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TRANSLATION, MODULATION AND DILATION SYSTEMS IN SET-VALUED SIGNAL PROCESSING

In this paper, we investigate a very important function space consists of set-valued functions defined on the set of real numbers with values on the space of all compact-convex subsets of complex numbers for which the pth power of their norm is integrable. In general, this space is denoted by $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ for $1 \leq p < \infty$ and it has an algebraic structure named as a quasilinear space which is a generalization of a classical linear space. Further, we introduce an inner-product (set-valued inner product) on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ and we think it is especially important to manage interval-valued data and interval-based signal processing. This also can be used in imprecise expectations. The definition of inner-product on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is based on Aumann integral which is ready for use integration of set-valued functions and we show that the space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Hilbert quasilinear space. Finally, we give translation, modulation and dilation operators which are three foundational set-valued operators on Hilbert quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$.

Key words and phrases: Hilbert quasilinear space, set-valued function, Aumann integral, translation, modulation, dilation.

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INTRODUCTION

The translation, modulation and dilation operators play an important role in signal processing. These operators are usually applied to electromagnetic signals such as radio, lasers, optics and computer networks. For example, the translation operator provides parallel displacement for a discret-time signal. The modulation operator changes the wealths of a sound wave. As it is well known, converting an analog signal to a digital signal leads to ambiguous computation errors. In such circumstances to perform signal processing we need the area of interval-valued signal processing, more generally set-valued signal processing (see [1–3]). In this work, we introduce translation, modulation and dilation operators on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ which is a special space of set-valued functions.

Unfortunately, the space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ have an algebraic structure which is not a linear space. This structure is called as a "quasilinear space" by Aseev in 1986 [5]. Therefore, he present an approach for the function spaces of set-valued mappings. Let us give the definition of a quasilinear space which is presented by Aseev [5].

A set X is called a quasilinear space if a partial order relation "$\preceq$", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that

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the following conditions hold for all elements $x, y, z, v \in X$ and all $\alpha, \beta \in \mathbb{R}$:

$$
\begin{aligned}
&x \preceq x, \\
&x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \\
&x = y \text{ if } x \preceq y \text{ and } y \preceq x, \\
&x + y = y + x, \\
&x + (y + z) = (x + y) + z, \\
&\text{there exists an element (zero) } \theta \in X \text{ such that } x + \theta = x, \\
&\alpha(\beta x) = (\alpha\beta)x, \\
&\alpha(x + y) = \alpha x + \alpha y, \\
&1x = x, \\
&0x = \theta, \\
&(\alpha + \beta)x \preceq \alpha x + \beta x, \\
&x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \\
&\alpha x \preceq \alpha y \text{ if } x \preceq y.
\end{aligned}
$$

Note that the concept of quasilinear space has been only introduced over the field $\mathbb{R}$. As distinct from Aseev's definition, in next section we will introduce the quasilinear spaces over general field $\mathbb{K}$ which consists of real or complex numbers.

Any linear space is a quasilinear space with the partial order relation "$x \preceq y \Longleftrightarrow x = y$".

Perhaps the most popular example of a nonlinear quasilinear space is the set of all nonempty closed intervals of real numbers sembolized by $\Omega_C(\mathbb{R})$, and it is a quasilinear space with the inclusion relation "$\subseteq$", the algebraic sum operation

$$A + B = \{a + b : a \in A,\ b \in B\}$$

and the real-scalar multiplication $\lambda A = \{\lambda a : a \in A\}$.

In fact $\Omega_C(\mathbb{R})$ is the set of all nonempty compact convex subsets of real numbers and it is a subset of $\Omega(\mathbb{R})$, the set of all nonempty compact subsets of real numbers which is an another important example of a nonlinear quasilinear space. In general, $\Omega(E)$ and $\Omega_C(E)$ are the sets of all nonempty closed bounded and nonempty convex closed bounded subsets of any normed linear space E, respectively. Both are a quasilinear space with the inclusion relation, the real-scalar multiplication and with a slight modification of addition as follows:

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}},$$

where the closure is taken on the norm topology of E.

The investigation of $\Omega_C(\mathbb{R})$ or more general $\Omega(\mathbb{C})$ contributes interval and convex analysis and they are excellent tools for mathematical formulation of many real-life situations, for example signal processing. Therefore we are interested in the space of $\Omega(\mathbb{C})$-valued functions in this article.

We know the Banach space $L^p(\mathbb{R})$ for $1 \leq p < \infty$ the space of all functions f for which $|f|^p$ is integrable, is one of the fundamental vector spaces in functional analysis. In this paper we will try to investigate the space $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ of all functions $F : \mathbb{R} \to \Omega(\mathbb{C})$ such that

$\int\limits_{\mathbb{R}} \|F(x)\|_{\Omega}^{p}\, dx$ do exist ($1 \le p < \infty$). We can see that the set $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ is a normed quasilinear space and the special case $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Hilbert quasilinear space. We use a new kind inner-product for set-valued functions to construct a norm structure of $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. The inner-product will be introduced by an integral in the sense of Aumann [8].

1 Preliminaries

We will start by giving the definition of quasilinear space which is different from Aseev's definition. In this definition we will consider the quasilinear spaces over a general field $\mathbb{K}$. The elements of $\mathbb{K}$ are real or complex numbers. We think that this approach is suitable mathematical background of some applications, e.g., interval analysis and signal processing.

A set X is called a quasilinear space over field $\mathbb{K}$ if a partial order relation "$\preceq$", an algebraic sum operation, and an operation of multiplication by real or complex numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{K}$:

$$
\begin{aligned}
&x \preceq x, \\
&x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \\
&x = y \text{ if } x \preceq y \text{ and } y \preceq x, \\
&x + y = y + x, \\
&x + (y + z) = (x + y) + z, \\
&\text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \\
&\alpha(\beta x) = (\alpha\beta)x, \\
&\alpha(x + y) = \alpha x + \alpha y, \\
&1x = x, \\
&0x = \theta, \\
&(\alpha + \beta)x \preceq \alpha x + \beta x, \\
&x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \\
&\alpha x \preceq \alpha y \text{ if } x \preceq y.
\end{aligned}
\tag{1}
$$

$\mathbb{K}$ is called the scalar field of the quasilinear space X, and X is called a real quasilinear space if $\mathbb{K} = \mathbb{R}$ and is called a complex quasilinear space if $\mathbb{K} = \mathbb{C}$. Mostly $\mathbb{K}$ will be $\mathbb{C}$ in this work.

Any real linear space is a quasilinear space with the partial order relation defined by "$x \preceq y$ if and only if $x = y$". In this case, quasilinear space axioms is the linear space axioms.

Lemma 1 ([5]). *Suppose that each element x in quasilinear space X has an inverse element $x' \in X$. Then the partial order in X is determined by equality, the distributivity conditions hold, and consequently X is a linear space.*

Hence in a real linear space, the equality is the only way to define a partial order such that conditions (1) hold.

It will be assumed in what follows that $-x = (-1) \cdot x$. Also, note that $-x$ may not be x'. Any element x in a quasilinear space is regular if and only if $x - x = \theta$, that is, if and only if $x' = -x$.

Now, let us record some basic necessary results from [5]. In a quasilinear space X, the element θ is minimal, i.e., $x = \theta$ if $x \preceq \theta$. An element x' is called *inverse* of $x \in X$ if $x + x' = \theta$. The inverse is unique whenever it exists. An element x possessing inverse is called *regular*, otherwise is called *singular*.

Definition 1 ([6]). *Suppose that X is a quasilinear space and $Y \subseteq X$. Then Y is called a* subspace of X *whenever Y is a quasilinear space with the same partial order on X.*

Theorem 1 ([6]). *Y is subspace of quasilinear space X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{K}$, $\alpha \cdot x + \beta \cdot y \in Y$.*

Proof of this theorem is quite similar to its classical linear algebraic analogue.

Let X be a quasilinear space and Y be a subspace of X. Suppose that each element x in Y has inverse element $x' \in Y$ then by Lemma 1 the partial order on Y is determined by the equality. In this case Y is a linear subspace of X. An element x in quasilinear space X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. Also, X_r stands for the set of all regular elements of X while X_s stands for the sets of all singular elements and zero in X. Further, it can be easily shown that X_r, X_{sym} and X_s are subspaces of X. They are called *regular, symmetric* and *singular subspaces* of X, respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of X is a linear space while the singular one is nonlinear at all.

Example 1. *In $\Omega_C(\mathbb{R})$,*

$$\{\{0\}\} \cup \{[a,b] : a, b \in \mathbb{R} \text{ and } a < b\}$$

is the singular subspace of $\Omega_C(\mathbb{R})$. Further $\{\{a\} : a \in \mathbb{R}\}$ is the set of all degenerate intervals or the set of all singletons of $\mathbb{R}$ constitutes the regular subspace X_r. It is a linear subspace of $\Omega_C(\mathbb{R})$ and $(\Omega_C(\mathbb{R}))_r$ is the copy of $\mathbb{R}$ in $\Omega_C(\mathbb{R})$. In fact, for any normed linear space E, each singleton $\{a\}$, $a \in E$, can be identified with the element a and hence E can be considered as the (regular) subspace of both $\Omega_C(E)$ and $\Omega(E)$. Further, the regular subspace of both $\Omega_C(E)$ and $\Omega(E)$ is isometrically isomorphic to E, namely, $(\Omega_C(E))_r \equiv E$ and $(\Omega(E))_r \equiv E$.

Let X be a real or complex quasilinear space. The real-valued function on X is called a *norm* if the following conditions hold:

$$\begin{aligned}
&\|x\| > 0 \text{ if } x \neq 0, \\
&\|x + y\| \leq \|x\| + \|y\|, \\
&\|\alpha x\| = |\alpha| \, \|x\|, \\
&\text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|, \\
&\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \\
&x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon \text{ then } x \preceq y,
\end{aligned}$$

here x, y, x_ε are arbitrary element in X and α is any scalar.

A quasilinear space X with a norm defined on it, is called *normed quasilinear space*. It follows from Lemma 1 that if any $x \in X$ has inverse element $x' \in X$, then the concept of normed quasilinear space coincides with the concept of real normed linear space. Notice again that x'

may not be exist but if x' exists then $x' = -x$. Hausdorff metric or norm metric on X is defined by the equality

$$h(x,y) = \inf\left\{r \geq 0 : x \preceq y + a_1^{(r)}, y \preceq x + a_2^{(r)} \text{ and } \left\|a_i^{(r)}\right\| \leq r, i = 1,2\right\}.$$

Since $x \preceq y + (x-y)$ and $y \preceq x + (y-x)$, the quantity $h(x,y)$ is well-defined for any elements $x, y \in X$, and it is not hard to see that the function h satisfies all the metric axioms. Also we should note that $h(x,y)$ may not equal to $\|x-y\|$ if X is not a linear space; however $h(x,y) \leq \|x-y\|$ for every $x, y \in X$.

Lemma 2 ([5]). *The operations of algebraic sum and multiplication by real or complex numbers are continuous with respect to the Hausdorff metric. The norm is continuous with respect to the Hausdorff metric.*

Example 2 ([5]). *For a normed linear space E, a norm on $\Omega(E)$ is defined by*

$$\|A\|_{\Omega} = \sup_{a \in E} \|a\|_E .$$

Hence $\Omega_C(E)$ and $\Omega(E)$ are normed quasilinear spaces. In this case the Hausdorff (norm) metric is defined as usual:

$$h(x,y) = \inf\{r \geq 0 : x \subseteq y + S_r(\theta), y \subseteq x + S_r(\theta)\},$$

where $S_r(\theta)$ is a closed ball of E and x, y are elements of $\Omega_C(E)$ or $\Omega(E)$. Further, $\Omega_C(E)$ is a closed subspace of $\Omega(E)$.

Definition 2 ([5]). *A normed quasilinear space X is called an Ω-space if there exists an element $B_X \neq \theta$ such that*

$$\text{if } \|x\|_X \leq \|B_X\|_X, \text{ then } x \preceq B_X.$$

If X is a real normed linear space, then $\Omega(X)$ is an Ω-space.

Now, let us give a useful type of quasilinear spaces called *consolidate* quasilinear space.

Definition 3 ([6]). *Let X be a quasilinear space, $M \subseteq X$ and $x \in M$. The set*

$$F_x^M = \{z \in M_r : z \preceq x\}$$

is called floor in M of x. *In the case of $M = X$ it is called only* floor of x *and written briefly F_x instead of F_x^X.*

Floor of an element x in linear spaces is the singleton $\{x\}$. Therefore, it is nothing to discuss the notion of floor of an element in a linear space.

Definition 4 ([6]). *A quasilinear space X is called* consolidate *quasilinear space whenever* $\sup F_y$ *do exists for every $y \in X$ and*

$$y = \sup F_y = \sup\{z \in X_r : z \preceq y\}.$$

Otherwise, X is called non-consolidate quasilinear space.

Especially, we should note that the supremum in this definition is defined according to the partial order relation "$\preceq$" on X. Hence, we will use the notion of "$\sup_{''\preceq''}$" in place of general notation "sup" to emphasize this case.

Example 3 ([6]). *For any normed linear space E, $\Omega(E)$ and $\Omega_C(E)$ are consolidate normed quasilinear space.*

Aseev launched a theory in [5] that we see it as the beginning of quasilinear functional analysis. However, there was a lot of deficiencies in the theory. One of them is the definition of inner-product. Now we will give the definition of inner-product in a quasilinear space which coinsides with its linear analogue [6,7]. Later we will present some fundamental properties of inner-product and Hilbert quasilinear spaces. Firstly, let us introduce a definition.

Definition 5. *For two quasilinear spaces $(X, \leq)$ and $(Y, \preceq)$, Y is called compatible contains X whenever $X \subseteq Y$ and the partial order relation $\leq$ on X is the restriction of the partial order relation $\preceq$ on Y. We briefly use the symbol $X \overline{\subseteq} Y$ in this case. We write $X \lessapprox Y$ whenever $X \overline{\subseteq} Y$ and $Y \overline{\subseteq} X$.*

Remark 1. *$X \lessapprox Y$ means X and Y are the same sets with the same partial order relations which make them quasilinear spaces. However, we may write $X = Y$ for $X \lessapprox Y$ whenever the relations are clear from context.*

Definition 6. *Let X be a quasilinear space.* Consolidation of X *is the smallest consolidate quasilinear space $\widehat{X}$ which compatible contains X, that is, if there exists another consolidate quasilinear space Y which compatible contains X then $\widehat{X} \subseteq Y$.*

Clearly, $\widehat{X} = X$ for some consolidate quasilinear space X. We do not know yet whether each quasilinear space has a consolidation. This notion is unnecessary in consolidate quasilinear spaces, hence it is redundant in linear spaces. Further, $\widehat{\Omega_C(\mathbb{R}^n)_s} = \Omega_C(\mathbb{R}^n)$.

For a quasilinear space X, the set $F_y^{\widehat{X}} = \left\{ z \in \left(\widehat{X}\right)_r : z \preceq y \right\}$ is the *floor of y in $\widehat{X}$*.

Now, let us give an extended definition of inner-product given in [7]. We can say that the inner product in the following definition may be seen a set-valued inner product on quasilinear spaces.

Definition 7. *Let X be a quasilinear space having a consolidation $\widehat{X}$. A mapping $\langle \, , \, \rangle : X \times X \to \Omega(\mathbb{K})$ is called an inner-product on X if for any $x, y, z \in X$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied :*

$$\text{If } x, y \in X_r \text{ then } \langle x, y \rangle \in \Omega_C(\mathbb{K})_r \equiv \mathbb{K},$$
$$\langle x + y, z \rangle \subseteq \langle x, z \rangle + \langle y, z \rangle,$$
$$\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \text{ and } \langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle,$$
$$\langle x, y \rangle = \langle y, x \rangle,$$
$$\langle x, x \rangle \geq 0 \text{ for } x \in X_r \text{ and } \langle x, x \rangle = 0 \Leftrightarrow x = 0,$$
$$\| \langle x, y \rangle \|_\Omega = \sup \left\{ \| \langle a, b \rangle \|_\Omega : a \in F_x^{\widehat{X}}, b \in F_y^{\widehat{X}} \right\},$$
$$\text{if } x \preceq y \text{ and } u \preceq v \text{ then } \langle x, u \rangle \subseteq \langle y, v \rangle,$$
$$\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that}$$
$$x \preceq y + x_\varepsilon \text{ and } \langle x_\varepsilon, x_\varepsilon \rangle \subseteq S_\varepsilon(\theta) \text{ then } x \preceq y.$$

A quasilinear space with an inner product is called an *inner product quasilinear space*.

Remark 2. *For some $x \in X_r$, $\langle x, x \rangle \geq 0$ means $\langle x, x \rangle$ is non-negative, that is, the order "$\geq$" in the definition is the usual order on $\Omega_C(\mathbb{K})_r \equiv \mathbb{K}$. It should not be confused with the order "$\preceq$" on X.*

Example 4 ([6,7]). *Let X be a linear Hilbert space. Then the space $\Omega(X)$ is a Hilbert quasilinear space by the inner product defined by*

$$\langle A, B \rangle_\Omega = \overline{\{\langle a, b \rangle_X : a \in A, b \in B\}}$$

for $A, B \in \Omega(X)$. Further, there is no need the closure for the definition of inner product on $\Omega(\mathbb{C})$, since $\{\langle a, b \rangle_\mathbb{C} : a \in A, b \in B\}$ is closed subset of $\mathbb{C}$. Namely, the inner product on $\Omega(\mathbb{C})$ is given by

$$\langle A, B \rangle_\Omega = \{\langle a, b \rangle_\mathbb{C} : a \in A, b \in B\}.$$

Every inner product quasilinear space X is a normed quasilinear space with the norm defined by

$$\|x\| = \sqrt{\|\langle x, x \rangle\|_\Omega}$$

for every $x \in X$. This norm is called *inner product norm*. Further $x_n \to x$ and $y_n \to y$ in a inner product quasilinear space then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Lemma 3 ([6]). *Let X be a inner product quasilinear space. Then*

$$\|\langle x, y \rangle\|_\Omega \leq \|x\|_X \|y\|_X$$

for $x, y \in X$.

A inner product quasilinear space is called *Hilbert quasilinear space* if it is complete according to the inner-product (norm) metric. For example, $\Omega(\mathbb{C})$ is a Hilbert quasilinear space.

Definition 8 ([5]). *Let X and Y be quasilinear spaces. A mapping $T : X \to Y$ is called a* quasilinear operator *if it satisfies the following conditions:*

$$\begin{aligned}
&T(x_1 + x_2) \preceq T(x_1) + T(x_2), \\
&T(\alpha x) = \alpha T(x) \quad \text{for any } \alpha \in \mathbb{R}, \\
&\text{if } x_1 \preceq x_2, \text{ then } T(x_1) \preceq T(x_2).
\end{aligned}$$

Definition 9. *Let X and Y be quasilinear spaces. A mapping $T : X \to Y$ is called a* linear operator *if it satisfies the following conditions:*

$$\begin{aligned}
&T(x_1 + x_2) = T(x_1) + T(x_2), \\
&T(\alpha x) = \alpha T(x) \quad \text{for any } \alpha \in \mathbb{R}, \\
&\text{if } x_1 \preceq x_2, \text{ then } T(x_1) \preceq T(x_2).
\end{aligned}$$

Hence linear operators can be obtained by adding an extra condition to the first condition of quasilinear operators.

Remark 3. *We will see that quasilinear operators may not conserve quasilinear structure. Due to this obstacle we introduce the linear operator notion acting on quasilinear spaces. Obviously, any linear operator between quasilinear spaces is a quasilinear operator, but not conversely. If X and Y are linear spaces then the definition of quasilinear operators coincides with the usual definition of a linear operators.*

Definition 10 ([5]). *Let X and Y be a normed quasilinear spaces. A quasilinear operator $T : X \to Y$ is said to be* bounded *if there exists a number $k > 0$ such that $\|Tx\| \leq k\|x\|$ for any $x \in X$.*

2 Aumann Integral

We will need the integral of set-valued functions when we deal with the space $L^p(\mathbb{R}, \Omega(\mathbb{C}))$, $1 \leq p < \infty$. For this purpose we will introduce the integral of a set-valued function and give some properties of this integral.

Integrals of set-valued functions are given by Robert J. Aumann in 1965. It is as follows [8]:

Let I be the unit interval $[0, 1]$. For any t in I, let $F(t)$ be a nonempty subset of $\mathbb{R}^n$. Suppose that $\mathcal{L}$ be the set of all point-valued functions f from I to $\mathbb{R}^n$ such that f is integrable over I and $f(t) \in F(t)$ for all t in I. Define

$$\int_I F(t)dt = \left\{ \int_I f(t)dt : f \in \mathcal{L} \right\}$$

i.e., the set of all integrals of members of $\mathcal{L}$.

Throughout the section we will use the notations: The triple $(\Gamma, \mathcal{A}, \mu)$ is a complete σ-finite measure space, X is a complete separable metric space and $F : \Gamma \rightsquigarrow X$ represents a set-valued function that assigns to each $t \in \Gamma$ a subset $F(t) \subseteq X$.

Let us give the main definitions and theorems with respect to the integral of a measurable set-valued function.

Definition 11 ([9]). *A set-valued function $F : \Gamma \rightsquigarrow X$ is called with* closed, open *or* compact valued *if $F(x)$ is a closed, open or compact set in X, for each $x \in \Gamma$, respectively.*

Definition 12 ([9]). *A set-valued function $F : \Gamma \rightsquigarrow X$ is called measurable if for any open subset $\mathcal{O} \subset X$,*

$$F^{-1}(\mathcal{O}) = \{x \in \Gamma : F(x) \cap \mathcal{O} \neq \varnothing\}$$

is element of $\mathcal{A}$.

Measurability of set-valued functions is closely associated with the concept of measurability of its selections.

Definition 13 ([9]). *For a given set-valued function $F : \Gamma \rightsquigarrow X$, a measurable function $f : \Gamma \to X$ satisfying*

$$\text{for all } x \in \Gamma, f(x) \in F(x)$$

is called a measurable selection of F.

Theorem 2 ([9]). *Let $F : \Gamma \rightsquigarrow X$ be closed valued. Then there exists a measurable selection of F.*

We denote by $L^p(\Gamma, X, \mu)$, $1 \leq p < \infty$ the Banach space of all measurable functions $f : \Gamma \to X$ such that $\int_\Gamma \|f\|^p \, d\mu < \infty$. If $\Gamma = \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and μ is the Lebesque measure then we find $L^p(\Gamma, X, \mu) = L^p(\mathbb{R})$. For $1 \leq p < \infty$, $S^p(F)$ is the set of all selections $f \in L^p(\Gamma, X, \mu)$ of a measurable set-valued function $F : \Gamma \rightsquigarrow X$ [10], i.e.,

$$S^p(F) = \{f : \Gamma \to X : \int_\Gamma \|f\|^p \, d\mu < \infty, \text{ and } f(x) \in F(x) \text{ for } x \in \Gamma\}.$$

Definition 14 ([4]). *A set-valued function $F : \Gamma \rightsquigarrow X$ is called* integrably bounded *if there exists a nonnegative function $f \in L^1(\Gamma, \mathbb{R}, \mu)$ such that*

$$F(x) \subset f(x)B \text{ almost everywhere in } \Gamma,$$

where B is the unit ball of X.

Aumann gave the definition of an integral of a set-valued function in the following way:

Definition 15 ([4]). *The integral of F on Γ is the set of integrals of integrable selections of F :*

$$\int_\Gamma F d\mu = \left\{ \int_\Gamma f d\mu : f \in S^1(F) \right\}.$$

We will say that F is integrable set-valued function in the sense of Aumann *if the set $\{\int_\Gamma f d\mu : f \in S^1(F)\}$ is not empty. Aumann integral of F will be shown as $\overset{(A)}{\int_\Gamma} F d\mu$.*

Proposition 1 ([9]). *If $G : \Gamma \rightsquigarrow X$ is Aumann integrable and $G(x) \subseteq F(x)$ almost everywhere on Γ. Then the set-valued function F is also Aumann integrable and*

$$\overset{(A)}{\int_\Gamma} G(x)dx \subseteq \overset{(A)}{\int_\Gamma} F(x)dx.$$

Proposition 2 ([9]). *If $F, F_1, F_2 : \Gamma \rightsquigarrow X$ are Aumann integrable then $F_1 + F_1$ and λF are Aumann integrable and*

$$\overset{(A)}{\int_\Gamma} (F_1 + F_2)(x)dx = \overset{(A)}{\int_\Gamma} F_1(x)dx + \overset{(A)}{\int_\Gamma} F_2(x)dx$$

and

$$\overset{(A)}{\int_\Gamma} (\lambda F)(x)dx = \lambda \overset{(A)}{\int_\Gamma} F(x)dx.$$

Proposition 3 ([9]). *If $F : \mathbb{R} \to \Omega(X)$ is Aumann integrable and the integral of F is compact then*

$$\left\| \overset{(A)}{\int_\Gamma} F(x)dx \right\|_\Omega \leq \overset{(A)}{\int_\Gamma} \|F(x)\|_\Omega \, dx.$$

Theorem 3 ([4]). *Let $F:\Gamma \rightsquigarrow \mathbb{R}^n$ be a measurable and closed-valued function. If μ is nonatomic and F is integrably bounded, then the Aumann integral of F is compact.*

Now let us present the *Dominated Convergence Theorem* for the Aumann integrals.

Theorem 4 ([11]). *If $F_n : \Gamma \to \Omega(\mathbb{C})$ $n = 1,2,...$ are measurable closed valued functions, $\{\|F_n(.)\|\}_{n=1}^{\infty}$ is uniformly integrable and $F_n(x) \to F(x)$ with respect to the Hausdorff metric then*

$$\overline{\int_{\Gamma}^{(A)} F_n(x)dx} \to \overline{\int_{\Gamma}^{(A)} F(x)dx}.$$

3 THE HILBERT QUASILINEAR SPACE $L^2(\mathbb{R},\Omega(\mathbb{C}))$

In this chapter we will concentrate on the quasilinear structure of the $L^p(\mathbb{R},\Omega(\mathbb{C}))$ space, $1 \le p < \infty$. We will show that $L^p(\mathbb{R},\Omega(\mathbb{C}))$ spaces are normed quasilinear space over the field $\mathbb{C}$ and later we construct a set-valued inner-product on $L^2(\mathbb{R},\Omega(\mathbb{C}))$ by way of Aumann integral.

For $1 \le p < \infty$, the space $L^p(\mathbb{R},\Omega(\mathbb{C}))$ consists of all set-valued measurable functions $F:\mathbb{R} \to \Omega(\mathbb{C})$ such that the Lebesque integral

$$\int_{\mathbb{R}} \|F(x)\|_{\Omega}^{p}\, dx$$

is well defined, where the notion of measurability of F is the measurability in Definition 12. Note that this integral is a classical Lebesque integral.

Among the $L^p(\mathbb{R},\Omega(\mathbb{C}))$ spaces, the case $p = 2$ has a special importance: We will say that $L^2(\mathbb{R},\Omega(\mathbb{C}))$ is an inner-product quasilinear space with respect to the inner-product which is defined via Aumann integral

$$\langle F,G\rangle = \int_{\mathbb{R}}^{(A)} \langle F(x),G(x)\rangle_{\Omega}\, dx. \tag{2}$$

Firstly, let us indicate $L^2(\mathbb{R},\Omega(\mathbb{C}))$ is a consolidate quasilinear space and so it has a consolidation. Therefore, we can define a set-valued inner-product function on this space. After the definition of inner-product on $L^2(\mathbb{R},\Omega(\mathbb{C}))$ we will denote the norm on $L^2(\mathbb{R},\Omega(\mathbb{C}))$ with

$$\|F\| = \left\| \overline{\{\int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}^{1/2}$$

and we will show that this norm comes from the inner-product given by the equality (2). Thereafter, we will show that the inner-product norm on $L^2(\mathbb{R},\Omega(\mathbb{C}))$ coincides with the expression

$$\left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^{2}\, dx\right)^{1/2},$$

namely, the equality

$$\|F\| = \left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^{2} \, dx \right)^{1/2}$$

is also a norm on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Further, we will prove that $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Banach quasilinear space with this norm. Thus, we will say that $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Hilbert quasilinear space.

The operations of algebraic sum, multiplication by a complex scalar and the partial order relation are defined as follows:

$$(F_1 + F_2)(x) = F_1(x) + F_2(x), \quad (\lambda F)(x) = \lambda F(x)$$

and

$$F_1 \preceq F_2 \Leftrightarrow F_1(x) \subseteq F_1(x) \text{ for any } x \in \mathbb{R}.$$

By a similar way given in [5], it is easy to verify that $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ is a quasilinear space over the field $\mathbb{C}$ by the above algebraic operations and the relation.

Now let us determine the regular elements of $L^p(\mathbb{R}, \Omega(\mathbb{C}))$, $1 \leq p < \infty$:

$$\begin{aligned} F \in (L^p(\mathbb{R}, \Omega(\mathbb{C})))_r &\Leftrightarrow F - F = \theta \Leftrightarrow F(x) - F(x) = \{0\}, \text{ for all } x \in \mathbb{R} \\ &\Leftrightarrow F(x) \in \Omega(\mathbb{C})_r \equiv \mathbb{C}, \text{ for all } x \in \mathbb{R}. \end{aligned}$$

By $\Omega(\mathbb{C})_r \equiv \mathbb{C}$ we mean there exist an isometric isomorphism (equivalence) between these normed linear spaces. Recall again that the regular subspace of a quasilinear space is just a linear space. Hence we can give the following corollaries.

Corollary 1. $(L^p(\mathbb{R}, \Omega(\mathbb{C})))_r = L^p(\mathbb{R}, \Omega(\mathbb{C})_r) \equiv L^p(\mathbb{R}, \mathbb{C}) = L^p(\mathbb{R})$ *for* $1 \leq p < \infty$. *Further, if* $F \in L^p((\mathbb{R}, \Omega(\mathbb{C})))_r$ *then there exists only one selection of* F *and this selection is equal to itself.*

Now we will prove that the $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ space is a inner-product quasilinear space.

Theorem 5. *The quasilinear space* $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ *is an inner-product quasilinear space with respect to the inner-product*

$$\langle F, G \rangle = \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), G(x) \rangle_{\Omega} \, dx \tag{3}$$

for $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ *and using the Aumann integral gives the equality*

$$\langle F, G \rangle = \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), G(x) \rangle_{\Omega} \, dx = \{ \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} \, dx : f \in S^2(F), g \in S^2(G) \}. \tag{4}$$

Proof. Previously, we shall verify that the equality (3) is well-defined, i.e., that the function

$$U_{F,G} : \mathbb{R} \to \Omega(\mathbb{C}), \ \ U_{F,G}(x) = \langle F(x), G(x) \rangle_{\Omega}$$

is integrable according to Aumann and this integral belongs to $\Omega(\mathbb{C})$ (see, Definition 7). If we consider the Theorem 2 then we can say that $U_{F,G}$ has a measurable selection, since $U_{F,G}$ is closed valued. Thus, this function is integrable according to Aumann. Now we will show that the Aumann integral of $U_{F,G}$ is an element of $\Omega(\mathbb{C})$: Firstly, let us show that $U_{F,G}$ is integrably bounded, namely, there exists a nonnegative integrable function $f:\mathbb{R}\to\mathbb{R}$ such that $U_{F,G}(x)\subseteq f(x)B$ for any $x\in\mathbb{R}$, where $B=\{a\in\mathbb{C}:|a|\leq 1\}$. By the definition of norm on $\Omega(\mathbb{C})$,

$$\|U_{F,G}(x)\|_{\Omega}=\|\langle F(x),G(x)\rangle_{\Omega}\|=\sup\{|\langle a_x,b_x\rangle_{\mathbb{C}}|:a_x\in F(x),b_x\in G(x)\}.$$

Since for each $x\in\mathbb{R}$, $U_{F,G}(x)$ is a compact set, there exists the elements $a_x^0\in F(x)$ and $b_x^0\in G(x)$ which are dependent on x such that

$$\|U_{F,G}(x)\|_{\Omega}=\|\langle F(x),G(x)\rangle_{\Omega}\|=\left|\left\langle a_x^0,b_x^0\right\rangle_{\mathbb{C}}\right|.$$

By reason of the fact that each of the elements x corresponds to the element $a_x^0\in F(x)$ and $b_x^0\in G(x)$, we can define the function $f:\mathbb{R}\to\mathbb{R}$ such that

$$f(x)=\left|\left\langle a_x^0,b_x^0\right\rangle_{\mathbb{C}}\right|.$$

Further,

$$\left|\int_{\mathbb{R}}f(x)dx\right|\leq\int_{\mathbb{R}}|f(x)|\,dx=\int_{\mathbb{R}}\left|\left\langle a_x^0,b_x^0\right\rangle_{\mathbb{C}}\right|dx=\int_{\mathbb{R}}\|U_{F,G}(x)\|_{\Omega}\,dx=\int_{\mathbb{R}}\|\langle F(x),G(x)\rangle_{\Omega}\|\,dx.$$

By Lemma 3 and Holder inequality we observe that

$$\int_{\mathbb{R}}\|\langle F(x),G(x)\rangle_{\Omega}\|\,dx\leq\int_{\mathbb{R}}(\|F(x)\|_{\Omega}\,\|G(x)\|_{\Omega})dx\leq(\int_{\mathbb{R}}\|F(x)\|_{\Omega}^2\,dx)^{1/2}(\int_{\mathbb{R}}\|G(x)\|_{\Omega}^2\,dx)^{1/2}.$$

The last inequality implies f is integrable since $F,G\in L^2(\mathbb{R},\Omega(\mathbb{C}))$. Furthermore,

$$\|U_{F,G}(x)\|_{\Omega}=\left|\left\langle a_x^0,b_x^0\right\rangle_{\mathbb{C}}\right|=\left|\left\langle a_x^0,b_x^0\right\rangle_{\mathbb{C}}\right|\|B\|=\|f(x)B\|.$$

Since $\Omega(\mathbb{C})$ is an Ω-space, we have that $U_{F,G}(x)\subseteq f(x)B$ for any $x\in\mathbb{R}$ and so $U_{F,G}$ is integrably bounded. Consequently, by the Theorem 3 we say that the Aumann integral of $U_{F,G}$

$$\langle F,G\rangle=\overset{(A)}{\int_{\mathbb{R}}}U_{F,G}(x)dx=\overset{(A)}{\int_{\mathbb{R}}}\langle F(x),G(x)\rangle_{\Omega}\,dx$$

is a compact set. The next step is to verify the equality (4): If we apply the definition of Aumann integral to the set-valued function $U_{F,G}$ then we write

$$\langle F,G\rangle=\overset{(A)}{\int_{\mathbb{R}}}\langle F(x),G(x)\rangle_{\Omega}\,dx=\overset{(A)}{\int_{\mathbb{R}}}U_{F,G}(x)dx=\{\int_{\mathbb{R}}h(x)dx:h\in S(U_{F,G})\}.$$

Now let us research the selections of $U_{F,G}$. By the definition of norm on $\Omega(\mathbb{C})$ we write

$$\langle F(x),G(x)\rangle_{\Omega}=\{\langle z,w\rangle_{\mathbb{C}}:z\in F(x),w\in G(x)\}.$$

If we remember that $h(x) \in U_{F,G}(x)$ for every $x \in \mathbb{R}$ then for the determined elements $z_x^0 \in F(x)$ and $w_x^0 \in G(x)$ it is written that

$$h(x) = \left\langle z_x^0, w_x^0 \right\rangle_{\mathbb{C}},$$

where z_x^0 and w_x^0 are depend on the element x. Let us describe the functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ such that $f(x) = z_x^0$ and $g(x) = w_x^0$. The functions f and g are well-defined due to the fact that h is a function. It is obvious that $f \in S^2(F)$ and $g \in S^2(G)$ and so $f, g \in L^2(\mathbb{R})$. Also we can see that

$$h(x) = \langle f(x), g(x) \rangle_{\mathbb{C}}$$

for any element x. The equality

$$\left| \int_{\mathbb{R}} h(x) dx \right| = \left| \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx \right| \leq \int_{\mathbb{R}} \left| \langle f(x), g(x) \rangle_{\mathbb{C}} \right| dx = \int_{\mathbb{R}} \left| f(x)\overline{g(x)} \right| dx$$

and from the Cauchy-Shwarz inequality give

$$\int_{\mathbb{R}} \left| f(x)\overline{g(x)} \right| dx \leq (\int_{\mathbb{R}} |f(x)|^2 dx)^{1/2} (\int_{\mathbb{R}} |g(x)|^2 dx)^{1/2} < \infty$$

and so $h \in S(U_{F,G})$. Hence,

$$\langle F, G \rangle = \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), G(x) \rangle_{\Omega} dx = \{ \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx : f \in S^2(F), g \in S^2(G) \}.$$

Now we shall show that the expression (3) defines an inner product on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ in the meaning of the Definition 7.

1. *If $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ then $\langle F, G \rangle \in \Omega(\mathbb{C})_r \equiv \mathbb{C}$.*
   If $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ then by the Corollary 1

$$\langle F, G \rangle = \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), G(x) \rangle_{\Omega} dx = \int_{\mathbb{R}} \langle F(x), G(x) \rangle_{\mathbb{C}} dx = \int_{\mathbb{R}} F(x)\overline{G(x)} dx.$$

   Also if we remember that the equality

$$\langle F, G \rangle = \int_{\mathbb{R}} F(x)\overline{G(x)} dx$$

   is complex-valued inner product on $L^2(\mathbb{R})$ then we say that $\langle F, G \rangle \in \Omega(\mathbb{C})_r \cong \mathbb{C}$.

2. $\langle F + G, H \rangle = \langle F, H \rangle + \langle G, H \rangle$:
   By the second condition of inner product on $\Omega(\mathbb{C})$ and the Proposition 1 we have that

$$\langle F + G, H \rangle = \overset{(A)}{\int_{\mathbb{R}}} \langle F(x) + G(x), H(x) \rangle_{\Omega} dx \subseteq \overset{(A)}{\int_{\mathbb{R}}} (\langle F(x), H(x) \rangle_{\Omega} + \langle G(x), H(x) \rangle_{\Omega}) dx$$

   and from the Proposition 2 we obtain that

$$\langle F + G, H \rangle \subseteq \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), H(x) \rangle_{\Omega} dx + \overset{(A)}{\int_{\mathbb{R}}} \langle G(x), H(x) \rangle_{\Omega} dx = \langle F, H \rangle + \langle G, H \rangle .$$

3. $\langle \lambda F, G\rangle = \lambda \langle F, G\rangle$ and $\langle F, \lambda G\rangle = \bar{\lambda} \langle F, G\rangle$:
   By the third condition of inner product on $\Omega(\mathbb{C})$ and the Proposition 2 we have that

$$\langle \lambda F, G\rangle = \int_{\mathbb{R}}^{(A)} \langle (\lambda F)(x), G(x)\rangle_{\Omega}\, dx = \int_{\mathbb{R}}^{(A)} \lambda \langle F(x), G(x)\rangle_{\Omega}\, dx$$
$$= \lambda \int_{\mathbb{R}}^{(A)} \langle F(x), G(x)\rangle_{\Omega}\, dx = \lambda \langle F, G\rangle .$$

   It can be easily shown that $\langle F, \lambda G\rangle = \bar{\lambda} \langle F, G\rangle$.

4. $\langle F, G\rangle = \langle G, F\rangle$:
   By the fourth condition of inner product on $\Omega(\mathbb{C})$,

$$\langle F, G\rangle = \int_{\mathbb{R}}^{(A)} \langle F(x), G(x)\rangle_{\Omega}\, dx = \int_{\mathbb{R}}^{(A)} \langle G(x), F(x)\rangle_{\Omega}\, dx = \langle G, F\rangle .$$

5. $\langle F, F\rangle \geq 0$ *for* $F \in (L^2(\mathbb{R}, \Omega(\mathbb{C})))_r$ *and* $\langle F, F\rangle = \{0\} \Leftrightarrow F = \theta$:
   If $F \in (L^2(\mathbb{R}, \Omega(\mathbb{C})))_r$ then $f \in L^2(\mathbb{R})$ by the Corollary (1) and so

$$\langle F, F\rangle = \int_{\mathbb{R}}^{(A)} \langle F(x), F(x)\rangle_{\Omega}\, dx = \{\int_{\mathbb{R}} \langle F(x), F(x)\rangle_{\mathbb{C}}\, dx\}$$
$$= \{\int_{\mathbb{R}} F(x)\overline{F(x)} dx\} = \{\int_{\mathbb{R}} |F(x)|^2\, dx\}.$$

   Since the inner-product on $L^2(\mathbb{R})$ is non-negative we have that $\int_{\mathbb{R}} |F(x)|^2\, dx \geq 0$ and so $\langle F, F\rangle \geq 0$.
   Now let us assume that $\langle F, F\rangle = 0$. Then

$$\int_{\mathbb{R}}^{(A)} \langle F(x), F(x)\rangle_{\Omega}\, dx = \{\int_{\mathbb{R}} \langle f(x), g(x)\rangle_{\mathbb{C}}\, dx : f, g \in S^2(F)\} = \{0\}.$$

   This implies $\int_{\mathbb{R}} f(x)\overline{f(x)} dx = \int_{\mathbb{R}} |f(x)|^2\, dx = 0$. Hence, by the Corollary (1) and the norm on $L^2(\mathbb{R})$ we say that $f = 0$. Since the any selection f of F is equal to 0, we say that $F = \theta$.

6. $\|\langle F, G\rangle\|_{\Omega} = \sup\{\|\langle f, g\rangle\|_{\Omega} : f \in F_F, g \in F_G\}$:
   Firstly, it is not hard to see that $F_F \subseteq S^2(F)$ for $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$. By this way we say that

$$\sup\{\|\langle f, g\rangle\|_{\Omega} : f \in F_F, g \in F_G\} = \sup\{\left|\int_{\mathbb{R}} \langle f(x), g(x)\rangle_{\mathbb{C}}\, dx\right| : f \in F_F, g \in F_G\}$$
$$= \sup\{\left|\int_{\mathbb{R}} \langle f(x), g(x)\rangle_{\mathbb{C}}\, dx\right| : f \in S^2(F), g \in S^2(G)\} = \|\langle F, G\rangle\|_{\Omega} .$$

7. $\langle F_1, G_1\rangle \subseteq \langle F_2, G_2\rangle$ *if* $F_1 \preceq F_2$ *and* $G_1 \preceq G_2$:

   If $F_1 \preceq F_2$ and $G_1 \preceq G_2$ then $F_1(x) \subseteq F_2(x)$ and $G_1(x) \subseteq G_2(x)$ for a.e. $x \in \mathbb{R}$. By the seventh condition of inner product on $\Omega(\mathbb{C})$ we say that

$$\langle F_1(x), G_1(x)\rangle \subseteq \langle F_2(x), G_2(x)\rangle .$$

   Therefore, using the Proposition 1 implies the equality

$$\langle F_1, G_1\rangle = \overset{(A)}{\int_{\mathbb{R}}} \langle F_1(x), G_1(x)\rangle\, dx \subseteq \overset{(A)}{\int_{\mathbb{R}}} \langle F_2(x), G_2(x)\rangle\, dx = \langle F_2, G_2\rangle .$$

8. *We show that if for any* $\varepsilon > 0$ *there exists an element* $F_\varepsilon \in L^2(R, \Omega(C))$ *such that* $F \preceq G + F_\varepsilon$ *and* $\langle F_\varepsilon, F_\varepsilon\rangle \subseteq S_\varepsilon(\theta)$ *then* $F \preceq G$:

   Suppose that for any $\varepsilon > 0$ there exists an element $F_\varepsilon \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ such that $F \preceq G + F_\varepsilon$ and $\langle F_\varepsilon, F_\varepsilon\rangle \subseteq S_\varepsilon(\theta)$. Then

$$\|\langle F_\varepsilon, F_\varepsilon\rangle\|_\Omega \le \|S_\varepsilon(\theta)\|_\Omega = \varepsilon. \tag{5}$$

   Further,

$$\|\langle F_\varepsilon, F_\varepsilon\rangle\|_\Omega = \left\| \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), F(x)\rangle_\Omega\, dx \right\|_\Omega = \left\| \{ \int_{\mathbb{R}} \langle f_\varepsilon(x), g_\varepsilon(x)\rangle_{\mathbb{C}}\, dx : f_\varepsilon, g_\varepsilon \in S^2(F_\varepsilon)\} \right\|_\Omega$$
$$= \sup\{ \left| \int_{\mathbb{R}} \langle f_\varepsilon(x), f_\varepsilon(x)\rangle_{\mathbb{C}}\, dx \right| : f_\varepsilon \in S^2(F_\varepsilon)\}$$
$$= \sup\{ \left| \int_{\mathbb{R}} |f_\varepsilon(x)|^2\, dx \right| : f_\varepsilon \in S^2(F_\varepsilon)\} = \|F_\varepsilon\|^2 .$$

   Hence by the inequality (5) we say that $\|F_\varepsilon\|^2 \le \varepsilon$. The last condition of norm on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ indicates $F \preceq G$.

□

For $1 \le p < \infty$, the expression

$$\|F\| = \left\| \overline{\{ \int_{\mathbb{R}} |f(x)|^p\, dx : f \in S^p(F)\}} \right\|_\Omega^{1/p}$$

defines a norm on $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ and so this space is a normed quasilinear space and this norm is an inner-product norm obtained from the inner-product (3). Notably,

$$\|F\|^2 = \|\langle F, F\rangle\| = \left\| \overset{(A)}{\int_{\mathbb{R}}} \langle F(x), F(x)\rangle_\Omega\, dx \right\|_\Omega = \left\| \{ \int_{\mathbb{R}} \langle f(x), g(x)\rangle_{\mathbb{C}}\, dx : f, g \in S^2(F)\} \right\|_\Omega$$
$$= \sup\{ \left| \int_{\mathbb{R}} \langle f(x), g(x)\rangle_{\mathbb{C}}\, dx \right| : f, g \in S^2(F)\} = \sup\{ \left| \int_{\mathbb{R}} \langle f(x), f(x)\rangle_{\mathbb{C}}\, dx \right| : f \in S^2(F)\}$$
$$= \sup\{ \int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}.$$

Since for bounded subset $A \subset \mathbb{C}$ we have that $\sup |A| = \sup \left|\overline{A}\right|$ where $|A| = \{|a| : a \in A\}$ then

$$\|F\|^2 = \|\langle F, F\rangle\| = \sup\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\} = \sup \overline{\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}}$$
$$= \left\| \overline{\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}$$

and so

$$\|F\| = \left\| \overline{\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}^{1/2}.$$

Lemma 4. *For $p = 2$, the inner-product norm is equivalents to $(\int\limits_{\mathbb{R}} \|F(x)\|_{\Omega}^2\, dx)^{1/2}$ i.e., if $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ then*

$$\|F\| = \left\| \overline{\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}^{1/2} = (\int\limits_{\mathbb{R}} \|F(x)\|_{\Omega}^2\, dx)^{1/2}. \tag{6}$$

Proof. By the Proposition 3 and the norm of inner-product on $\Omega(\mathbb{C})$, we write

$$\|F\|^2 = \|\langle F, F\rangle\| = \left\| \int\limits_{\mathbb{R}}^{(A)} \langle F(x), F(x)\rangle_{\Omega}\, dx \right\| \leq \int\limits_{\mathbb{R}} \|\langle F(x), F(x)\rangle_{\Omega}\|\, dx = \int\limits_{\mathbb{R}} \|F(x)\|_{\Omega}^2\, dx \tag{7}$$

and

$$\|F\|^2 = \|\langle F, F\rangle\| = \left\| \overline{\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}. \tag{8}$$

Using the (7) and (8) we have the inequality

$$\|F\|^2 = \|\langle F, F\rangle\| = \left\| \overline{\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega} \leq \int\limits_{\mathbb{R}} \|F(x)\|_{\Omega}^2\, dx. \tag{9}$$

It is obvious that for any $x \in \mathbb{R}$, $F(x)$ is a compact subset of $\mathbb{C}$. Hence, there exists an element t_0^x in $F(x)$ such that

$$\sup\{|t| : t \in F(x)\} = |t_0^x|.$$

Let us define the function $g : \mathbb{R} \to \mathbb{C}$ with $g(x) = t_0^x$. It is not hard to see that g is well-defined. The function g is an element of $S^2(F)$ due to the fact that for $x \in \mathbb{R}$, $g(x) = t_0^x \in F(x)$ and

$$\int\limits_{\mathbb{R}} \|F(x)\|_{\Omega}^2\, dx = \int\limits_{\mathbb{R}} (\sup\{|t| : t \in F(x)\})^2 dx = \int\limits_{\mathbb{R}} |t_0^x|^2\, dx = \int\limits_{\mathbb{R}} |g(x)|^2\, dx. \tag{10}$$

Since

$$\int\limits_{\mathbb{R}} |g(x)|^2\, dx \leq \sup\{\int\limits_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}$$

and

$$\sup\{\int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\} = \sup \overline{\{\int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}}$$
$$= \left\| \overline{\{\int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}.$$

This implies

$$\int_{\mathbb{R}} |g(x)|^2\, dx \leq \left\| \overline{\{\int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}. \tag{11}$$

By the (10) and (11) we say that

$$\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2\, dx = \left\| \overline{\{\int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}. \tag{12}$$

Therefore, from the (9) and (12) we obtain that

$$\|F\| = \left\| \overline{\{\int_{\mathbb{R}} |f(x)|^2\, dx : f \in S^2(F)\}} \right\|_{\Omega}^{1/2} = \left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2\, dx \right)^{1/2}.$$

□

Theorem 6. *The quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is complete with the norm given by (6), i.e., $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Banach quasilinear space.*

Proof. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ such that $\sum_{k=1}^{\infty} \|F_k\| < \infty$. We will show that the series $\sum_{k=1}^{\infty} F_k$ is convergent. For this we need to find a function F in $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ such that

$$\lim_{n\to\infty} h_{L^2}(\sum_{k=1}^{n} F_k, F) = 0$$

where h_{L^2} is the Hausdorff metric on the normed quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Now we define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \sum_{k=1}^{\infty} (\|F_k(x)\|_{\Omega})^2.$$

Applying the Monoton Convergence Theorem and Minkowski inequality prove that

$$\begin{aligned}
\int_{\mathbb{R}} g(x)dx &= \int_{\mathbb{R}} (\sum_{k=1}^{\infty} (\|F_k(x)\|_{\Omega})^2) dx = \int_{\mathbb{R}} \lim_{n\to\infty} (\sum_{k=1}^{n} (\|F_k(x)\|_{\Omega})^2) dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} (\sum_{k=1}^{n} (\|F_k(x)\|_{\Omega})^2) dx \leq \lim_{n\to\infty} (\sum_{k=1}^{n} (\int_{\mathbb{R}} \|F_k(x)\|_{\Omega}^2\, dx)^{1/2})^2 \\
&= \lim_{n\to\infty} (\sum_{k=1}^{n} \|F_k\|)^2 = (\sum_{k=1}^{\infty} \|F_k\|)^2.
\end{aligned}$$

This shows that g is integrable function, since $\sum_{k=1}^{\infty} \|F_k\|$ is convergent. Thus, $g(x)$ is finite for any $x \in \mathbb{R}$ and the series $\sum_{k=1}^{\infty} \|F_k(x)\|_\Omega$ is convergent for any $x \in \mathbb{R}$. Due to the fact that $\Omega(\mathbb{C})$ is complete, we say that the series $\sum_{k=1}^{\infty} F_k(x)$ is convergent. Let us consider that the function $F : \mathbb{R} \to \Omega(\mathbb{C})$ defined by

$$F(x) = \begin{cases} \sum_{k=1}^{\infty} F_k(x) & , g(x) < \infty \\ \{0\} & , g(x) = \infty \end{cases}.$$

From the Proposition 6.1.13 in [12] we say that the set-valued function F is measurable. Since

$$\|F(x)\|_\Omega^2 = \left\| \sum_{k=1}^{\infty} F_k(x) \right\|_\Omega^2 \leq (\sum_{k=1}^{\infty} \|F_k(x)\|_\Omega)^2 = g(x)$$

and

$$\int_{\mathbb{R}} g(x)dx < \infty$$

we have that $\int_{\mathbb{R}} \|F(x)\|_\Omega^2 dx < \infty$. This implies F belongs to $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Further, for a.e. $x \in \mathbb{R}$

$$\begin{aligned} 0 \leq \lim_{n\to\infty} h_\Omega(\sum_{k=1}^{n} F_k(x), F(x)) &\leq \lim_{n\to\infty} \left\| \sum_{k=1}^{n} F_k(x) - F(x) \right\|_\Omega \\ &= \left\| \lim_{n\to\infty} \sum_{k=1}^{n} F_k(x) - \lim_{n\to\infty} F(x) \right\|_\Omega = \left\| \sum_{k=1}^{\infty} F_k(x) - F(x) \right\|_\Omega = 0 \end{aligned}$$

and so

$$\lim_{n\to\infty} h_\Omega(\sum_{k=1}^{n} F_k(x), F(x)) = 0. \tag{13}$$

Now we shall prove that the function series $\sum_{k=1}^{\infty} F_k$ converges to F in $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ to complete the proof. In accordance with this purpose we will show that $\lim_{n\to\infty} h_{L^2}(\sum_{k=1}^{n} F_k, F) = 0$. Firstly, if we use the Hausdorff metric on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$, we say that for any $\varepsilon > 0$ there exist elements $F_r^i \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$, $i = 1, 2$ such that $\sum_{k=1}^{n} F_k \preceq F + F_r^1$, $F \preceq \sum_{k=1}^{n} F_k + F_r^2$ and $\left\|F_r^i\right\| \leq r$. Hence $\sum_{k=1}^{n} F_k(x) \subseteq F(x) + F_r^1(x)$, $F(x) \subseteq \sum_{k=1}^{n} F_k(x) + F_r^2(x)$ for a.e. $x \in \mathbb{R}$. Further, by the Hausdorff metric on $\Omega(\mathbb{C})$ we have that $\left\|F_r^i(x)\right\| \leq h_\Omega(\sum_{k=1}^{n} F_k(x), F(x)) + r$ for a.e. $x \in \mathbb{R}$ and $i = 1, 2$. Moreover, for any $r > 0$

$$h_{L^2}(\sum_{k=1}^{n} F_k, F) \leq \left\|F_r^i\right\| = (\int_{\mathbb{R}} \left\|F_r^i(x)\right\|_\Omega^2 dx)^{1/2} \leq (\int_{\mathbb{R}} (h_\Omega(\sum_{k=1}^{n} F_k(x), F(x)) + r)^2 dx)^{1/2}.$$

Hence, we have proved the inequality

$$h_{L^2}(\sum_{k=1}^{n} F_k, F) \leq (\int_{\mathbb{R}} (h_\Omega(\sum_{k=1}^{n} F_k(x), F(x)))^2 dx)^{1/2}.$$

Consequently,

$$\lim_{n\to\infty}(h_{L^2}(\sum_{k=1}^{n}F_k,F))^2 \le \lim_{n\to\infty}\int_{\mathbb{R}} h_\Omega(\sum_{k=1}^{n}F_k(x),F(x))^2dx.$$

Using the Theorem 4 the above inequality gives

$$\lim_{n\to\infty}(h_{L^2}(\sum_{k=1}^{n}F_k,F))^2 \le \int_{\mathbb{R}}(\lim_{n\to\infty}(h_\Omega(\sum_{k=1}^{n}F_k(x),F(x)))^2)dx$$

and this implies $\lim\limits_{n\to\infty}h_{L^2}(\sum\limits_{k=1}^{n}F_k,F)=0$ by the equality (13). So the proof is complete. □

Theorem 7. *The quasilinear space $L^2(\mathbb{R},\Omega(\mathbb{C}))$ is a Hilbert quasilinear space with the inner-product given by (4).*

Proof. We know that $L^2(\mathbb{R},\Omega(\mathbb{C}))$ is a inner-product quasilinear space with respect to the inner-product given by (4). By the definition of norm obtained this inner-product we have that

$$\|F\| = \left\|\overline{\{\int_{\mathbb{R}}|f(x)|^2\,dx : f\in S^2(F)\}}\right\|_\Omega^{1/2}.$$

Using the Lemma 4 and Theorem 6 show that $L^2(\mathbb{R},\Omega(\mathbb{C}))$ is complete. Thus, $L^2(\mathbb{R},\Omega(\mathbb{C}))$ is Hilbert quasilinear space. □

4 TRANSLATION, MODULATION AND DILATION OPERATORS ON $L^2(\mathbb{R},\Omega(\mathbb{C}))$

In this section we introduce some important operators on $L^2(\mathbb{R},\Omega(\mathbb{C}))$.

Definition 16. ***(Translation)*** *For $a\in\mathbb{R}$ and $F\in L^2(\mathbb{R},\Omega(\mathbb{C}))$, the operator $\mathcal{T}_a$ is defined by*

$$(\mathcal{T}_aF)(x) = F(x-a) = \overline{\{(T_af_n)(x) = f_n(x-a) : f_n\in S^1(F), n=1,2,...\}} \tag{14}$$

and is called translation by a, where T_a is the translation operator on $L_2(\mathbb{R})$.

Note that $\mathcal{T}_aF$ is defined by the set of translations of countable measurable selections of F. By using the Castaing's theorem (see, [13]) we say that there exists a sequence (f_n) of measurable selections of F such that

$$F(x-a) = \overline{\bigcup_{n\ge1}(T_af_n)(x)} = \overline{\bigcup_{n\ge1}f_n(x-a).}$$

This implies that $F(x-a)$ which is the translation by $a\in\mathbb{R}$ of a set-valued function $F\in L^2(\mathbb{R},\Omega(\mathbb{C}))$ can be written as (14). Hence the translation operator $\mathcal{T}_a$ is a natural generalization of classical translation operator T_a in this way.

Notation: We will often write $\mathcal{T}_aF(x)$ instead of $(\mathcal{T}_aF)(x)$ and similarly for the other operators.

Translation operator $\mathcal{T}_a$ is a bounded linear operator between quasilinear spaces: Actually, given any $F,G\in L^2(\mathbb{R},\Omega(\mathbb{C}))$ and $\lambda\in\mathbb{C}$ we write

$$\mathcal{T}_a(F+G)(x) = (F+G)(x-a) = F(x-a)+G(x-a) = \mathcal{T}_aF(x)+\mathcal{T}_aG(x),$$

$$\mathcal{T}_a(\lambda F)(x) = (\lambda F)(x-a) = \lambda F(x-a) = \lambda \mathcal{T}_a F(x).$$

These show that

$$\mathcal{T}_a(F+G) = \mathcal{T}_a F + \mathcal{T}_a G \text{ and } \mathcal{T}_a(\lambda F) = \lambda \mathcal{T}_a F. \tag{15}$$

Also, if $F \preceq G$ then $F(x) \subseteq G(x)$ for each $x \in \mathbb{R}$. Hence, for any $x \in \mathbb{R}$

$$\mathcal{T}_a F(x) = F(x-a) \subseteq G(x-a) = \mathcal{T}_a G(x). \tag{16}$$

This implies $\mathcal{T}_a F \preceq \mathcal{T}_a G$. By the (15) and (16) we say that $\mathcal{T}_a$ is linear in the meaning of Definition 9. Furthermore, if $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$, the change of variable $z = x - a$ shows that

$$\int_{\mathbb{R}} \|\mathcal{T}_a F(x)\|_\Omega^2 \, dx = \int_{\mathbb{R}} \|F(x-a)\|_\Omega^2 \, dx = \int_{\mathbb{R}} \|F(z)\|_\Omega^2 \, dz \tag{17}$$

and so

$$\|\mathcal{T}_a F\| = \|F\|,$$

namely, $\mathcal{T}_a$ is bounded.

Now we will define the modulation and dilation operators in analogy to the definition of translation operator.

Definition 17. ***(Modulation, Dilation)*** *For a set-valued function $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ we define the following operators:*

(i) *For $b \in \mathbb{R}$, the operator $\mathcal{E}_b$ is defined by*

$$(\mathcal{E}_b F)(x) = e^{2\pi i b x} F(x) = \overline{\{(E_b f_n)(x) = e^{2\pi i b x} f_n(x) : f_n \in S^1(F), n = 1, 2, ...\}}$$

and is called modulation by b, where E_b is the modulation operator on $L_2(\mathbb{R})$. This definition shows that the modulation operator $\mathcal{E}_b$ is a natural generalization of classical modulation operator E_b.

(ii) *For $c \in \mathbb{R}$, the operator $\mathcal{D}_c$ is defined by*

$$(\mathcal{D}_c F)(x) = \frac{1}{\sqrt{c}} F(\frac{x}{c}) = \overline{\{D_c f_n(x) = \frac{1}{\sqrt{c}} f_n(\frac{x}{c}) : f_n \in S^1(F), n = 1, 2, ...\}}$$

and is called dilation by c, where D_c is the dilation operator on $L_2(\mathbb{R})$. Thus, we say that the modulation operator $\mathcal{D}_c$ is a natural generalization of classical modulation operator D_c.

It can be easily shown that $\mathcal{E}_b$ and $\mathcal{D}_c$ are bounded linear operators as per above.

Definition 18. *Let X_1 and X_2 be Hilbert quasilinear spaces and $T : X_1 \to X_2$ be a bounded linear operator. The operator $T^* : X_2 \to X_1$ is called the adjoint operator of T such that for any $x \in X_1$ and $y \in X_2$,*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle .$$

Definition 19. *Let X be a Hilbert quasilinear space and $T : X \to X$ a bounded linear operator. The operator T is self-adjoint if $T = T^*$ and is unitary if $TT^* = T^*T = I$.*

Proposition 4. *The operators $\mathcal{T}_a, \mathcal{E}_b$ and $\mathcal{D}_c$ are unitary operators from space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ to $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Further,*

- $\mathcal{T}_a^{-1} = \mathcal{T}_{-a} = (\mathcal{T}_a)^*$,
- $\mathcal{E}_b^{-1} = \mathcal{E}_{-b} = (\mathcal{E}_b)^*$,
- $\mathcal{D}_c^{-1} = \mathcal{D}_{1/c} = (\mathcal{D}_c)^*$.

Proof. We give a complete proof for the operator $\mathcal{T}_a$, since the proof is similar to the cases of $\mathcal{E}_b$ and $\mathcal{D}_c$ Due to the assumption $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ and the equality (17) we say that $\mathcal{T}_a$ maps $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ into $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Now we will prove that $\mathcal{T}_a$ is unitary: The change of variable $z = x - a$ yields that

$$\begin{aligned}\langle \mathcal{T}_a F, G\rangle &= \overset{(A)}{\int_{\mathbb{R}}} \langle \mathcal{T}_a F(x), G(x)\rangle_{\Omega}\, dx = \overset{(A)}{\int_{\mathbb{R}}} \langle F(x-a), G(x)\rangle_{\Omega}\, dx \\ &= \overset{(A)}{\int_{\mathbb{R}}} \langle F(z), G(z+a)\rangle_{\Omega}\, dz = \langle F, \mathcal{T}_{-a} G\rangle .\end{aligned}$$

Hence by the definition of the adjoint operator $\mathcal{T}_a^*$ we prove that $\mathcal{T}_a^* = \mathcal{T}_{-a}$. Moreover, it is not hard to show that $\mathcal{T}_a\mathcal{T}_a^* = \mathcal{T}_a\mathcal{T}_{-a} = I$ and $\mathcal{T}_a^*\mathcal{T}_a = \mathcal{T}_{-a}\mathcal{T}_a = I$. The calculations show that $\mathcal{T}_a$ is unitary and $\mathcal{T}_a^{-1} = \mathcal{T}_{-a} = (\mathcal{T}_a)^*$. □

Operators denoted by composition of some of the translation, modulation and dilation operators appear in mathematics and engineering. For this purpose, the following Proposition is useful.

Proposition 5. *For any $a, b \in \mathbb{R}$ and $c > 0$, the following commutation relations hold:*

(i) $(\mathcal{T}_a\mathcal{E}_b F)(x) = e^{2\pi i b(x-a)} F(x-a) = e^{-2\pi ba}(\mathcal{E}_b\mathcal{T}_a F)(x)$,

(ii) $(\mathcal{T}_a\mathcal{D}_c F)(x) = \frac{1}{\sqrt{c}} F(\frac{x}{c} - \frac{a}{c}) = (\mathcal{D}_c\mathcal{T}_{a/c} F)(x)$,

(iii) $(\mathcal{D}_{ca}\mathcal{E}_b F)(x) = \frac{1}{\sqrt{c}} e^{2\pi i b/c} F(\frac{x}{c}) = (\mathcal{E}_{b/c}\mathcal{D}_c F)(x)$.

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Левент Х., Їлмаз Й. *Системи перенесення, модуляції та затримки у множиннозначній обробці сигналу* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 143–164.

У цій статті досліджується важливий простір функцій, який складається з множиннозначних функцій, визначених на множині дійсних чисел зі значеннями у просторі всіх компактних опуклих підмножин комплексних чисел, для яких p-тий степінь їхньої норми інтегровний. Загалом цей простір позначають $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ при $1 \leq p < \infty$ і він має алгебраїчну структуру, його називають квазілінійним простором, що є узагальненням класичного лінійного простору. Далі вводиться скалярний добуток (множиннозначний скалярний добуток) на $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ і, на наш погляд, це важливо для управління інтервальнозначними даними та інтервальною обробкою сигналів. Також це можна використати в терії нечітких сподівань. Визначення скалярного добутку в $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ базується на понятті інтегралу Аумана, який застосовується для інтегрування множиннозначних функцій. Ми показуємо, що простір $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ є гільбертовим квазілінійним простором. Насамкінець ми означаємо оператори перенесення, модуляції та затримки, які є трьома основоположними множиннозначними операторами у гільбертовому квазілінійному просторі $L^2(\mathbb{R}, \Omega(\mathbb{C}))$.

Ключові слова і фрази: гільбертів квазілінійний простір, множиннозначна функція, інтеграл Аумана, перенесення, модуляція, затримка.