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SOME PROPERTIES OF SHIFT OPERATORS ON ALGEBRAS GENERATED BY $*$-POLYNOMIALS

A $*$-polynomial is a function on a complex Banach space X, which is a sum of so-called (p,q)-polynomials. In turn, for non-negative integers p and q, a (p,q)-polynomial is a function on X, which is the restriction to the diagonal of some mapping, defined on the Cartesian power X^{p+q}, which is linear with respect to every of its first p arguments and antilinear with respect to every of its other q arguments. The set of all continuous $*$-polynomials on X form an algebra, which contains the algebra of all continuous polynomials on X as a proper subalgebra. So, completions of this algebra with respect to some natural norms are wider classes of functions than algebras of holomorphic functions. On the other hand, due to the similarity of structures of $*$-polynomials and polynomials, for the investigation of such completions one can use the technique, developed for the investigation of holomorphic functions on Banach spaces.

We investigate the Fréchet algebra of functions on a complex Banach space, which is the completion of the algebra of all continuous $*$-polynomials with respect to the countable system of norms, equivalent to norms of the uniform convergence on closed balls of the space. We establish some properties of shift operators (which act as the addition of some fixed element of the underlying space to the argument of a function) on this algebra. In particular, we show that shift operators are well-defined continuous linear operators. Also we prove some estimates for norms of values of shift operators. Using these results, we investigate one special class of functions from the algebra, which is important in the description of the spectrum (the set of all maximal ideals) of the algebra.

Key words and phrases: (p,q)-polynomial, $*$-polynomial, shift operator.

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INTRODUCTION

$*$-Polynomials (see definition below) are natural generalizations of usual polynomials on complex Banach spaces. Such objects were firstly studied in [4]. It is known that completions of the algebra of all continuous polynomials on some complex Banach space with respect to topologies of uniform convergence on some bounded subsets of the space are algebras of holomorphic functions. On the other hand, the analogical completions of the algebra of all continuous $*$-polynomials contain wider classes of continuous functions. Except of holomorphic functions, they can contain functions, which are complex-conjugate to holomorphic. Also, as it is shown in [3], such algebras can contain functions, which cannot be represented as linear combination of products of holomorphic functions and complex-conjugate to holomorphic functions. Thus, such algebras can contain the wide enough class of continuous functions on a complex Banach space. The algebraic structure gives the opportunity to consider the elements

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of the algebra as continuous functions on the spectrum (the set of maximal ideals) of the algebra. In the description of spectra of algebras of functions on Banach spaces the so-called convolution operation on the spectrum plays an important role. In turn, the convolution operation on the spectrum is defined with aid of the so-called shift operators, defined on the algebra. Shift operators for algebras of holomorphic functions on Banach spaces and their applications for the spectra were investigated in [1], [2], [7], [8].

In this work we establish some properties of shift operators on the Fréchet algebra of functions on a complex Banach space, which is the completion of the algebra of all continuous $*$-polynomials with respect to the countable set of norms, which are equivalent to norms of the uniform convergence on closed balls with rational radii, centered at 0. We show that shift operators are well-defined continuous linear operators. Also we investigate one special class of functions from the algebra, constructed by using of the composition of continuous linear functionals with shift operators. Such classes of functions play an important role in the description of spectra of algebras of functions on Banach spaces.

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{Q}_+$ be the set of all positive rationals. Let X be a complex Banach space. A mapping $A : X^{p+q} \to \mathbb{C}$, where $p, q \in \mathbb{N} \cup \{0\}$ are such that $p \neq 0$ or $q \neq 0$, is called a (p, q)-linear mapping, if A is linear with respect to every of first p arguments and it is antilinear with respect to every of last q arguments. A (p, q)-linear mapping, which is invariant with respect to permutations of its first p arguments and last q arguments separately, is called symmetric. A mapping $P : X \to \mathbb{C}$ is called a (p, q)-polynomial if there exists a symmetric (p, q)-linear mapping $A_P : X^{p+q} \to \mathbb{C}$ such that P is the restriction to the diagonal of A_P, i.e.,

$$P(x) = A_P(\underbrace{x, \ldots, x}_{p+q})$$

for every $x \in X$. The mapping A_P is called the symmetric (p, q)-linear mapping, associated with P. (p, q)-polynomials and (p, q)-linear mappings were studied in [5] and [6].

Note that for (p, q)-polynomials the following analog of the Binomial formula holds:

$$P(x+y) = \sum_{j=0}^{p} \sum_{k=0}^{q} \frac{p!q!}{j!(p-j)!k!(q-k)!} A_P(x^j, y^{p-j}, x^k, y^{q-k}), \tag{1}$$

where

$$A_P(x^j, y^{p-j}, x^k, y^{q-k}) = A_P(\underbrace{x, \ldots, x}_{j}, \underbrace{y, \ldots, y}_{p-j}, \underbrace{x, \ldots, x}_{k}, \underbrace{y, \ldots, y}_{q-k})$$

for every $x, y \in X$. Let us denote by $\mathcal{P}({}^{pq}X)$ the space of all continuous (p, q)-polynomials with norm

$$\|P\| = \sup_{\|x\| \leq 1} |P(x)|.$$

Also, for convenience, let $\mathcal{P}({}^{00}X) = \mathbb{C}$.

A mapping $P : X \to \mathbb{C}$ is called a $*$-polynomial if it can be represented in the form

$$P = \sum_{p=0}^{M} \sum_{q=0}^{N} P_{pq},$$

where $M, N \in \mathbb{N} \cup \{0\}$ and $P_{pq} \in \mathcal{P}({}^{pq}X)$. Denote $\mathcal{P}_*(X)$ the algebra of all continuous $*$-polynomials on the space X.

1 The Main Result

Let

$$\{\|\cdot\|_r :\ r \in (0,+\infty)\} \tag{2}$$

be the set of norms on $\mathcal{P}_*(X)$ such that

1. $\|PQ\|_r \le \|P\|_r \|Q\|_r$ for every $P, Q \in \mathcal{P}_*(X)$ and $r \in (0,+\infty)$.
2. There exist functions $(0,+\infty) \ni t \mapsto c_t \in (0,+\infty)$ and $(0,+\infty) \ni t \mapsto C_t \in (0,+\infty)$ such that $\inf_{t\in[a,b]} c_t > 0$ and $\sup_{t\in[a,b]} C_t < +\infty$ for every $b > a > 0$, and

$$c_r \sup_{\|x\|\le r} |P(x)| \le \|P\|_r \le C_r \sup_{\|x\|\le r} |P(x)|$$

for every $r \in (0,+\infty)$ and $P \in \mathcal{P}_*(X)$.

Let

$$\{\|\cdot\|_r :\ r \in \mathbb{Q}_+\} \tag{3}$$

be the subset of the set of norms (2). Note that the set (3) is countable. Let $\mathcal{A}(X)$ be the completion of $\mathcal{P}_*(X)$ with respect to the metric, generated by the set of norms (3). It can be checked that $\mathcal{A}(X)$ is a Fréchet algebra of functions on X. By the continuity of norms from (3),

$$c_r \sup_{\|x\|\le r} |f(x)| \le \|f\|_r \le C_r \sup_{\|x\|\le r} |f(x)| \tag{4}$$

for every $r \in \mathbb{Q}_+$ and $f \in \mathcal{A}(X)$.

Theorem 1. *(i). For every $x \in X$ the operator $T_x : \mathcal{A}(X) \to \mathcal{A}(X)$, defined by*

$$(T_x f)(y) = f(x+y),$$

where $f \in \mathcal{A}(X)$ and $y \in X$, is a well-defined continuous linear operator such that

$$\|T_x f\|_r \le C_r c^{-1}_{r+\|x\|} \|f\|_{r+\|x\|},$$

for every $f \in \mathcal{A}(X)$ and $r \in \mathbb{Q}_+$.

(ii). For every $f \in \mathcal{A}(X)$ and for every continuous linear functional $\varphi : \mathcal{A}(X) \to \mathbb{C}$, the function $h_{\varphi,f} : X \to \mathbb{C}$, defined by

$$h_{\varphi,f}(x) = \varphi(T_x f),$$

belongs to $\mathcal{A}(X)$, and

$$|h_{\varphi,f}(x)| \le K C_s c^{-1}_{s+\|x\|} \|f\|_{s+\|x\|},$$

for every $x \in X$ and for every $s \in \mathbb{Q}_+$ such that φ is continuous with respect to $\|\cdot\|_s$, where $K = \sup_{\|f\|_s \le 1} |\varphi(f)|$.

Proof. (i). Let $x \in X$. For every $f \in \mathcal{A}(X)$, since $(T_x f)(y) = f(x+y)$ and f is well-defined at $x+y$, it follows that $T_x f$ is well-defined at y. Also note that for every $r \in \mathbb{Q}_+$

$$\sup_{\|y\|\le r} |f(x+y)| \le \sup_{\|z\|\le r+\|x\|} |f(z)| \le c^{-1}_{r+\|x\|} \|f\|_{r+\|x\|},$$

i.e.,

$$\sup_{\|y\|\le r} |(T_x f)(y)| \le c_{r+\|x\|}^{-1} \|f\|_{r+\|x\|}. \tag{5}$$

Let $P \in \mathcal{P}_*(X)$. Let us show that $T_x P \in \mathcal{P}_*(X)$. Let $P = \sum_{p=0}^{M} \sum_{q=0}^{N} P_{pq}$, where $P_{pq} \in \mathcal{P}({}^{pq}X)$. By (1),

$$(T_x P)(y) = \sum_{p=0}^{M} \sum_{q=0}^{N} \sum_{j=0}^{p} \sum_{k=0}^{q} \frac{p!q!}{j!(p-j)!k!(q-k)!} A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k}),$$

where $A_{P_{pq}}$ is the symmetric (p,q)-linear mapping, associated with the (p,q)-polynomial P_{pq} for every $p \in \{0,\dots,M\}$ and $q \in \{0,\dots,N\}$. Note that for fixed $x \in X$ the function $A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k})$ is a continuous $(p-j, q-k)$-polynomial with respect to y. Therefore, $T_x P$ is a continuous $*$-polynomial.

Let $f \in \mathcal{A}(X)$. Let us show that $T_x f \in \mathcal{A}(X)$. Since $\mathcal{P}_*(X)$ is dense in $\mathcal{A}(X)$, it follows that there exists the sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{P}_*(X)$, which converges to f with respect to every norm from (3). Consider the sequence $\{T_x f_n\}_{n=1}^{\infty}$. Since $f_n \in \mathcal{P}_*(X)$, it follows that $T_x f_n \in \mathcal{P}_*(X)$. Thus, $\{T_x f_n\}_{n=1}^{\infty} \subset \mathcal{P}_*(X) \subset \mathcal{A}(X)$. Let us show that the sequence $\{T_x f_n\}_{n=1}^{\infty}$ is fundamental in $\mathcal{A}(X)$. Let $r \in \mathbb{Q}_+$. For $m, n \in \mathbb{N}$, by (4),

$$\|T_x f_m - T_x f_n\|_r \le C_r \sup_{\|y\|\le r} |(T_x f_m)(y) - (T_x f_n)(y)|.$$

By (5),

$$\sup_{\|y\|\le r} |(T_x f_m)(y) - (T_x f_n)(y)| \le c_{r+\|x\|}^{-1} \|f_m - f_n\|_{r+\|x\|}.$$

Thus,

$$\|T_x f_m - T_x f_n\|_r \le C_r c_{r+\|x\|}^{-1} \|f_m - f_n\|_{r+\|x\|}.$$

Since the sequence $\{f_n\}_{n=1}^{\infty}$ is fundamental, it follows that the sequence $\{T_x f_n\}_{n=1}^{\infty}$ is fundamental. Since the algebra $\mathcal{A}(X)$ is complete, it follows that there exists $g \in \mathcal{A}(X)$ such that the sequence $\{T_x f_n\}_{n=1}^{\infty}$ converges to g. Let $y \in X$. Let us show that $(T_x f)(y) = g(y)$. Let $\rho \in \mathbb{Q}_+$ be such that $\rho > \|y\|$. Since the sequence $\{T_x f_n\}_{n=1}^{\infty}$ converges to g, it follows that $\{\|T_x f_n - g\|_\rho\}_{n=1}^{\infty}$ converges to 0. By (4),

$$\sup_{\|z\|\le \rho} |(T_x f_n)(z) - g(z)| \le c_\rho^{-1} \|T_x f_n - g\|_\rho.$$

Therefore,

$$|(T_x f_n)(y) - g(y)| \le c_\rho^{-1} \|T_x f_n - g\|_\rho.$$

Consequently, the sequence $\{(T_x f_n)(y)\}_{n=1}^{\infty}$ converges to $g(y)$. On the other hand, by (5),

$$\sup_{\|z\|\le \rho} |(T_x f)(z) - (T_x f_n)(z)| \le c_{\rho+\|x\|}^{-1} \|f - f_n\|_{\rho+\|x\|}.$$

Therefore,

$$|(T_x f)(y) - (T_x f_n)(y)| \le c_{\rho+\|x\|}^{-1} \|f - f_n\|_{\rho+\|x\|}.$$

Since $\|f - f_n\|_{\rho+\|x\|} \to 0$ as $n \to \infty$, it follows that the sequence $\{(T_x f_n)(y)\}_{n=1}^{\infty}$ converges to $(T_x f)(y)$. Therefore, $(T_x f)(y) = g(y)$. Thus, $T_x f = g$ and, consequently, $T_x f \in \mathcal{A}(X)$.

By (4) and (5),

$$\|T_x f\|_r \leq C_r \sup_{\|y\| \leq r} |(T_x f)(y)| \leq C_r c_{r+\|x\|}^{-1} \|f\|_{r+\|x\|} \tag{6}$$

for every $r \in \mathbb{Q}_+$.

(ii). Let $f \in \mathcal{A}(X)$ and $\varphi \in \mathcal{A}(X)'$. Note that the function $h_{\varphi,f}(x) = \varphi(T_x f)$ is well-defined at every point $x \in X$, because $T_x f$ belongs to $\mathcal{A}(X)$ and φ is well-defined on $\mathcal{A}(X)$.

Since φ is a continuous linear functional on $\mathcal{A}(X)$, there exists $s \in \mathbb{Q}_+$ such that φ is continuous with respect to the norm $\| \cdot \|_s$. Therefore, for every $f \in \mathcal{A}(X)$,

$$|\varphi(f)| \leq K\|f\|_s, \tag{7}$$

where $K = \sup_{\|f\|_s \leq 1} |\varphi(f)|$. By (7) and (6),

$$|\varphi(T_x f)| \leq K\|T_x f\|_s \leq K C_s c_{s+\|x\|}^{-1} \|f\|_{s+\|x\|},$$

i.e.,

$$|h_{\varphi,f}(x)| \leq K C_s c_{s+\|x\|}^{-1} \|f\|_{s+\|x\|}. \tag{8}$$

Let $P = \sum_{p=0}^{M} \sum_{q=0}^{N} P_{pq}$ be a continuous $*$-polynomial. Let us show that a function $h_{\varphi,P}(x) = \varphi(T_x P)$ is a continuous $*$-polynomial. By (1), taking into account the linearity of φ, we have

$$h_{\varphi,P}(x) = \sum_{p=0}^{M} \sum_{q=0}^{N} \sum_{j=0}^{p} \sum_{k=0}^{q} \frac{p!q!}{j!(p-j)!k!(q-k)!} \varphi(y \mapsto A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k})).$$

Note that the function

$$w_{p,q,j,k}(x) = \varphi(y \mapsto A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k}))$$

is the restriction to the diagonal of (j,k)-linear symmetric mapping

$$B(x_1, \ldots, x_j, x_{j+1}, \ldots, x_{j+k}) = \varphi(y \mapsto A_{P_{pq}}(x_1, \ldots, x_j, y^{p-j}, x_{j+1}, \ldots, x_{j+k}, y^{q-k})),$$

therefore, $w_{p,q,j,k}$ is a continuous (j,k)-polynomial. Hence, $h_{\varphi,P}$ is a continuous $*$-polynomial.

Let us show that $h_{\varphi,f} \in \mathcal{A}(X)$ for every $f \in \mathcal{A}(X)$. Since $f \in \mathcal{A}(X)$ and $\mathcal{P}_*(X)$ is dense in $\mathcal{A}(X)$, there exists the sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{P}_*(X)$, which converges to f. Since $f_n \in \mathcal{P}_*(X)$, it follows that $h_{\varphi,f_n} \in \mathcal{P}_*(X)$. Therefore, the sequence $\{h_{\varphi,f_n}\}_{n=1}^{\infty}$ is contained in $\mathcal{P}_*(X)$. Let us show that this sequence is fundamental in $\mathcal{A}(X)$. Let $r \in \mathbb{Q}_+$. For $m, n \in \mathbb{N}$, by (8),

$$|h_{\varphi,f_m}(x) - h_{\varphi,f_n}(x)| = |h_{\varphi,f_m - f_n}(x)| \leq K C_s c_{s+\|x\|}^{-1} \|f_m - f_n\|_{s+\|x\|}.$$

Therefore,

$$\|h_{\varphi,f_m} - h_{\varphi,f_n}\|_r \leq C_r \sup_{\|x\| \leq r} |h_{\varphi,f_m}(x) - h_{\varphi,f_n}(x)| \leq K C_r C_s \sup_{\|x\| \leq r} c_{s+\|x\|}^{-1} \|f_m - f_n\|_{s+\|x\|}.$$

Note that

$$\sup_{\|x\| \leq r} c_{s+\|x\|}^{-1} \|f_m - f_n\|_{s+\|x\|} \leq \Big(\sup_{\|x\| \leq r} c_{s+\|x\|}^{-1} \Big) \Big(\sup_{\|x\| \leq r} \|f_m - f_n\|_{s+\|x\|} \Big)$$

and
$$\sup_{\|x\|\le r} c_{s+\|x\|}^{-1} = \Big(\inf_{\|x\|\le r} c_{s+\|x\|}\Big)^{-1} = \Big(\inf_{t\in[s,s+r]} c_t\Big)^{-1},$$
which is finite, because $\inf_{t\in[s,s+r]} c_t > 0$. By (4),
$$\sup_{\|x\|\le r} \|f_m - f_n\|_{s+\|x\|} \le \sup_{\|x\|\le r} C_{s+\|x\|} \sup_{\|y\|\le s+\|x\|} |f_m(y) - f_n(y)| \le$$
$$\le \sup_{\|x\|\le r} C_{s+\|x\|} \sup_{\|y\|\le s+r} |f_m(y) - f_n(y)| = \Big(\sup_{t\in[s,s+r]} C_t\Big)\|f_m - f_n\|_{s+r}.$$
Thus,
$$\|h_{\varphi,f_m} - h_{\varphi,f_n}\|_r \le KC_rC_s\Big(\inf_{t\in[s,s+r]} c_t\Big)^{-1}\Big(\sup_{t\in[s,s+r]} C_t\Big)\|f_m - f_n\|_{s+r}.$$
Therefore, since the sequence $\{f_n\}_{n=1}^{\infty}$ is fundamental, it follows that the sequence $\{h_{\varphi,f_n}\}_{n=1}^{\infty}$ is fundamental. Since $\mathcal{A}(X)$ is complete, there exists $v \in \mathcal{A}(X)$ such that the sequence $\{h_{\varphi,f_n}\}_{n=1}^{\infty}$ converges to v. Let us show that $h_{\varphi,f} = v$. Let $x \in X$. Let $\rho \in \mathbb{Q}_+$ be such that $\rho > \|x\|$. By (4),
$$\sup_{\|z\|\le\rho} |h_{\varphi,f_n}(z) - v(z)| \le c_\rho^{-1}\|h_{\varphi,f_n} - v\|_\rho.$$
Therefore,
$$|h_{\varphi,f_n}(x) - v(x)| \le c_\rho^{-1}\|h_{\varphi,f_n} - v\|_\rho.$$
Since $\|h_{\varphi,f_n} - v\|_\rho \to 0$ as $n \to \infty$, it follows that the sequence $\{h_{\varphi,f_n}(x)\}_{n=1}^{\infty}$ converges to $v(x)$. On the other hand, by the continuity of φ and T_x, since $f_n \to f$ as $n \to \infty$, we have $\varphi(T_x(f_n)) \to \varphi(T_x(f))$ as $n \to \infty$, i.e., $h_{\varphi,f_n}(x) \to h_{\varphi,f}(x)$ as $n \to \infty$. Therefore, $h_{\varphi,f}(x) = v(x)$. Thus, $h_{\varphi,f} = v$ and, consequently, $h_{\varphi,f} \in \mathcal{A}(X)$. $\square$

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Василишин Т.В. *Деякі властивості операторів зсуву на алгебрах, породжених $*$-поліномами* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 206–212.

$*$-Поліном — це функція на комплексному банаховому просторі X, яка є сумою так званих (p,q)-поліномів. У свою чергу, для невід'ємних чисел p і q, (p,q)-поліном — це функція на просторі X, яка є звуженням на діагональ деякого відображення, визначеного на декартовому степені X^{p+q}, яке є лінійним відносно кожного із своїх перших p аргументів і антилінійним відносно кожного із решти q своїх аргументів. Множина всіх неперервних $*$-поліномів на просторі X утворює алгебру, яка містить алгебру всіх неперервних поліномів на просторі X як власну підалгебру. Таким чином, поповнення цієї алгебри відносно деяких природних норм є ширшими класами функцій, ніж алгебри аналітичних функцій. З іншого боку, завдяки подібності будови $*$-поліномів і поліномів, для дослідження таких поповнень можна використовувати техніку, розроблену для дослідження аналітичних функцій на банахових просторах.

У роботі досліджується алгебра Фреше функцій на комплексному банаховому просторі, яка є поповненням алгебри всіх неперервних $*$-поліномів відносно зліченної системи норм, еквівалентних до норм рівномірної збіжності на замкнених кулях простору. Встановлено деякі властивості оператора зсуву (який діє як додавання деякого фіксованого елемента простору до аргументу функції) на цій алгебрі. Зокрема, показано, що оператори зсуву є добре визначеними неперервними лінійними операторами. Також доведено деякі оцінки для норм значень операторів зсуву. Використовуючи ці результати, досліджено один спеціальний клас функцій із алгебри, який є важливим для опису спектра (множини всіх максимальних ідеалів) алгебри.

Ключові слова і фрази: (p,q)-поліном, $*$-поліном, оператор зсуву.