_{E_1 \cap (0,1)} \frac{dr_1}{1-r_1} < +\infty, \quad \int_{E_2 \cap (1,+\infty)} \frac{dr_2}{r_2} < +\infty,$$

follows from classical Wiman's inequality [6]

$$\mathfrak{M}_{f_2}(r_2) \le \mu_{f_2}(r_2) (\ln \mu_{f_2}(r_2))^{1/2} \ln^{1/2} r_2 \quad \left(r_2 \in (r_2^0, +\infty) \setminus E_2, \int_{E_2 \cap (1,+\infty)} \frac{(\ln r)^{1/2} dr}{r} < +\infty \right)$$

for entire function f_2 and Kővari inequality [8]

$$\mathfrak{M}_{f_1}(r_1) \leq \frac{\mu_{f_1}(r_1)}{(1-r_1)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_{f_1}(r_1)}{1-r_1} \quad \left(r_1 \in (r_1^0, 1) \setminus E_1, \int_{E_1 \cap (0,1)} \frac{dr}{1-r} < +\infty\right)$$

for analytic in $\mathbb{D}$ function f_1, where $\mathfrak{M}_g(t) = \sum_{n=0}^{+\infty} |g_n| t^n$, $\mu_g(t) = \max\{|g_n| t^n \colon n \geq 0\}$ and function $g(\tau) = \sum_{n=0}^{+\infty} g_n \tau^n$ and $t > 0$. Moreover, $\nu_{\ln}(E \cap \triangle_R) < +\infty$ for every $R = (R_1, R_2) \in T, R_1 > 0, R_2 > 0$.

Inequality (5) for the class $\mathcal{A}^2$ is proved in [10].

Theorem 1 ([10]). *Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \in \Upsilon$ such that for $r \in T \setminus E$ inequality* (5) *holds.*

None of the exponents $1+\delta$ of (5) can not be replaced by a number less than 1 (see [10]).

Remark 2. Remark, that inequality (5) follows from Proposition 1 also in the cases $f \in \mathcal{A}_0^2 \setminus \left(\mathcal{A}_1^2 \cup \mathcal{A}_2^2\right)$, $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_1^2$, i.e., analog of Wiman's inequality is not considered only in the case $f \in \mathcal{A}_1^2 \setminus \mathcal{A}_2^2$.

Let $\Omega = [0,1]$ and P be the Lebesgue measure on $\mathbb{R}$. We consider the Steinhaus probability space $(\Omega, \mathcal{A}, P)$, where $\mathcal{A}$ is the σ-algebra of Lebesgue measurable subsets of Ω.

Let $Z = (Z_{nm}(t))$ be some sequence of complex valued random variables defined in this space. For $f \in \mathcal{A}^2$ by $\mathcal{K}(f, Z)$ we denote the class of random analytic functions of the form

$$f(z,t) = \sum_{n+m=0}^{+\infty} a_{nm} Z_{nm}(t) z_1^n z_2^m. \tag{6}$$

In the sequel, the notion *"almost surely"* will be used in the sense that the corresponding property holds *almost everywhere* with respect to Lebesgue measure P on Ω. We say that some relation holds *almost surely in the class* $\mathcal{K}(f, Z)$ if it holds for each analytic function $f(z,t)$ of the form (6) almost surely in t.

Let $Z = (Z_{nm}(t))$ be some sequence of random variables defined in this space. $Z_{nm}(t) = X_{nm}(t) + iY_{nm}(t)$ such that both $X = X_{nm}(t)$ and $Y = Y_{nm}(t)$ are real multiplicative system (MS). For $f \in \mathcal{A}^2$ by $\mathcal{K}(f, Z)$ we denote the class of random analytic functions of the form

$$f(z,t) = \sum_{n+m=0}^{+\infty} a_{nm} Z_{nm}(t) z_1^n z_2^m. \tag{7}$$

For such functions in [11] it is proved following statement (Levy's phenomenon).

Theorem ([11]). *If $f \in \mathcal{A}^2$, and $Z = (Z_{nm}(t))$, $Z_{nm}(t) = X_{nm}(t) + iY_{nm}(t)$ such that $X = (X_{nm}(t))$ and $Y = (Y_{nm}(t))$ are real multiplicative systems uniformly bounded by the number* 1, *then for every $\delta > 0$ almost surely in $\mathcal{K}(f, Z)$ there exists a set $E = E(f, t, \delta), E \in \Upsilon$, such that for all $r \in T \setminus E$*

$$M_f(r,t) := \max\{|f(z,t)| : |z_1| \leq r_1, |z_2| \leq r_2\} \leq \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \cdot \ln^{1/4+\delta} r_2. \tag{8}$$

In the case when $\mathcal{R} = (R_n(t))$ is the Rademacher sequence, i.e. $(R_n(t))$ is a sequence of independent uniformly distributed random variables on $[0,1]$ such that $P\{t\colon R_n(t) = \pm 1\} = 1/2$, P. Levy [12] proved that for any entire function f of one complex variable we can replace the exponent $1/2$ by $1/4$ in the classical Wiman's inequality almost surely in the class $\mathcal{K}(f, \mathcal{R})$ *(Levy's fenomenon)*. Later P. Erdős and A. Rényi [13] proved the same result for the class $\mathcal{K}(f, H)$, where $H = (e^{2\pi i \omega_n(t)})$ is the Steinhaus sequence, i.e. $(\omega_n(t))$ is a sequence of independent uniformly distributed random variables on $[0,1]$. This statement is true also for any class $\mathcal{K}(f, X)$, where $X = (X_n(t))$ is multiplicative system (MS) uniformly bounded by the number 1. That is for all $n \in \mathbb{N}$ and $t \in [0,1]$ we have $|X_n(t)| \le 1$ and

$$\text{for all } 1 \le i_1 < i_2 < \cdots < i_k\colon\ \mathbf{M}(X_{i_1} X_{i_2} \cdots X_{i_k}) = 0,$$

where $\mathbf{M}\xi$ is the expectation of a random variable ξ ([14,15]). The same holds for $Z = (Z_n)$, $Z_n = X_n + iY_n$, and $X = (X_n), Y = (Y_n)$ are both MS.

In the spring of 1996 during the report of P. V. Filevych at the Lviv seminar of the theory of analytic functions professors A. A. Goldberg and M. M. Sheremeta posed the following question (see [16]). Does Levy's fenomenon take place for analogues of Wiman's inequality for entire functions of several complex variables?

In the papers [16,17] we have found an affirmative answer to this question about Fenton's inequality [18] for random entire functions of two complex variables, in [19] about a inequality from [21] for random entire functions of several complex variables, in [27] in the case of analytic functions in the polydisc.

In this paper we consider the class $\mathcal{K}(f, \theta)$ of analytic functions

$$f(z,t) = f(z_1, z_2, t) = \sum_{n+m=0}^{+\infty} a_{nm} e^{2\pi i \theta_{nm} t} z_1^n z_2^m. \tag{9}$$

Here (θ_{nm}) is a sequence of positive integer such that its arrangement (θ_k^*) by increasing $\{\theta_{nm} : (n,m) \in \mathbb{Z}_+^2\} = \{\theta_k^* : k \in \mathbb{Z}_+\}, \theta_{k+1}^* > \theta_k^*$, satisfies the condition (θ *is Hadamard sequence*)

$$\theta_{k+1}^* / \theta_k^* \ge q > 1, k > 0. \tag{10}$$

Remark, that in the case $q \ge 2$ analytic functions of the form (9) satisfy the assumptions of previous theorem from [11], because $(\cos \theta_n t)$, $(\sin \theta_n t)$ are MS. But in the case $q > 1$ the sequence of random variables $(\cos \theta_n t)_{n \in \mathbb{Z}_+}$ need not be a MS (see [16]). So the following question arrives naturally: *does Levy's phenomenon hold for the class $\mathcal{K}(f, \theta)$ with $f \in \mathcal{A}^2$ and a Hadamard sequence θ?*

2 MAIN RESULT

Theorem 2. *Let $\delta > 0, f \in \mathcal{K}(f, \theta)$ be an analytic function of the from (9) and a sequence of a positive integer $(\theta_{nm})_{(n,m) \in \mathbb{Z}_+^2}$ satisfies condition (10). Then almost surely for $t \in \mathbb{R}$ there exists $E(\delta, t) \in \Upsilon$ such that for all $r \in T \setminus E$ we have*

$$M_f(r,t) = \max_{|z|=r} |f(z,t)| \le \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1/4+\delta} r_2. \tag{11}$$

Similar inequalities for entire functions of one complex variable one can find in [13,26], for analytic functions in the unit disc in [9], for entire functions of two variables [11,17,19,20,22, 23,27], for analytic functions without exceptional sets [15,24].

3 AUXILIARY LEMMAS

Lemma 1 ([25]). *Let $\theta = (\theta_{nm})$ be a sequence of integers which satisfies (10). Then for any $\beta > 0, l \in \mathbb{N}, l \geq 2$ and $\{c_{n,m} : (nm) \in \mathbb{Z}_+^2\} \subset \mathbb{C}$ there exits $A > 0,\ B > 0$ such that*

$$P\left\{t\colon \max\left\{\left|\sum_{n+m=0}^{l} c_{nm}e^{in\psi_1}e^{im\psi_2}e^{2\pi i\theta_{nm}t}\right| : \psi \in [0,2\pi]^2\right\} \geq A_\beta S_l \ln^{1/2} l\right\} \leq \frac{B}{l^\beta}, \tag{12}$$

where $S_l^2 = \sum_{n+m=0}^{l} |c_{nm}|^2$.

Lemma 2 ([10]). *Let $\delta > 0$ and $h\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ be an increasing function on each variable such that*

$$\int_1^{+\infty}\int_1^{+\infty} \frac{du_1du_2}{h(u_1,u_2)} < +\infty.$$

Then there exists a set $E \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T\backslash E$ we have

$$\frac{\partial}{\partial r_1}\ln \mathfrak{M}_f(r) \leq \frac{1}{1-r_1}h\Big(\ln \mathfrak{M}_f(r), \ln r_2\Big), \ \frac{\partial}{\partial r_2}\ln \mathfrak{M}_f(r) \leq \frac{1}{r_2(1-r_1)^\delta}\Big(\ln \mathfrak{M}_f(r)\Big)^{1+\delta}. \tag{13}$$

Lemma 3. *There exits set $E \in \Upsilon$ such that for all $r \in T \setminus E$ we have*

$$\sum_{n+m=0}^{+\infty}(n+m)|a_{nm}|r_1^n r_2^m \leq \frac{\mu_f(r)}{(1-r_1)^{2+\delta}}\ln^{2+3\delta}\frac{\mu_f(r)}{(1-r_1)}\ln^{3/2+3\delta} r_2.$$

Proof. Let $h(r) = (r_1r_2)^{1+\delta}$. Then by Lemma 2, there exist set $E \in \Upsilon$ such that for all $r \in T \setminus E_1$ we have

$$\begin{aligned}
\frac{\partial}{\partial r_1}\ln \mathfrak{M}_f(r) &= \frac{1}{\mathfrak{M}_f(r)}\sum_{n+m=0}^{+\infty} n|a_{nm}|r_1^{n-1}r_2^m = \frac{1}{r_1\mathfrak{M}_f(r)}\sum_{n+m=0}^{+\infty} n|a_{nm}|r_1^n r_2^m\\
&\leq \frac{1}{1-r_1}\ln^{1+\delta}\mathfrak{M}_f(r)\ln r_2^{1+\delta}\sum_{n+m=0}^{+\infty} n|a_{nm}|r_1^n r_2^m \leq \frac{r_1}{1-r_1}\mathfrak{M}_f(r)\ln^{1+\delta}\mathfrak{M}_f(r)\ln r_2^{1+\delta}\\
&\leq \frac{1}{1-r_1}\mathfrak{M}_f(r)\ln^{1+\delta}\mathfrak{M}_f(r)\ln r_2^{1+\delta}\sum_{n+m=0}^{+\infty} m|a_{nm}|r_1^n r_2^m \leq \frac{r_2}{r_2(1-r_1)^\delta}\mathfrak{M}_f(r)\ln^{1+\delta}\mathfrak{M}_f(r)\\
&\leq \frac{1}{(1-r_1)^\delta}\mathfrak{M}_f(r)\ln^{1+\delta}\mathfrak{M}_f(r)\sum_{n+m=0}^{+\infty}(n+m)|a_{nm}|r_1^n r_2^m \leq \frac{2\mathfrak{M}_f(r)}{1-r_1}\ln^{1+\delta}\mathfrak{M}_f(r)\ln^{1+\delta} r_2.
\end{aligned}$$

By Theorem 1 we obtain for all $r \in T \setminus E_2$

$$\begin{aligned}
\sum_{n+m=0}^{+\infty}(n+m)|a_{nm}|r_1^n r_2^m &\leq \frac{\mu_f(r)}{(1-r_1)^{2+\delta}}\ln^{1+\delta}\frac{\mu_f(r)}{1-r_1}\ln^{1/2+\delta} r_2 \ln^{1+2\delta}\frac{\mu_f(r)}{1-r_1}\ln^{1+2\delta} r_2\\
&\leq \frac{\mu_f(r)}{(1-r_1)^{2+\delta}}\ln^{2+3\delta}\frac{\mu_f(r)}{1-r_1}\ln^{3/2+3\delta} r_2.
\end{aligned}$$

□

Proof of Theorem 2. For $k, m \in \mathbb{Z}_+$ and $l \in \mathbb{Z}$ such that $k > -l$ we denote

$$G_{kl} = \Big\{ r = (r_1, r_2) \in T \colon k \le \ln \frac{1}{1-r_1} \le k+1,\ l \le \ln \mu_f(r) \le l+1 \Big\},$$
$$G_{klm} = \Big\{ r = (r_1, r_2) \in G_{kl} \colon m \le \ln r_2 \le m+1 \Big\},\ \ G^+_{kl} = \bigcup_{i=k}^{+\infty} \bigcup_{j=l}^{+\infty} G_{ij}.$$

Remark that the set

$$E_0 = \Big\{ r \in T \colon\ \ln \frac{1}{1-r_1} + \ln \mu_f(r) < 1 \Big\} = \Big\{ r \in T \colon \frac{\mu_f(r)}{1-r_1} < e \Big\} \in \Upsilon,$$

because there exists r_0 such that $E_0 \cap \Delta_{r_0} = \varnothing$. By Lemma 3 for all $r \in T \backslash E_1$ we have

$$\begin{aligned} \sum_{n+m \ge d} |a_{nm}| r_1^n r_2^m &\le \sum_{n+m \ge d} \frac{n+m}{d} |a_{nm}| r_1^n r_2^m \le \frac{1}{d} \sum_{n+m=0}^{+\infty} (n+m) |a_{nm}| r_1^n r_2^m \\ &\le \frac{1}{d} \frac{\mu_f(r)}{(1-r_1)^{2+\delta}} \cdot \ln^{2+3\delta} \frac{\mu_f(r)}{1-r_1} \ln^{3/2+3\delta} r_2 \le \mu_f(r), \end{aligned} \tag{14}$$

where $d = d(r) = \frac{e^{2+\delta}}{(1-r_1)^{2+\delta}} \cdot \ln^{2+3\delta} \frac{\mu_f(r)}{1-r_1} \ln^{3/2+3\delta} r_2$.

Let $G^*_{kl} = G_{kl} \setminus E_2$, $I = \{(i;j) \colon G^*_{ij} \ne \varnothing\}$, $E_2 = E_0 \cup E_1 \cup \left(\bigcup_{(i,j) \notin I} G_{ij} \right)$. Then $\# I = +\infty$. For $(k, l) \in I$ we choose a sequence $r^{(k,l)} \in G^*_{kl}$ such that $M_f(r^{(k,l)}) = \min\limits_{r \in G^*_{kl}} M_f(r)$. So, for all $r \in G^*_{kl}$ we get

$$\begin{aligned} &\frac{1}{e} \mu_f(r^{(k,l)}) \le \mu_f(r) \le e \mu_f(r^{(k,l)}),\ \frac{1}{e} \frac{1}{1-r_1^{(k,l)}} \le \frac{1}{1-r_1} \le e \frac{1}{1-r_1^{(k,l)}}, \\ &\frac{1}{e^2} \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \le \frac{\mu_f(r)}{1-r_1} \le \frac{e^2 \mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \end{aligned} \tag{15}$$

and also $\bigcup_{(k,l) \in I} G^*_{kl} = \bigcup_{(k,l) \in I} G_{kl} \setminus E_2 = \bigcup_{k,l=1}^{+\infty} G_{kl} \setminus E_2 = T \setminus E_2$. Denote $N_{kl} = [2 d_1(r^{(k,l)})]$, where

$$d_1(r) = \frac{e^{2+\delta}}{(1-r_1)^{2+\delta}} \cdot \ln^{2+3\delta} \frac{e^2 \mu_f(r)}{1-r_1} \ln^{3/2+3\delta}(e r_2).$$

For $r \in G^*_{kl}$ we put

$$W_{N_{kl}}(r, t) = \max \left\{ \left| \sum_{n+m \le N_{kl}} a_{nm} r_1^n r_2^m e^{i n_1 \psi_1 + i n_2 \psi_2 + 2\pi i \theta_{nm} t} \right| \colon \psi \in [0, 2\pi]^2 \right\}.$$

For a Lebesgue measurable set $G \subset G^*_{kl}$ and for $(k, l) \in I$ we denote $\nu_{kl}(G) = \frac{\text{meas}(G)}{\text{meas}(G^*_{kl})}$, where meas denotes the Lebesgue measure on $\mathbb{R}^2$.

Remark that ν_{kl} is a probability measure defined on the family of Lebesgue measurable subsets of G^*_k ([19]). Let $\Omega = \bigcup_{(k,l) \in I} G^*_{kl}$ and for all $i, j \in \mathbb{Z}_+$ $k_i, l_{i,j} \colon (k_i, l_{i,j}) \in I$, $k_i < k_{i+1}$,

$l_{i,j} < l_{i,j+1}$. For Lebesgue measurable subsets G of Ω we denote

$$\nu(G) = 2^{k_0}\sum_{i=0}^{+\infty}\left(\frac{1}{2^{k_i}}\left(1-\left(\frac{1}{2}\right)^{k_{i+1}-k_i}\right)\right.$$
$$\left.\times\sum_{j=0}^{N_i}\frac{2^{l_{i,0}}}{2^{l_{i,j}}}\frac{\left(1-\left(\frac{1}{2}\right)^{l_{i,j+1}-l_{i,j}}\right)}{1-\left(\frac{1}{2}\right)^{l_{i,N_{i+1}}+l_{i,0}}}\nu_{k_{i+1}l_{i+1,j+1}}(G\cap G^*_{k_{j+1}l_{i+1,j+1}})\right),$$

where $N_i = \max\{j\colon (k_i, l_{ij}) \in I\}$. Remark that $\nu_{k_{j+1}l_{j+1}}(G^*_{k_{j+1}l_{j+1}}) = \nu(\Omega) = 1$.

Thus ν is a probability measure, which is defined on measurable subsets of Ω. On $[0,1]\times\Omega$ we define the probability measure $P_0 = P\otimes\nu$, which is a direct product of the probability measures P and ν. Now for $(k;l)\in I$ we define

$$F_{kl} = \{(t,r)\in[0,1]\times\Omega\colon W_{N_{kl}}(r,t) > AS_{N_{kl}}(r)\ln^{1/2}N_{kl}\},$$
$$F_{kl}(r) = \{t\in[0,1]\colon W_{N_{kl}}(r,t) > AS_{N_{kl}}(r)\ln^{1/2}N_{kl}\},$$

where $S^2_{N_{kl}}(r) = \sum_{n+m=0}^{N_{kl}}|a_{nm}|^2 r_1^{2n}r_2^{2m}$ and A is the constant from Lemma 1 with $\beta = 1$. Using Fubini's theorem and Lemma 1 with $c_{nm} = a_{nm}r_1^n r_2^m$ and $\beta = 1$, we get for $(k,l)\in I$

$$P_0(F_{kl}) = \int\limits_{\Omega}\left(\int\limits_{F_{kl}(r)} dP\right)d\nu = \int\limits_{\Omega} P(F_{kl}(r))d\nu \le \frac{1}{N_{kl}}\nu(\Omega) = \frac{1}{N_{kl}}.$$

Note that $N_{kl} > \frac{1}{(1-r_1^{(k,l)})^{2+\delta}}\ln^{2+3\delta}\frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\ln^{3/2+3\delta} r_2^{(k,l)} \ge e^{2k}(l+k)^3$. Therefore

$$\sum_{(k,l)\in I} P_0(F_{kl}) \le \sum_{k=1}^{+\infty}\sum_{l=-k+1}^{+\infty}\frac{1}{e^{2k}(l+k)^3} < +\infty.$$

By Borel-Cantelli's lemma the infinite quantity of the events $\{F_{kl}\colon (k,l)\in I\}$ may occur with probability zero. So,

$$P_0(F) = 1, \quad F = \bigcup_{s=1}^{+\infty}\bigcup_{m=1}^{+\infty}\bigcap_{\substack{k\ge s,\ l\ge m\\ (k,l)\in I}}\overline{F_{kl}} \subset [0,1]\times\Omega.$$

Then for any point $(t,r)\in F$ there exist $k_0 = k_0(t,r)$ and $l_0 = l_0(t,r)$ such that for all $k\ge k_0$, $l\ge l_0$, $(k,l)\in I$ we have $W_{N_{kl}}(r,t)\le AS_{N_{kl}}(r)\ln^{1/2}N_{kl}$.

So, $\nu(F^\wedge(t)) = 1$ (see [19]).

For any $t\in F_1$([19]) and $(k,l)\in I$ we choose a point $r_0^{(k,l)}(t)\in G^*_{kl}$ such that

$$W_{N_{kl}}(r_0^{(k,l)}(t),t) \ge \frac{3}{4}M_{kl}(t),\ M_{kl}(t) \stackrel{\text{def}}{=} \sup\{W_{N_{kl}}(r,t)\colon r\in G^*_{kl}\}.$$

Then from $\nu_{kl}(F^\wedge(t)\cap G^*_{kl}) = 1$ for all $(k,l)\in I$ it follows that there exists a point $r^{(k,l)}(t)\in G^*_{kl}\cap F^\wedge(t)$ such that $|W_{N_{kl}}(r_0^{(k,l)}(t),t) - W_{N_{kl}}(r^{(k,l)}(t),t)| < \frac{1}{4}M_{kl}(t)$ or

$$\frac{3}{4}M_{kl}(t) \le W_{N_{kl}}(r_0^{(k,l)}(t),t) \le W_{N_{kl}}(r^{(k,l)}(t),t) + \frac{1}{4}M_{kl}(t).$$

Since $(t, r^{(k,l)}(t)) \in F$, from inequality (3) we obtain

$$\frac{1}{2}M_{kl}(t) \le W_{N_{kl}}(r^{(k,l)}(t), t) \le AS_{N_{kl}}(r^{(k,l)}(t)) \ln^{1/2} N_{kl}.$$

Now for $r^{(k,l)} = r^{(k,l)}(t)$ we get

$$S^2_{N_{kl}}(r^{(k,l)}) \le \mu_f(r^{(k,l)})\mathfrak{M}_f(r^{(k,l)}) \le \frac{\mu_f^2(r^{(k,l)})}{(1-r_1^{(k,l)})^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \ln^{1/2+\delta} r_2^{(k,l)}.$$

So, for $t \in F_1$ and all $k \ge k_0(t)$, $l \ge l_0(t)$, we obtain

$$S_N(r^{(k,l)}) \le \mu_f(r^{(k,l)}) \left(\frac{1}{1-r_1^{(k,l)}} \ln \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \sqrt{\ln r_2^{(k,l)}}\right)^{1/2+\delta/2}. \tag{16}$$

It follows from (15) that $d_1(r^{(k,l)}) \ge d(r)$ for $r \in G^*_{kl}$. Then for $t \in F_1$, $r \in F^\wedge(t) \cap G^*_{kl}$, $(k,l) \in I$, $k \ge k_0(t), l \ge l_0(t)$ we get

$$M_f(r,t) \le \sum_{n+m \ge 2d_1(r^{(k,l)})} |a_{nm}| r_1^n r_2^m + W_{N_{kl}}(r,t) \le \sum_{n+m\ge 2d(r)} |a_{nm}| r_1^n r_2^m + M_{kl}(t).$$

Finally for $t \in F_1, r \in F^\wedge(t) \cap G^*_{kl}$, $l \ge l_0(t)$ and $k \ge k_0(t)$ we obtain

$$\begin{aligned}
M_f(r^{(k,l)}, t) &\le \mu_f(r^{(k,l)}) + 2AS_{N_{kl}}(r^{(k,l)}) \ln^{1/2} N_{kl} \\
&\le \mu_f(r^{(k,l)}) + 2A\mu_f(r^{(k,l)}) \left(\frac{1}{1-r_1^{(k,l)}} \ln \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}}\right)^{1/2+\delta/2} \ln^{1/4+\delta} r_2^{(k,l)} \\
&\times \ln\left(\frac{2e^{2+\delta}}{(1-r_1^{(k,l)})^{2+\delta}} \cdot \ln^{2+3\delta} \frac{e^2\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \ln^{3/2+3\delta}(er_2^{(k,l)})\right), \\
M_f(r,t) &\le \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \cdot \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1/4+\delta} r_2.
\end{aligned} \tag{17}$$

Therefore inequality (17) holds almost surely ($t \in F_1$, $P(F_1) = 1$) for all

$$r \in \Big(\bigcup_{(k,l)\in I} (G^*_{kl} \cap F^\wedge(t)) \cap G^+_{kl}\Big) \setminus E^* = (T \cap G^+_{kl}) \setminus (E^* \cup G^* \cup E_1) = T \setminus E_2,$$

where $G^+_{kl} = \bigcup_{i=k}^{+\infty} \bigcup_{j=l}^{+\infty} G_{kl}$, $E_2 = E_1 \cup G^* \cup E^*$, $G^* = \bigcup_{(k,l)\in I}(G^*_{kl} \setminus F^\wedge(t))$.

It remains to remark that $\nu(G^*)$ satisfies $\nu(G^*) = \sum_{(k,l)\in I}(\nu_{kl}(G^*_{kl}) - \nu_{kl}(F^\wedge(t))) = 0$. Then for all $(k,l) \in I$ we obtain

$$\nu_{kl}(G^*_{kl} \setminus F^\wedge(t)) = \frac{\mathrm{meas}(G^*_{kl} \setminus F^\wedge(t))}{\mathrm{meas}(G^*_{kl})} = 0, \ \ \mathrm{meas}(G^*_{kl} \setminus F^\wedge(t)) = \iint\limits_{G^*_{kl} \setminus F^\wedge(t)} \frac{dr_1 dr_2}{(1-r_1)r_2} = 0.$$

$\square$

REFERENCES

[1] Valiron G. *Sur les fonctions entieres d'ordre fini et d'ordre null et en particulier les fonctions a correspondance reguliere*. Ann. Fac. Sci. Toulouse Math. (3) 1913, **5** (3), 117–257. doi:10.5802/afst.287 (in French)

[2] Wiman A. *Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Function und dem grössten Gliebe der zugehörigen Taylorischen Reiche*. Acta Math. 1914, **37**, 305–326. doi:10.1007/BF02401837 (in German)

[3] Wittich G. Recent studies on unique analytical functions. Fizmatgiz, Moscow, 1960. (in Russian)

[4] Goldberg A.A., Levin B.Ya., Ostrovsky I.V. Entire and meromorphic functions. In: Results of scientific and technical. modern. probl. mat. fundam. guide, 85. VINITI, Moscow, 1991. (in Russian)

[5] Pólya G., Szegő G. Aufgaben und Lehrsätze aus der Analysis. Springer, Berlin, 1925. (in German)

[6] Skaskiv O.B., Filevych P.V. *On the size of an exceptional set in the Wiman theorem*. Mat. Stud. 1999, **12** (1), 31–36. (in Ukrainian)

[7] Skaskiv O.B., Zrum O.V. *On an exceptional set in the Wiman inequalities for entire functions*. Mat. Stud. 2004, **21** (1), 13–24. (in Ukrainian)

[8] Kovari T. *On the maximum modulus and maximum term of functions analytic in the unit disc*. J. Lond. Math. Soc. (2) 1966, **41** (1), 129–137. doi:10.1112/jlms/s1-41.1.129

[9] Skaskiv O.B., Kuryliak A.O. *Direct analogues of Wiman's inequality for analytic functions in the unit disc*. Carpathian Math. Publ. 2010, **2** (1), 109–118. (in Ukrainian)

[10] Kuryliak A.O., Shapovalovska L.O., Skaskiv O.B. *Wiman's type inequality for some double power series*. Mat. Stud. 2013, **39** (2), 134–141.

[11] Kuryliak A., Skaskiv O., Tsvigun V. *Levy's phenomenon for analytic functions in* $\mathbb{D} \times \mathbb{C}$. Mat. Stud. 2016, **46** (2), 121–129. doi:10.15330/ms.46.2.121-129

[12] Lévy P. *Sur la croissance de fonctions entière*. Bull. Soc. Math. France 1930, **58**, 29–59. doi: 10.24033/bsmf.1157 (in French)

[13] Erdős P., Rényi A. *On random entire function*. Zastosowania mat. 1969, **10**, 47–55.

[14] Filevych P.V. *Some classes of entire functions in which the Wiman-Valiron inequality can be almost certainly improved*. Mat. Stud. 1996, **6**, 59–66. (in Ukrainian)

[15] Filevych P.V. *Wiman-Valiron type inequalities for entire and random entire functions of finite logarithmic order*. Sib. Math. J. 2003, **42** (3), 579–586. (translation of Sib. Mat. Zhurn. 2003, **42** (3), 683–692. (in Russian))

[16] Zrum O.V., Skaskiv O.B. *On Wiman's inequality for random entire functions of two variables*. Mat. Stud. 2005, **23** (2), 149–160. (in Ukrainian)

[17] Skaskiv O.B., Zrum O.V. *On improvement of Fenton's inequality for entire functions of two complex variables*. Math. Bull. Shevchenko Sci. Soc. 2006, **3**, 56–68. (in Ukrainian)

[18] Fenton P.C. *Wiman-Valiron theory in two variables*. Trans. Amer. Math. Soc. 1995, **347** (11), 4403–4412. doi:10.2307/2155043

[19] Kuryliak A.O., Skaskiv O.B., Zrum O.V. *Levy's phenomenon for entire functions of several variables*. Ufa Math. J. 2014, **6** (2), 111–120. doi:10.13108/2014-6-2-111

[20] Skaskiv O.B., Zrum O.V. *Wiman's type inequality for entire functions of two complex variables with rapidly oscillating coefficient*. Mat. Metodi Fiz.-Mekh. Polya 2005, **48** (4), 78–87. (in Ukrainian)

[21] Krishna G.J., Rao N.I.H. *Generalised inverse and probability techniques and some fundamental growth theorems in* $\mathbb{C}^k$. J. Indian Math. Soc. (N.S.) 1977, **41**, 203–219.

[22] Bitlyan I.F., Goldberg A.A. *Wiman-Valiron's theorem for entire functions of several complex variables*. Vestn. St.-Peterbg. Univ. Mat. Mekh. Astron. 1959, **2** (13), 27–41. (in Russian)

[23] Shumitzky A. *A probabilistic approach to the Wiman—Valiron theory for functions of several complex entire*. Complex Var. Elliptic Equ. 1989, **13** (1–2), 85–98. doi:10.1080/17476938908814380

[24] Kuryliak A.O., Skaskiv O.B. *Wiman's type inequalities without exceptional sets for random entire functions of several variables*. Mat. Stud. 2012, **38** (1), 35–50.

[25] Kuryliak A.O., Shapovalovska L.O. *Wiman's inequality for entire functions of several complex varibles with rapidly oscillating coefficients*. Mat. Stud. 2015, **43** (1), 16–26. doi:10.15330/ms.43.1.16-26

[26] Steele J.M., *Sharper Wiman inequality for entire functions with rapidly oscillating coefficients*. J. Math. Anal. Appl. 1987, **123** (2), 550–558. doi:10.1016/0022-247X(87)90329-5

[27] Kuryliak A.O., Skaskiv O.B., Skaskiv S.R. *Levy's phenomenon for analytic functions in the polydisc*. arXiv: 1602.04756

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Нехай $\mathcal{A}^2$ клас аналітичних функцій f вигляду

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m$$

з областю збіжності $\mathbb{T} = \{z \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < +\infty\}$ таких, що $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ в $\mathbb{T}$ і існує $r_0 = (r_1^0, r_2^0) \in [0,1)\times[0,+\infty)$ таке, що для всіх $r \in (r_1^0, 1)\times(r_2^0, +\infty)$ маємо $r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) + \ln r_1 > 1$, де $\mathfrak{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m$. Нехай $\mathcal{K}(f,\theta) = \{f(z,t) = \sum_{n+m=0}^{+\infty} a_{nm} e^{2\pi i t(\theta_n+\theta_m)} : t \in \mathbb{R}\}$ — клас аналітичних функцій, де (θ_{nm}) — послідовність додатних цілих чисел така, що її впорядкування (θ_k^*) за зростанням задовольняє умову

$$\theta_{k+1}^*/\theta_k^* \ge q > 1, k > 0.$$

Для аналітичних функцій з класу $\mathcal{K}(f,\theta)$ уточнено нерівність типу Вімана.

Ключові слова і фрази: нерівність типу Вімана, аналітичні функції від декількох комплексних змінних.