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MIXED PROBLEM FOR THE SINGULAR PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE

The scheme for solving of a mixed problem is proposed for a differential equation

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(c(x)\frac{\partial T}{\partial x}\right) - g(x)\,T$$

with coefficients $a(x)$, $g(x)$ that are the generalized derivatives of functions of bounded variation, $c(x) > 0$, $c^{-1}(x)$ is a bounded and measurable function. The boundary and initial conditions have the form

$$\begin{cases} p_1 T(0,\tau) + p_2 T_x^{[1]}(0,\tau) = \psi_1(\tau), \\ q_1 T(l,\tau) + q_2 T_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases}$$

$$T(x,0) = \varphi(x),$$

where $p_1 p_2 \leqslant 0$, $q_1 q_2 \geqslant 0$ and by $T_x^{[1]}(x,\tau)$ we denote the quasiderivative $c(x)\frac{\partial T}{\partial x}$. A solution of this problem we seek by the reduction method in the form of sum of two functions $T(x,\tau) = u(x,\tau) + v(x,\tau)$. This method allows to reduce solving of proposed problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions for the search of the function $u(x,\tau)$ and a mixed problem with zero boundary conditions for some inhomogeneous equation with an unknown function $v(x,\tau)$. The first of these problems is solved through the introduction of the quasiderivative. Fourier method and expansions in eigenfunctions of some boundary value problem for the second-order quasidifferential equation $\big(c(x)X'(x)\big)' - g(x)X(x) + \omega a(x)X(x) = 0$ are used for solving of the second problem. The function $v(x,\tau)$ is represented as a series in eigenfunctions of this boundary value problem. The results can be used in the investigation process of heat transfer in a multilayer plate.

Key words and phrases: mixed problem, quasiderivative, eigenfunctions, Fourier method.

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INTRODUCTION

Boundary problems for differential equations of heat conduction with smooth coefficients were studied quite comprehensively in the literature (e.g., see [7]). However, during the modeling of heat transfer processes, the boundary problems with piecewise continuous coefficients or coefficients that have generalized derivatives of discontinuous functions are often appeared. Such problems have already begun to be studied in the works [2, 5, 6].

The present paper deals with solving of a mixed problem for a partial differential equation of parabolic type with coefficients that are the generalized derivatives of functions of bounded variation. A reduction method [7] is used for solving of this problem. This method allows to

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reduce solving of this problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions and a mixed problem with zero boundary conditions for some inhomogeneous equation. Fourier method and expansions in eigenfunctions of some boundary value problem for the second-order quasidifferential equation are used for solving of the second of these problems. In this paper we consider a more general statement of the problem than in [2]. Moreover, it is proved the non-negativity of the eigenvalues, which is a necessary condition for the correctness of the description of the heat transfer process.

Quasidifferential equations are equations that contain terms of the form $(p(x)y^{(m)})^{(n)}$. These equations cannot be reduced to conventional differential equations by n-fold differentiation if the coefficient $p(x)$ is not sufficiently smooth. The introduction of quasiderivatives is used for their research [3].

1 Formulation of the problem

Consider the next mixed problem for a partial differential equation of parabolic type. It is necessary to find a solution $T(x,\tau)$ of the equation

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(c(x)\frac{\partial T}{\partial x}\right) - g(x)\,T \tag{1}$$

with boundary conditions

$$\begin{cases} p_1 T(0,\tau) + p_2 T_x^{[1]}(0,\tau) = \psi_1(\tau), \\ q_1 T(l,\tau) + q_2 T_x^{[1]}(l,\tau) = \psi_2(\tau) \end{cases} \tag{2}$$

and initial condition

$$T(x,0) = \varphi(x), \tag{3}$$

where $a(x) = b'(x)$, $g(x) = h'(x)$, $b(x)$, $h(x)$ are right continuous nondecreasing real functions of bounded variation on the interval $[0,l]$, $c(x) > 0$, $c^{-1}(x)$ is a bounded and measurable function on the interval $[0,l]$, $\varphi(x)$ is a continuous function on the interval $[0,l]$, $\psi_1(\tau)$ and $\psi_2(\tau)$ are continuously differentiable functions for $\tau \geqslant 0$, p_1, p_2, q_1, q_2 are real numbers, $p_1p_2 \leqslant 0$, $q_1q_2 \geqslant 0$. By $T_x^{[1]}(x,\tau) \overset{df}{=} c(x)\frac{\partial T}{\partial x}$ we denote the quasiderivative. The primes in the formulas $a(x) = b'(x)$, $g(x) = h'(x)$ stand for the generalized differentiation, and hence the functions $a(x)$, $g(x)$ are measures, i.e., a zero-order distributions on the space of continuous compactly supported functions [1].

A solution of problem (1)–(3) seek by the reduction method in the form of sum of two functions

$$T(x,\tau) = u(x,\tau) + v(x,\tau). \tag{4}$$

Any of functions u or v can be chosen by a special way, then another one will be determined uniquely.

2 Quasistationary boundary problem for $u(x,\tau)$

We define $u(x,\tau)$ as the solution of the boundary problem

$$\frac{\partial}{\partial x}\left(c(x)\frac{\partial u}{\partial x}\right) - g(x)\,u = 0, \tag{5}$$

$$\begin{cases} p_1 u(0,\tau) + p_2 u_x^{[1]}(0,\tau) = \psi_1(\tau), \\ q_1 u(l,\tau) + q_2 u_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases} \tag{6}$$

which is derived from problem (1)–(3), if τ is a parameter. Here the quasiderivative $u_x^{[1]}(x,\tau) \stackrel{df}{=} c(x)\frac{\partial u}{\partial x}$, then $\frac{\partial u}{\partial x} = \frac{u^{[1]}}{c(x)}$. With the help of the vector $\bar{u} = (u, u^{[1]})^T$ equation (5) is reduced to the system

$$\begin{pmatrix} u \\ u^{[1]} \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{c(x)} \\ g(x) & 0 \end{pmatrix} \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}. \tag{7}$$

Boundary conditions (6) are also represented in the vector form

$$P \cdot \bar{u}(0,\tau) + Q \cdot \bar{u}(l,\tau) = \bar{\Gamma}(\tau), \tag{8}$$

where

$$P = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ q_1 & q_2 \end{pmatrix}, \quad \bar{\Gamma}(\tau) = \begin{pmatrix} \psi_1(\tau) \\ \psi_2(\tau) \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}.$$

Suppose that we know the Cauchy matrix $B(x,s)$ of system (7). It can be constructed, for example, when the coefficients $c(x)$ and $g(x)$ are piecewise constant functions [4]. In the general case, it is necessary to know the fundamental system of solutions of system (7) for construction the Cauchy matrix [4]. By [4], equation (5) with arbitrary initial conditions $u(x_0,\tau) = u_0$, $u^{[1]}(x_0,\tau) = u_1$, $x_0 \in [0,l]$ has a unique solution in the class of absolutely continuous functions, and the quasiderivative $u^{[1]}$ of this solution has a bounded variation by the variable x on the interval $[0,l]$.

We have $\bar{u}(x,\tau) = B(x,0)\bar{u}_0$, where $\bar{u}_0 = \bar{u}(0,\tau)$. We shall determine $\bar{u}_0$. From boundary conditions (8) we obtain $P \cdot \bar{u}_0 + Q \cdot B(l,0) \cdot \bar{u}_0 = \bar{\Gamma}$ whence $\bar{u}_0 = (P + Q \cdot B(l,0))^{-1} \cdot \bar{\Gamma}$. Therefore,

$$\bar{u}(x,\tau) = B(x,0) \cdot (P + Q \cdot B(l,0))^{-1} \cdot \bar{\Gamma}(\tau). \tag{9}$$

3 MIXED PROBLEM FOR $v(x,\tau)$

We substitute $u(x,\tau)$ and $v(x,\tau)$ into equation (1)

$$a(x)\left(\frac{\partial u}{\partial \tau} + \frac{\partial v}{\partial \tau}\right) = \frac{\partial}{\partial x}\left(c(x)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)\right) - g(x)(u+v).$$

In consequence of (5) we have the equation

$$a(x)\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\left(c(x)\frac{\partial v}{\partial x}\right) - g(x)v - a(x)\frac{\partial u}{\partial \tau}. \tag{10}$$

According to formula (9) the derivative $\frac{\partial u}{\partial \tau}$ is a continuous function of the variable x on $[0,l]$ and so the last term in equation (10) is correct.

By taking into account formula (4), we define the boundary conditions for v from conditions (2)

$$p_1 u(0,\tau) + p_2 u_x^{[1]}(0,\tau) + p_1 v(0,\tau) + p_2 v_x^{[1]}(0,\tau) = \psi_1(\tau),$$
$$q_1 u(l,\tau) + q_2 u_x^{[1]}(l,\tau) + q_1 v(l,\tau) + q_2 v_x^{[1]}(l,\tau) = \psi_2(\tau).$$

By virtue of (6), we obtain

$$\begin{cases} p_1 v(0,\tau) + p_2 v_x^{[1]}(0,\tau) = 0, \\ q_1 v(l,\tau) + q_2 v_x^{[1]}(l,\tau) = 0. \end{cases} \tag{11}$$

The initial condition is determined similarly

$$v(x,0) = \varphi(x) - u(x,0) \stackrel{df}{=} \tilde{\varphi}(x). \tag{12}$$

4 FOURIER METHOD AND EIGENVALUE PROBLEM

We search for non-trivial solutions of the homogeneous differential equation

$$a(x)\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\left(c(x)\frac{\partial v}{\partial x}\right) - g(x)v$$

with boundary conditions (11) in the form

$$v(x,\tau) = e^{-\omega\tau}X(x), \tag{13}$$

where ω is a parameter, and $X(x)$ is a function. Then

$$-\omega a(x)e^{-\omega\tau}X(x) = \big(c(x)X'(x)\big)'e^{-\omega\tau} - g(x)X(x)e^{-\omega\tau},$$

whence we get the quasidifferential equation

$$\big(c(x)X'(x)\big)' - g(x)X(x) + \omega a(x)X(x) = 0. \tag{14}$$

Substituting formula (13) in boundary conditions (11), we obtain

$$\begin{cases} p_1 X(0) + p_2 X^{[1]}(0) = 0, \\ q_1 X(l) + q_2 X^{[1]}(l) = 0. \end{cases} \tag{15}$$

We denote by ω_k the eigenvalues of boundary problem (14), (15). Let $X_k(\omega_k, x)$ be the corresponding eigenfunctions, $k = 1, 2, \ldots, \infty$.

By [8], all eigenvalues ω_k of boundary problem (14), (15) are real, there are a countable number of them, and their set has not a finite limit point. The eigenfunctions $X_k(\omega_k, x)$ that are corresponded to the different eigenvalues are orthogonal in the sense

$$\int_0^l X_m(\omega_m, x)X_n(\omega_n, x)db(x) = 0, \quad \omega_m \neq \omega_n.$$

We now prove that all of the eigenvalues ω_k of boundary value problem (14), (15) are non-negative with the coefficients imposed in section 1.

To do this, we multiply both parts of the equation

$$\big(c(x)X_k'(x)\big)' - g(x)X_k(x) + \omega_k a(x)X_k(x) = 0$$

by $X_k(x)$

$$\big(c(x)X_k'(x)\big)'X_k(x) - g(x)X_k^2(x) + \omega_k a(x)X_k^2(x) = 0.$$

Then, taking into account that $X_k^{[1]}(x) = c(x)X_k'(x)$, after transformations we obtain

$$\omega_k a(x)X_k^2(x) = -\big(X_k^{[1]}(x)X_k(x)\big)' + X_k^{[1]}(x)X_k'(x) + g(x)X_k^2(x).$$

By integrating both parts of the obtained ratio in the range from 0 to l, we will have

$$\begin{aligned}\omega_k \int_0^l X_k^2(x)db(x) &= -X_k^{[1]}(l)X_k(l) + X_k^{[1]}(0)X_k(0)\\ &\quad + \int_0^l c(x)(X_k'(x))^2dx + \int_0^l X_k^2(x)dh(x).\end{aligned} \tag{16}$$

Since the functions $X_k(x)$ are absolutely continuous as a result of [4], and their quasiderivatives $X_k^{[1]}(x)$ have a bounded variation on the interval $[0,l]$, we see that all of the above transformations have sense.

All integrals in formula (16) are non-negative. If $p_1 = 0$ or $p_2 = 0$, then $X_k^{[1]}(0)X_k(0) = 0$. If $p_1p_2 < 0$, then from the first condition of system (15) we have $X_k^{[1]}(0) = -\frac{p_1}{p_2}X_k(0)$. Then $X_k^{[1]}(0)X_k(0) = -\frac{p_1}{p_2}X_k^2(0) \geqslant 0$. Similarly, if $q_1 = 0$ or $q_2 = 0$, then $X_k^{[1]}(l)X_k(l) = 0$, otherwise $X_k^{[1]}(l)X_k(l) = -\frac{q_1}{q_2}X_k^2(l) \leqslant 0$. Consequently, it follows from formula (16) that all $\omega_k \geqslant 0$.

5 METHOD OF THE EIGENFUNCTIONS

We seek $v(x,\tau)$ in the form of the series

$$v(x,\tau) = \sum_{k=1}^{\infty} t_k(\tau)X_k(\omega_k, x), \tag{17}$$

where $X_k(\omega_k, x)$ are the eigenfunctions of boundary problem (14), (15). We substitute formula (17) into equation (10)

$$\begin{aligned}a(x)\frac{\partial}{\partial\tau}\left(\sum_{k=1}^{\infty} t_k(\tau)X_k\right) &= \frac{\partial}{\partial x}\left(c(x)\frac{\partial}{\partial x}\left(\sum_{k=1}^{\infty} t_k(\tau)X_k\right)\right)\\ &\quad - g(x)\sum_{k=1}^{\infty} t_k(\tau)X_k - a(x)\frac{\partial u}{\partial\tau},\end{aligned}$$

whence, under the assumption of uniform convergence of series (17) and series derived from it by differentiation by x or τ, we have

$$a(x)\sum_{k=1}^{\infty} t_k'(\tau)X_k = \sum_{k=1}^{\infty} t_k(\tau)\Big(\big(c(x)X_k'\big)' - g(x)X_k\Big) - a(x)\frac{\partial u}{\partial\tau}.$$

As a result of equation (14) there is equality

$$\big(c(x)X_k'\big)' - g(x)X_k = -\omega_k a(x)X_k,$$

then

$$a(x)\sum_{k=1}^{\infty} t_k'(\tau)X_k = -\sum_{k=1}^{\infty} t_k(\tau)\omega_k a(x)X_k - a(x)\frac{\partial u}{\partial\tau}.$$

Therefore,

$$\sum_{k=1}^{\infty}\left(t_k'(\tau)+\omega_k t_k(\tau)\right)X_k=-\frac{\partial u}{\partial\tau}. \tag{18}$$

We expand the known function $\frac{\partial u}{\partial \tau}$ in a series in the eigenfunctions of boundary problem (14), (15):

$$\frac{\partial u}{\partial\tau}=\sum_{k=1}^{\infty}d_k(\tau)X_k(\omega_k,x), \tag{19}$$

where

$$d_k(\tau)=\frac{1}{\|X_k\|}\int_0^l\frac{\partial u}{\partial\tau}X_k(\omega_k,x)db(x),\quad \|X_k\|=\int_0^l X_k^2(\omega_k,x)db(x).$$

By substituting formula (19) into (18), we obtain

$$t_k'(\tau)+\omega_k t_k(\tau)=-d_k(\tau),\quad k=1,2,\dots,\infty. \tag{20}$$

Since formulas (12) and (17), we have

$$v(x,0)=\sum_{k=1}^{\infty}t_k(0)X_k(\omega_k,x)\equiv\tilde{\varphi}(x).$$

We expand the function $\tilde{\varphi}(x)$ in a series in the eigenfunctions

$$\tilde{\varphi}(x)=\sum_{k=1}^{\infty}\varphi_k X_k(\omega_k,x),\quad \varphi_k=\frac{1}{\|X_k\|}\int_0^l\tilde{\varphi}(x)X_k(\omega_k,x)db(x).$$

Consequently,

$$t_k(0)=\varphi_k,\quad k=1,2,\dots,\infty. \tag{21}$$

Then for all positive integer k we have Cauchy problems (20), (21) for ordinary differential equations.

General solutions of linear inhomogeneous equations (20) acquire the formulas

$$t_k(\tau)=\left(C_k-\int_0^{\tau}d_k(s)e^{\omega_k s}ds\right)e^{-\omega_k\tau},$$

where C_k are arbitrary constants. Therefore, by using initial conditions (21), we find for each positive integer k the solution of the corresponding Cauchy problem

$$t_k(\tau)=\varphi_k e^{-\omega_k\tau}-\int_0^{\tau}d_k(s)e^{\omega_k(s-\tau)}ds.$$

Then, by virtue of formula (17), we obtain

$$v(x,\tau)=\sum_{k=1}^{\infty}\left(\varphi_k e^{-\omega_k\tau}-\int_0^{\tau}d_k(s)e^{\omega_k(s-\tau)}ds\right)X_k(\omega_k,x).$$

Thus, by using the reduction method, Fourier method and the expansion in a series in eigenfunctions, we built the solution of the boundary problem for the parabolic type partial differential equation with distributions. The results can be used in the investigation of the process of heat transfer in a multilayer plate.

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Запропоновано схему розв'язування мішаної задачі для диференціального рівняння

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(c(x)\frac{\partial T}{\partial x}\right) - g(x)\,T$$

з коефіцієнтами $a(x)$, $g(x)$, які є узагальненими похідними функцій обмеженої варіації, $c(x) > 0$, $c^{-1}(x)$ — обмежена і вимірна функція. Крайові і початкова умови мають вигляд

$$\begin{cases} p_1 T(0,\tau) + p_2 T_x^{[1]}(0,\tau) = \psi_1(\tau), \\ q_1 T(l,\tau) + q_2 T_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases}$$

$$T(x,0) = \varphi(x),$$

де $p_1p_2 \leqslant 0$, $q_1q_2 \geqslant 0$, а через $T_x^{[1]}(x,\tau)$ позначено квазіпохідну $c(x)\frac{\partial T}{\partial x}$. Розв'язок цієї задачі шукається методом редукції у вигляді суми двох функцій $T(x,\tau) = u(x,\tau) + v(x,\tau)$. Цей метод дає змогу звести розв'язування поставленої задачі до розв'язування двох задач: крайової квазістаціонарної задачі з початковими і крайовими умовами для відшукання функції $u(x,\tau)$ і мішаної задачі з нульовими крайовими умовами для деякого неоднорідного рівняння з невідомою функцією $v(x,\tau)$. Перша з цих задач розв'язується з допомогою введення квазіпохідної. Для розв'язування другої задачі застосовується метод Фур'є і розвинення за власними функціями деякої крайової задачі для квазідиференціального рівняння другого порядку $\big(c(x)X'(x)\big)' - g(x)X(x) + \omega a(x)X(x) = 0$. Функція $v(x,\tau)$ подається у вигляді ряду за власними функціями цієї крайової задачі. Отримані результати можна використовувати для дослідження процесу теплопередачі в багатошаровій плиті.

Ключові слова і фрази: мішана задача, квазіпохідна, власні функції, метод Фур'є.