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THE INVERSE AND DERIVATIVE CONNECTING PROBLEMS FOR SOME HYPERGEOMETRIC POLYNOMIALS

Given two polynomial sets $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ such that $\deg(P_n(x)) = \deg(Q_n(x)) = n$. The so-called connection problem between them asks to find coefficients $\alpha_{n,k}$ in the expression $Q_n(x) = \sum_{k=0}^{n} \alpha_{n,k} P_k(x)$. The connection problem for different types of polynomials has a long history, and it is still of interest. The connection coefficients play an important role in many problems in pure and applied mathematics, especially in combinatorics, mathematical physics and quantum chemical applications. For the particular case $Q_n(x) = x^n$ the connection problem is called the inversion problem associated to $\{P_n(x)\}_{n\geq 0}$. The particular case $Q_n(x) = P'_{n+1}(x)$ is called the derivative connecting problem for polynomial family $\{P_n(x)\}_{n\geq 0}$. In this paper, we give a closed-form expression of the inversion and the derivative coefficients for hypergeometric polynomials of the form

$$
{}_2F_1\left[\begin{matrix}-n, a\\ b\end{matrix}\,\middle|\, z\right], \quad {}_2F_1\left[\begin{matrix}-n, n+a\\ b\end{matrix}\,\middle|\, z\right], \quad {}_2F_1\left[\begin{matrix}-n, a\\ \pm n+b\end{matrix}\,\middle|\, z\right],
$$

where ${}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, z\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$ is the Gauss hypergeometric function and $(x)_n$ denotes the Pochhammer symbol defined by $(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n > 0. \end{cases}$

All polynomials are considered over the field of real numbers.

Key words and phrases: connection problem, inversion problem, derivative connecting problem, connecting coefficients, hypergeometric functions, hypergeometric polynomials.

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INTRODUCTION

Given two polynomial sets $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ such that

$$
\deg(P_n(x)) = \deg(Q_n(x)) = n.
$$

The connection problem between them consists in finding the coefficients $\alpha_{n,k}$ in the expansion

$$
Q_n(x) = \sum_{k=0}^{n} \alpha_{n,k} P_k(x).
$$

For the particular case $Q_n(x) = x^n$ the connection problem is called the inversion problem associated to $\{P_n(x)\}_{n\geq 0}$. The particular case $Q_n(x) = P'_{n+1}(x)$ is called the derivative connecting problem for polynomial family $\{P_n(x)\}_{n\geq 0}$.

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The study of such a problem has attracted lot of interest in the last few years. The inverse problem for classical orthogonal polynomials are considered in [6], for more general case see [7]. The connection coefficients have been computed explicitly for classical orthogonal polynomials in [6] and [8].

The derivation connection problem (with respect to parameter derivatives) for hypergeometric polynomials ${}_2F_1\left[\begin{matrix}-n & a\\ & b\end{matrix}\middle| z\right]$ was solved in [9]. In [10, 11] the first author solved the derivation connection problem for the Fibonacci, Lucas and Kravchuk polynomials and use the solutions to produce new combinatorial identities for these polynomials.

Our aim in this paper is to compute the inversion and derivative connection coefficients for hypergeometric polynomials of the forms

$${}_2F_1\left[\begin{matrix}-n & a\\ & b\end{matrix}\middle| z\right],\quad {}_2F_1\left[\begin{matrix}-n & n+a\\ & b\end{matrix}\middle| z\right],\quad {}_2F_1\left[\begin{matrix}-n & a\\ & \pm n+b\end{matrix}\middle| z\right],$$

where

$${}_2F_1\left[\begin{matrix}a & b\\ & c\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

is the Gauss hypergeometric function.

The main results of this paper are gathered together in the following two theorems.

Theorem 1. *The following identities hold:*

$$\begin{aligned}
&(i)\quad z^n=\frac{(b)_n}{(a)_n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\,{}_2F_1\left[\begin{matrix}-i & a\\ & b\end{matrix}\middle| z\right],\\
&(ii)\quad z^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}(a+2i)\frac{(b)_n}{(a+i)_{n+1}}\,{}_2F_1\left[\begin{matrix}-i & i+a\\ & b\end{matrix}\middle| z\right],\\
&(iii)\quad z^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}(b+2n-1)\frac{(b+i)_{n-1}}{(a)_n}\,{}_2F_1\left[\begin{matrix}-i & a\\ & i+b\end{matrix}\middle| z\right],\\
&(iv)\quad z^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}(b-1)\frac{(b-i)_{n-1}}{(a)_n}\,{}_2F_1\left[\begin{matrix}-i & a\\ & -i+b\end{matrix}\middle| z\right].
\end{aligned}$$

Theorem 2. *The following identities hold:*

$$\begin{aligned}
&(i)\quad \frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n & a\\ & b\end{matrix}\middle| z\right]=\sum_{i=0}^{n-2}\frac{n!}{i!}\frac{a-b}{(a+i)_{n-i}}\,{}_2F_1\left[\begin{matrix}-i & a\\ & b\end{matrix}\middle| z\right]-n\frac{b+n-1}{a+n-1}\,{}_2F_1\left[\begin{matrix}-(n-1) & a\\ & b\end{matrix}\middle| z\right],\\
&(ii)\quad \frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n & n+a\\ & b\end{matrix}\middle| z\right]=\sum_{i=0}^{n-1}\left((-n+i)\binom{n}{i}\frac{(a+2i)(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}}\right.\\
&\qquad\qquad \left.\times\,{}_3F_2\left[\begin{matrix}-n+i+1 & b+i & a+i+n+1\\ & b+i+1 & a+2i+1\end{matrix}\middle| 1\right]\right){}_2F_1\left[\begin{matrix}-i & i+a\\ & b\end{matrix}\middle| z\right],\\
&(iii)\quad \frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n & a\\ & -n+b\end{matrix}\middle| z\right]=\sum_{i=0}^{n-2}(-1)^{n+i}\frac{n!}{i!}\frac{(b-1)}{(b-n)_{n-i}}\,{}_2F_1\left[\begin{matrix}-i & a\\ & -i+b\end{matrix}\middle| z\right]\\
&\qquad\qquad +\frac{n(a+n-1)}{(n-b)}\,{}_2F_1\left[\begin{matrix}-(n-1) & a\\ & -(n-1)+b\end{matrix}\middle| z\right].
\end{aligned}$$

1 BASIC DEFINITIONS AND IDENTITIES

The generalized hypergeometric series is defined by

$$ {}_pF_q\left[\begin{matrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_q \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{z^k}{k!}, $$

where a_i, b_i are complex parameters and $(x)_n$ denotes the Pochhammer symbol (or shifted factorial) defined by

$$ (x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n > 0. \end{cases} $$

It is assumed that b_i are not negative integers or zero.

The partial case ${}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right]$ is called the Gauss hypergeometric function. The series converges when $|z| < 1$ and also when $z = 1$ provided that $\mathrm{Re}(c-a-b) > 0$. In this case the Gauss summation identity holds:

$$ {}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\middle| 1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1} $$

where $\Gamma(z)$ is the Gamma function defined by the equality $\Gamma(z+1) = z\Gamma(z)$.

When $a = -n$ or $b = -n$ is a negative integer the series terminates and reduces to a polynomial of degree n, called a hypergeometric polynomial:

$$ {}_2F_1\left[\begin{matrix} -n & a \\ & b \end{matrix}\middle| z\right] = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(a)_i}{(b)_i} z^i. $$

For the hypergeometric polynomial the summation identity becomes

$$ {}_2F_1\left[\begin{matrix} -n & a \\ & b \end{matrix}\middle| 1\right] = \frac{(b-a)_n}{(b)_n}, \tag{2} $$

and this is equivalent to Vandermonde's theorem. If the hypergeometric function is differentiated of z, it gives

$$ \frac{d}{dz}\,{}_2F_1\left[\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right] = \frac{ab}{c}\,{}_2F_1\left[\begin{matrix} a+1 & b+1 \\ & c+1 \end{matrix}\middle| z\right]. \tag{3} $$

We also need the following properties of the Pochhammer symbol

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(n)}, \tag{4} $$

$$ (x)_n = n!\binom{x+n-1}{n} (-x)_n = (-1)^n n! \binom{x}{n}, \tag{5} $$

$$ (-x)_n = (-1)^n (x-n+1)_n, \tag{6} $$

$$ (x)_{n+m} = (x)_m (x+m)_n, \tag{7} $$

(see [1,2] for more details). We will also often use the summation interchange formula

$$ \sum_{i=1}^{n} \sum_{j=1}^{i} a_i b_j = \sum_{j=1}^{n} \left(\sum_{i=j}^{n} a_i \right) b_j, \tag{8} $$

(see [3]).

2 INVERSE PROBLEM

A solution of the inverse problem for the family $P_n(z) = \sum\limits_{k=0}^{n} p_{n,k} z^k$, namely

$$z^n = \sum_{k=0}^{n} \alpha_i P_i(z) = \sum_{i=0}^{n} \alpha_i \left(\sum_{k=0}^{i} p_{i,k} z^k \right) = \sum_{k=0}^{n} \left(\sum_{i=k}^{n} \alpha_i p_{i,k} \right) z^k, \tag{9}$$

defines the orthogonal relation

$$\sum_{i=k}^{n} \alpha_i p_{i,k} = \delta_{n,k},$$

where $\delta_{n,k}$ is the Kronecker delta. Similar orthogonal relations are frequently encountered in combinatorial problems and have been extensively studied by Riordan [4]. Thus, to solve the inverse problems we will check whether the numbers α_i and the coefficients of the corresponding hypergeometric polynomials are orthogonal.

Let us prove Theorem 1. For the item (i) we just check an orthogonality. We have

$$\begin{aligned}
\frac{(b)_n}{(a)_n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} {}_2F_1 \left[\begin{matrix} -i, a \\ b \end{matrix} \middle| z \right] &= \frac{(b)_n}{(a)_n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{k=0}^{i} \frac{(-i)_k (a)_k}{(b)_k} \frac{z^k}{k!} \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \left(\sum_{i=k}^{n} (-1)^i \binom{n}{i} \frac{(-i)_k}{k!} \frac{(a)_k}{(b)_k} \right) z^k \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \frac{(a)_k}{(b)_k} \left(\sum_{i=k}^{n} (-1)^i \binom{n}{i} \frac{(-i)_k}{k!} \right) z^k \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \frac{(a)_k}{(b)_k} \left(\sum_{i=k}^{n} (-1)^{i+k} \binom{n}{i} \binom{i}{k} \right) z^k \\
&= \frac{(b)_n}{(a)_n} \sum_{k=0}^{n} \frac{(a)_k}{(b)_k} \delta_{n,k} z^k = z^n,
\end{aligned}$$

as required. Here we have used (8) and the well known (see [4]) orthogonal relation

$$\sum_{i=k}^{n} (-1)^{i+k} \binom{n}{i} \binom{i}{k} = \delta_{n,k}.$$

(ii) We have

$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(a+2i)(b)_n}{(a+i)_{n+1}} {}_2F_1 \left[\begin{matrix} -i, i+a \\ b \end{matrix} \middle| z \right] &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{(a+2i)(b)_n}{(a+i)_{n+1}} \sum_{k=0}^{i} \frac{(-i)_k (i+a)_k}{(b)_k} \frac{z^k}{k!} \\
&= \sum_{k=0}^{n} \left(\sum_{i=k}^{n} (-1)^i \binom{n}{i} (a+2i) \frac{(b)_n}{(a+i)_{n+1}} \frac{(-i)_k}{k!} \frac{(i+a)_k}{(b)_k} \right) z^k \\
&= \sum_{k=0}^{n} \frac{(b)_n}{(b)_k} \left(\sum_{i=k}^{n} (-1)^{i+k} \binom{n}{i} \binom{i}{k} \frac{a+2i}{(a+i+k)_{n-k+1}} \right) z^k \\
&= \sum_{k=0}^{n} \frac{(b)_n}{(b)_k (n-k+1)!} \left(\sum_{i=k}^{n} (-1)^{i+k} (a+2i) \binom{n}{i} \binom{i}{k} \binom{a+n+i}{n-k+1}^{-1} \right) z^k \\
&= \sum_{k=0}^{n} \frac{(b)_n}{(b)_k (n-k+1)!} \binom{n}{k} \left(\sum_{i=k}^{n} (-1)^{i+k} (a+2i) \binom{n-k}{n-i} \binom{a+n+i}{n-k+1}^{-1} \right) z^k.
\end{aligned}$$

Let us prove the orthogonal relation

$$\sum_{i=k}^{n}(-1)^{i+k}(a+2i)\binom{n-k}{n-i}\binom{a+n+i}{n-k+1}^{-1}=\delta_{n,k}.$$

Rewrite the relation in an equivalent form by shifting the index of summation from i to $i+k$:

$$\sum_{i=0}^{n-k}(-1)^{i}(a+2(i+k))\binom{n-k}{i}\binom{a+n+i+k}{n-k+1}^{-1}=\delta_{n,k}.$$

Now we again perform the shifts $n-k\mapsto n$ and $a+n+2k\to a$ and will get the relation in such a simplified form

$$\sum_{i=0}^{n}(-1)^{i}(a+2i)\binom{n}{i}\binom{a+n+i}{n+1}^{-1}=\delta_{n,0}.$$

For $n=0$ the both sides are equal to 1. Let us prove that the sum equals 0 for $n>0$. Indeed, we have

$$\begin{aligned}\sum_{i=0}^{n}(-1)^{i}(a+2i)\binom{n}{i}\binom{a+n+i}{n+1}^{-1}&=\sum_{i=0}^{n}\frac{(-1)^{i}(a+2i)n!(n+1)!(a+i-1)!}{i!(n-i)!(a+n+i)!}\\&=\frac{(a-1)!n!}{(a+n)!}\sum_{i=0}^{n}\frac{(a+2i)(-n)_i(a)_i}{i!(a+n+1)_i}.\end{aligned}$$

Now to calculate the last sum we divide it into two sums and then express them by hypergeometric functions

$$\begin{aligned}\sum_{i=0}^{n}\frac{(a+2i)(-n)_i(a)_i}{i!(a+n+1)_i}&=a\sum_{i=0}^{n}\frac{(-n)_i(a)_i}{i!(a+n+1)_i}+2\sum_{i=0}^{n}\frac{i(-n)_i(a)_i}{i!(a+n+1)_i}\\&=a\,{}_2F_1\left[\begin{matrix}-n,a\\n+a+1\end{matrix}\middle|1\right]+2\sum_{i=1}^{n}\frac{(-n)_i(a)_i}{(i-1)!(a+n+1)_i}\\&=a\,{}_2F_1\left[\begin{matrix}-n,a\\n+a+1\end{matrix}\middle|1\right]+\frac{2a\,(-n)}{(a+n+1)}\,{}_2F_1\left[\begin{matrix}-n+1,a+1\\n+a+2\end{matrix}\middle|1\right]\\&=a\left(\frac{\Gamma(a+n+1)\Gamma(2n+1)}{\Gamma(2n+a+1)\Gamma(n+1)}-\frac{\Gamma(a+n+1)\Gamma(2n+1)}{\Gamma(2n+a+1)\Gamma(n+1)}\right)=0.\end{aligned}$$

(iii) Since

$${}_2F_1\left[\begin{matrix}-n\ \ a\\n+b\end{matrix}\middle|z\right]=\sum_{k=0}^{n}\frac{(-i)_k(a)_k}{(i+b)_k}\frac{z^k}{k!}=(-1)^k\binom{i}{k}\frac{(a)_k}{(i+b)_k},$$

we have to prove the following orthogonal relation:

$$\sum_{i=k}^{n}(-1)^{i+k}\binom{n}{i}\binom{i}{k}(b+2n-1)\frac{(b+i)_{n-1}}{(a)_n}\frac{(a)_k}{(b+i)_k}=\delta_{n,k}. \tag{10}$$

After simplification we obtain

$$(b+2n-1)\sum_{i=k}^{n}(-1)^{i+k}\binom{n-k}{n-i}(b+i+k)_{n-1-k}=\delta_{n,k}.$$

By index shifting like as in (ii) we get the identity

$$(b-1)\sum_{i=0}^{n}(-1)^i\binom{n}{i}(b+i)_{n-1}=\delta_{n,0}.$$

For $n=0$ taking into account

$$(b)_{-1}=\frac{\Gamma(b-1)}{\Gamma(b)}=\frac{(b-2)!}{(b-1)!}=\frac{1}{b-1},$$

we have that the identity is true.

For $n>0$ taking into account

$${}_2F_1\left[\begin{matrix}-n & b\\ c\end{matrix}\middle| 1\right]=\frac{(c-b)_n}{c_n},$$

we get

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(b+i)_{n-1}&=\sum_{i=0}^{n}(-1)^i\frac{n!}{i!(n-i)!}(b+i)_{n-1}=\sum_{i=0}^{n}\frac{(-n)_i}{i!}(b+i)_{n-1}\\&=(b)_{n-1}\sum_{i=0}^{n}\frac{(-n)_i(b+n-1)_i}{(b)_i i!}\\&=(b)_{n-1}{}_2F_1\left[\begin{matrix}-n,b+n-1\\ b\end{matrix}\middle| 1\right]=(b)_{n-1}\frac{(-n+1)_n}{(b)_n}=0.\end{aligned}$$

This complete the proof of the item (iii).

(iv) Since

$${}_2F_1\left[\begin{matrix}-n & a\\ -n+b\end{matrix}\middle| z\right]=\sum_{k=0}^{n}\frac{(-i)_k(a)_k}{(-i+b)_k}\frac{z^k}{k!}=(-1)^k\binom{i}{k}\frac{(a)_k}{(-i+b)_k},$$

we have to prove the following orthogonal relation:

$$(b-1)\sum_{i=k}^{n}(-1)^{k+i}\binom{n}{i}\binom{i}{k}(b-i+k)_{n-1-k}=\delta_{n,k}.$$

The proof techniques are similar to the one of the identity (10) and we omit it.

3 THE DERIVATIVE CONNECTING PROBLEM

Let us prove Theorem 2.

Proof. (i) We first prove the auxiliary combinatorial identity:

$$S_{n,k}=\sum_{i=k}^{n-2}\frac{1}{(i-k)!(a+i)_{n-i}}=\frac{1}{(a+k)\,(a+n-1)\,(n-(k+2))!}. \tag{11}$$

Simplify

$$\begin{aligned} S_{n,k} &= \sum_{i=k}^{n-2} \frac{1}{(i-k)!(a+i)_{n-i}} = \sum_{i=k}^{n-2} \frac{1}{(i-k)!(n-i)!\binom{a+n-1}{n-i}} \\ &= \frac{1}{(n-k)!} \sum_{i=k}^{n-2} \frac{(n-k)!}{(i-k)!(n-i)!} \binom{a+n-1}{n-i}^{-1} \\ &= \frac{1}{(n-k)!} \sum_{i=k}^{n-2} \binom{n-k}{i-k} \binom{a+n-1}{n-i}^{-1} = \frac{1}{(n-k)!} \sum_{i=0}^{n-k-2} \binom{n-k}{i} \binom{a+n-1}{n-i-k}^{-1}. \end{aligned}$$

Put

$$S'_{n,k} = (n-k)! S_n = \sum_{i=0}^{n-k-2} \frac{\binom{n-k}{i}}{\binom{a+n-1}{n-i-k}}.$$

We prove by double induction on n and then on k that

$$S'_{n,k} = \frac{(n-k)!}{(a+k)\,(a+n-1)\,(n-(k+2))!} = \frac{(n-k)(n-k-1)}{(a+k)\,(a+n-1)}. \tag{12}$$

Firstly we fix k and use the induction on n. The base case $n = k+2$ is obviously true. Assume that the identity

$$S'_{n-1,k} = \sum_{i=0}^{n-k-3} \frac{\binom{n-k-1}{n-i-k-1}}{\binom{a+n-1}{n-i-k}} = \frac{(n-k-1)(n-k-2)}{(a+k)\,(a+n-2)},$$

holds. Then by standard combinatorial technique we have

$$\begin{aligned} S'_{n,k} = \sum_{i=0}^{n-k-2} \frac{\binom{n-k}{i}}{\binom{a+n-1}{n-i-k}} &= \frac{\binom{n-k}{n-k-2}}{\binom{a+n-1}{2}} + \sum_{i=0}^{n-k-3} \frac{\binom{n-k}{n-i-k}}{\binom{a+n-1}{n-i-k}} \\ &= \frac{\binom{n-k}{2}}{\binom{a+n-1}{2}} + \sum_{i=0}^{n-k-3} \frac{\frac{n-k}{n-i-k}\binom{n-k-1}{n-i-k-1}}{\frac{a+n-2}{n-i-k}\binom{a+n-2}{n-i-k-1}} \\ &= \frac{\binom{n-k}{2}}{\binom{a+n-1}{2}} + \frac{n-k}{a+n-1} \sum_{i=0}^{n-k-3} \frac{\binom{n-k-1}{n-i-k-1}}{\binom{a+n-1}{n-i-k}} \\ &= \frac{(n-k)(n-k-1)}{(a+n-1)(a+n-2)} + \frac{n-k}{a+n-1} S'_{n-1,k} \\ &= \frac{(n-k)(n-k-1)}{(a+n-1)(a+n-2)} + \frac{n-k}{a+n-1} \frac{(n-k-1)(n-k-2)}{(a+k)\,(a+n-2)} \\ &= \frac{(n-k)(n-k-1)}{(a+n-1)(a+n-2)} \left(1 + \frac{n-k-2}{a+k}\right) = \frac{(n-k)(n-k-1)}{(a+k)\,(a+n-1)}. \end{aligned}$$

Thus, for a fixed k and all n the following relation

$$S_{n,k} = \frac{1}{(n-k)!}\frac{(n-k)(n-k-1)}{(a+k)\,(a+n-1)} = \frac{1}{(a+k)\,(a+n-1)\,(n-(k+2))!}$$

holds.

Now let us fix n. The induction on k is true due to obvious identity $S'_{n,k+1} = S'_{n-1,k}$. This completes the proof of (12).

Let us show that for the coefficients α_i

$$\alpha_i = \frac{n!}{i!}\frac{a-b}{(a+i)_{n-i}}, \quad \alpha_{n-1} = -n\frac{b+n-1}{a+n-1},$$

the following identity holds:

$$\sum_{i=k}^{n-1} \alpha_i (-i)_k = \frac{(-n)_{k+1}(b+k)}{a+k}. \tag{13}$$

Indeed, by (11) we obtain

$$\begin{aligned}\sum_{i=k}^{n-2} \alpha_i (-i)_k &= (a-b)n! \sum_{i=k}^{n-2} \frac{(-i)_k}{i!(a+i)_{n-i}} = (a-b)n!(-1)^k \sum_{i=k}^{n-2} \frac{1}{(i-k)!(a+i)_{n-i}} \\ &= \frac{(a-b)n!(-1)^k}{(a+k)\,(a+n-1)\,(n-(k+2))!} = \frac{(k-n+1)\,(a-b)(-n)_{k+1}}{(a+k)\,(a+n-1)}.\end{aligned}$$

Taking into account the identity

$$\frac{(k-n+1)\,(a-b)}{(a+k)\,(a+n-1)} + \frac{b+n-1}{a+n-1} = \frac{b+k}{a+k},$$

we get

$$\begin{aligned}\sum_{i=k}^{n-1} \alpha_i (-i)_k &= \sum_{i=k}^{n-2} \alpha_i (-i)_k + (-n)\frac{b+n-1}{a+n-1}(-(n-1))_k \\ &= \frac{(k-n+1)\,(a-b)(-n)_{k+1}}{(a+k)\,(a+n-1)} + \frac{b+n-1}{a+n-1}(-n)_{k+1} = \frac{(-n)_{k+1}(b+k)}{a+k}.\end{aligned}$$

This establishes the identity (13).

Now we can prove Theorem 2, item (i). Taking into account

$$\alpha_i = \frac{n!}{i!}\frac{a-b}{(a+i)_{n-i}}, \quad \alpha_{n-1} = -n\frac{b+n-1}{a+n-1},$$

let us expand the sum

$$\begin{aligned}&\sum_{i=0}^{n-2} \frac{n!}{i!}\frac{a-b}{(a+i)_{n-i}}\, {}_2F_1\left[\begin{matrix}-i, a\\ b\end{matrix}\middle| z\right] - n\frac{b+n-1}{a+n-1}{}_2F_1\left[\begin{matrix}-(n-1), a\\ b\end{matrix}\middle| z\right] = \sum_{i=0}^{n-1} \alpha_i\, {}_2F_1\left[\begin{matrix}-i, a\\ b\end{matrix}\middle| z\right] \\ &= \sum_{k=0}^{n-1}\left(\sum_{i=k}^{n-1} \alpha_i \frac{(-i)_k (b)_k}{(a)_k}\right)\frac{z^k}{k!} = \sum_{k=0}^{n-1}\left(\sum_{i=k}^{n-1} \alpha_i (-i)_k\right)\frac{(b)_k}{(a)_k}\frac{z^k}{k!} \\ &= \sum_{k=0}^{n-1} \frac{(-n)_{k+1}(b+k)}{a+k}\frac{(b)_k}{(a)_k}\frac{z^k}{k!} = \frac{-nb}{a}\sum_{k=0}^{n-1} \frac{(-n+1)_k (b+1)_k}{(a+1)_k}\frac{z^k}{k!} \\ &= \frac{-nb}{a}\, {}_2F_1\left[\begin{matrix}-n+1, b+1\\ a+1\end{matrix}\middle| z\right] = \frac{d}{dz}\, {}_2F_1\left[\begin{matrix}-n, a\\ b\end{matrix}\middle| z\right].\end{aligned}$$

(ii). Let us find the differential connecting coefficients for the family of polynomials ${}_2F_1\left[\begin{matrix}-n & n+a\\ & b\end{matrix}\middle| z\right]$ by using the solution of the corresponding inverse problem (Theoren 1,(ii)):

$$z^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+2i)\frac{(b)_k}{(a+i)_{k+1}}\,{}_2F_1\left[\begin{matrix}-i & i+a\\ & b\end{matrix}\middle| z\right].$$

Taking into account

$$\frac{(a+1)_k}{(a)_k}=\frac{a+k}{a},\quad \frac{(b)_k}{(b+1)_k}=\frac{b}{b+k},\quad \frac{(-n)_{k+1}}{k!}=(-1)^{k+1}\binom{n}{k}(n-k),$$

we have

$$\begin{aligned}
\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,n+a\\ b\end{matrix}\middle| z\right] &= \frac{-n(n+a)}{b}\,{}_2F_1\left[\begin{matrix}-(n-1),n+a+1\\ b+1\end{matrix}\middle| z\right]\\
&= \frac{-n(n+a)}{b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(n+a+1)_k}{(b+1)_k}\frac{z^k}{k!}\\
&= \frac{-n(n+a)}{b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(n+a+1)_k}{k!(b+1)_k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(a+2i)\frac{(b)_k}{(a+i)_{k+1}}\,{}_2F_1\left[\begin{matrix}-i,i+a\\ b\end{matrix}\middle| z\right]\\
&= \frac{-n(n+a)}{b}\sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}(-1)^i\binom{k}{i}\frac{(-n+1)_k(n+a+1)_k}{k!(b+1)_k}\frac{(a+2i)(b)_k}{(a+i)_{k+1}}\right){}_2F_1\left[\begin{matrix}-i,i+a\\ b\end{matrix}\middle| z\right]\\
&= \sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}(-1)^{i+k+1}\binom{k}{i}\binom{n}{k}(n-k)\frac{(a+2i)(n+a)_{k+1}}{(b+k)(a+i)_{k+1}}\right){}_2F_1\left[\begin{matrix}-i,i+a\\ b\end{matrix}\middle| z\right]\\
&= \sum_{i=0}^{n-1}\binom{k}{i}\left(\sum_{k=i}^{n-1}(-1)^{i+k+1}\binom{n-i}{n-k}(n-k)\frac{(a+2i)(n+a)_{k+1}}{(b+k)(a+i)_{k+1}}\right){}_2F_1\left[\begin{matrix}-i,i+a\\ b\end{matrix}\middle| z\right].
\end{aligned}$$

In the internal sum we perform the shift of the index of summation as $k\mapsto k+i$:

$$\begin{aligned}
&\sum_{k=i}^{n-1}(-1)^{i+k+1}\binom{n-i}{n-k}(n-k)\frac{(a+2i)(n+a)_{k+1}}{(b+k)(a+i)_{k+1}}\\
&\qquad= \sum_{k=0}^{n-1-i}(-1)^{k+1}\binom{n-i}{n-(k+i)}(n-(k+i))\frac{(a+2i)(n+a)_{k+1+i}}{(b+k+i)(a+i)_{k+1+i}}.
\end{aligned}$$

By using the relations

$$\frac{(a+2i)(n+a)_{k+1+i}}{(a+i)_{k+1+i}}=\frac{(a+2i)\,(n+a)_{i+1}}{(a+i)_{i+1}}\cdot\frac{(a+n+i+1)_k}{(a+2i+1)_k},$$

$$\frac{1}{(b+k+i)}=\frac{1}{(b+i)}\cdot\frac{(b+i)_k}{(b+i+1)_k},$$

$$(-1)^{k+1}\binom{n-i}{n-(k+i)}(n-(k+i))=(-n+i)\,\frac{(-n+i+1)_k}{k!},$$

we rewrite the sum in the form

$$\begin{aligned}
&(-n+i)\,(a+2i)\,\frac{(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}}\sum_{k=0}^{n-1-i}\frac{(-n+i+1)_k(b+i)_k(a+n+i+1)_k}{k!(b+i+1)_k(a+2i+1)_k}\\
&= (-n+i)\,(a+2i)\,\frac{(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}}\,{}_3F_2\left[\begin{matrix}-n+i+1,b+i,,a+i+n+1\\ b+i+1,a+2i+1\end{matrix}\middle| 1\right].
\end{aligned}$$

Finally, we get

$$\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,n+a\\ b\end{matrix}\middle|\, z\right] = \sum_{i=0}^{n-1}\left((-n+i)\binom{n}{i}\frac{(a+2i)\,(n+a)_{i+1}}{(b+i)\,(a+i)_{i+1}}\right.$$
$$\left.\times\,{}_3F_2\left[\begin{matrix}-n+i+1,b+i,,a+i+n+1\\ b+i+1,a+2\,,i+1\end{matrix}\middle|\, 1\right]\right)\,{}_2F_1\left[\begin{matrix}-i,i+a\\ b\end{matrix}\middle|\, z\right],$$

as reguired.

(iii) We have to prove that

$$\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,a\\ -n+b\end{matrix}\middle|\, z\right] = \sum_{i=0}^{n-2}(-1)^{n+i}\frac{n!}{i!}\frac{(b-1)}{(b-n)_{n-i}}\,{}_2F_1\left[\begin{matrix}-i,a\\ -i+b\end{matrix}\middle|\, z\right]$$
$$-\frac{n(a+n-1)}{(b-n)}\,{}_2F_1\left[\begin{matrix}-(n-1),a\\ -(n-1)+b\end{matrix}\middle|\, z\right].$$

We find the differential connecting coefficients for the family of polynomials ${}_2F_1\left[\begin{matrix}-n\ \ a\\ -n+b\end{matrix}\middle|\, z\right]$ by using the solution of the corresponding inverse problem (Theoren 1, item (iii)):

$$z^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}\,{}_2F_1\left[\begin{matrix}-i\ \ a\\ -i+b\end{matrix}\middle|\, z\right].$$

We have

$$\frac{d}{dz}\,{}_2F_1\left[\begin{matrix}-n,a\\ -n+b\end{matrix}\middle|\, z\right] = \frac{-na}{-n+b}\,{}_2F_1\left[\begin{matrix}-n+1,a+1\\ -n+1+b\end{matrix}\middle|\, z\right] = \frac{-na}{-n+b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(a+1)_k}{(-n+b+1)_k}\frac{z^k}{k!}$$
$$= \frac{-na}{-n+b}\sum_{k=0}^{n-1}\frac{(-n+1)_k(a+1)_k}{k!(-n+b+1)_k}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}\,{}_2F_1\left[\begin{matrix}-i,a\\ -i+b\end{matrix}\middle|\, z\right]$$
$$= \frac{-na}{-n+b}\sum_{i=0}^{n-1}\left(\sum_{k=i}^{n-1}\frac{(-n+1)_k(a+1)_k}{k!(-n+b+1)_k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}\right)\,{}_2F_1\left[\begin{matrix}-i,a\\ -i+b\end{matrix}\middle|\, z\right].$$

Put

$$\alpha_{n,i} = \sum_{k=i}^{n-1}\frac{-na}{-n+b}\frac{(-n+1)_k(a+1)_k}{k!(-n+b+1)_k}(-1)^i\binom{k}{i}(b-1)\frac{(b-i)_{k-1}}{(a)_k}$$
$$= \frac{-n}{-n+b}\sum_{k=i}^{n-1}(-1)^i\binom{k}{i}(b-1)\frac{(a+k)(-n+1)_k(b-i)_{k-1}}{k!(-n+b+1)_k}$$

For $i = n-1$ we have

$$\alpha_{n,n-1} = \frac{-n}{-n+b}(-1)^{n-1}(b-1)\frac{(a+n-1)(-n+1)_{n-1}(b-(n-1))_{n-2}}{(n-1)!(-n+b+1)_{k-1}} = -\frac{n(a+n-1)}{(b-n)}.$$

For $i < n-1$ we have

$$\alpha_{n,i} = (-1)^{n+i}\frac{n!}{i!}\frac{(b-1)}{(b-n)_{n-i}},$$

thus

$$\frac{-n}{-n+b}\sum_{k=i}^{n-1}(-1)^i\binom{k}{i}(b-1)\frac{(a+k)(-n+1)_k(b-i)_{k-1}}{k!(-n+b+1)_k}=(-1)^{n+i}\frac{n!}{i!}\frac{(b-1)}{(b-n)_{n-i}}$$

or

$$\sum_{k=i}^{n-1}\binom{k}{i}\frac{(a+k)(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}=(-1)^n\frac{n!}{i!}\frac{(b-n)}{(b-n)_{n-i}}.$$

Divide the sum into the two sums

$$\sum_{k=i}^{n-1}\binom{k}{i}\frac{(a+k)(-n)_k(b-i)_{k-1}}{k!(-n+b+1)_k}=a\sum_{k=i}^{n-1}\binom{k}{i}\frac{(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}$$
$$+\sum_{k=i}^{n-1}\binom{k}{i}\frac{k(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}$$

and calculate them separately. Taking into account

$$(b-i)_{k-1}=(b-i)_i(b)_{k-1-i}=(b-i)_i(k-1-i)!\binom{b+k-i-2}{k-1-i},$$
$$(-n+b+1)_k=k!\binom{-n+b+1}{k},\quad (b)_{k-1}=\frac{(b+k-2)!}{(b-1)!},$$

for the first sum we have

$$\sum_{k=i}^{n-1}\binom{k}{i}\frac{(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k}=\sum_{k=i}^{n-1}(-1)^{k+1}\binom{k}{i}\binom{n}{k}\frac{(n-k)(b-i)_{k-1}}{(-n+b+1)_k}$$
$$=\binom{n}{i}\sum_{k=i}^{n-1}(-1)^{k+1}\binom{n-i}{n-k}(n-k)\frac{(b-i)_{k-1}}{(-n+b+1)_k}$$
$$=\binom{n}{i}(b-i)_i\sum_{k=i}^{n-1}(-1)^{k+1}\binom{n-i}{n-k}(n-k)\frac{(b)_{k-1-i}}{(-n+b+1)_k}.$$

Now we shift the summation indexes $k\mapsto k+i$ and $n\mapsto n+i$:

$$\begin{aligned}\sum_{k=i}^{n-1}(-1)^{k+1}\binom{n-i}{n-k}\frac{(n-k)(b)_{k-1-i}}{(-n+b+1)_k}&=\sum_{k=0}^{n-i-1}(-1)^{k+i+1}\binom{n-i}{k}\frac{(n-k-i)(b)_{k-1}}{(-n+b+1)_{k+i}}\\&=\sum_{k=0}^{n-1}(-1)^{k+i+1}\binom{n}{k}\frac{(n-k)(b)_{k-1}}{(-n-i+b+1)_{k+i}}\\&=(-1)^{i+1}(-n+b-i)!\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\frac{(n-k)(b)_{k-1}}{(-n+b+k)!}\\&=(-1)^{i+1}\frac{(-n+b-i)!}{(b-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)\frac{(b+k-2)!}{(-n+b+k)!}.\end{aligned}\tag{14}$$

Let $[z^n]f(z)$ denote the operation of extracting the coefficient of z^n in a formal power series $f(z)$. It is clear that $[z^n]$ is a linear operation and the following well known properties holds:

$$[z^p](1+z)^q=\binom{q}{p},\quad [z^{p-q}]f(z)=[z^p]z^qf(z),$$

see [5]. By using these properties let us prove that the sum (14) is equal to 0. We have

$$\begin{aligned}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)\frac{(b+k-2)!}{(b+k-n)!} &= \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k\frac{(b+n-k-2)!}{(b-k)!}\\ &= (-1)^n n(n-2)!\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\binom{b+n-k-2}{b-k}\\ &= (-1)^{n+1}n(n-2)!\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{b+n-k-3}{b-k-1}\\ &= (-1)^{n+1}n(n-2)!\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{-n+1}{b-k+1}\\ &= (-1)^{n+1}n(n-2)!\sum_{k=0}^{n-1}\binom{n-1}{k}[z^{b-k+1}](1+z)^{-n+1}\\ &= (-1)^{n+1}n(n-2)![z^{b+1}](1+z)^{-n+1}\sum_{k=0}^{n-1}\binom{n-1}{k}z^k\\ &= (-1)^{n+1}n(n-2)![z^{b+1}](1+z)^{-n+1}(1+z)^{n-1}\\ &= (-1)^{n+1}n(n-2)![z^{b+1}]1 = 0,\end{aligned}$$

and the claim follows.

The second identity

$$\sum_{k=i}^{n-1}\binom{k}{i}\frac{k(-n)_{k+1}(b-i)_{k-1}}{k!(-n+b+1)_k} = (-1)^n\frac{n!}{i!}\frac{(b-n)}{(b-n)_{n-i}}$$

can be proved using the same arguments used in (11), so we will omit it here. □

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Бедратюк Л.П., Бедратюк Г.І. *Обернена задача та задача диференційовної зв'язності для деяких гіпергеометричних многочленів* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 235–247.

Розглянемо послідовності многочленів $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ такі, що $\deg(P_n(x)) = n$, $\deg(Q_n(x)) = n$. Задача зв'язності для них полягає у знаходженні коефіцієнтів $\alpha_{n,k}$ у виразі $Q_n(x) = \sum\limits_{k=0}^{n} \alpha_{n,k} P_k(x)$. Задача зв'язності для різних типів многочленів має довгу історію і продовжує викликати інтерес в різних галузях математики, зокрема в комбінаториці, математичній фізиці, квантовій хімії. Для часткового випадку $Q_n(x) = x^n$ задача зв'язності називається оберненою задачею для $\{P_n(x)\}_{n\geq 0}$. Частковий випадок $Q_n(x) = P'_{n+1}(x)$ має назву диференціальної задачі зв'язності для послідовності многочленів $\{P_n(x)\}_{n\geq 0}$. В пропонованій статті ми знаходимо у замкненому вигляді коефіцієнти оберненої і диференціальної задач зв'язності для гіпергеометричних многочленів вигляду

$$ {}_2F_1\left[\begin{matrix} -n, a \\ b \end{matrix}\middle| z\right], \quad {}_2F_1\left[\begin{matrix} -n, n+a \\ b \end{matrix}\middle| z\right], \quad {}_2F_1\left[\begin{matrix} -n, a \\ \pm n + b \end{matrix}\middle| z\right], $$

де ${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix}\middle| z\right] = \sum\limits_{k=0}^{\infty} \dfrac{(a)_k (b)_k}{(c)_k} \dfrac{z^k}{k!}$ – гіпергеометрична функція Гауса, а $(x)_n$ позначає символ Похгаммера, який визначається формулою $(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1)(x+2)\cdots(x+n-1), & n > 0. \end{cases}$

Всі многочлени розглядаються над полем дійсних чисел.

Ключові слова і фрази: гіпергеометрична фунція, коефіцієнти зв'язності, обернена задача, задача диференціальної зв'язності, гіпергеометричний многочлен.