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SLIMANE A.

SPACES GENERATED BY THE CONE OF SUBLINEAR OPERATORS

This paper deals with a study on classes of non linear operators. Let $SL(X,Y)$ be the set of all sublinear operators between two Riesz spaces X and Y. It is a convex cone of the space $H(X,Y)$ of all positively homogeneous operators. In this paper we study some spaces generated by this cone, therefore we study several properties, which are well known in the theory of Riesz spaces, like order continuity, order boundedness etc. Finally, we try to generalise the concept of adjoint operator. First, by using the analytic form of Hahn-Banach theorem, we adapt the notion of adjoint operator to the category of positively homogeneous operators. Then we apply it to the class of operators generated by the sublinear operators.

Key words and phrases: Riesz space, Banach lattice, homogeneous operator, sublinear operator, order continuous operator.

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INTRODUCTION

The theory of Riesz spaces plays an important role in several branches of mathematics, in particular in the geometry of Banach spaces and the theory of linear operators where the notion of Banach lattice play a central role. In this work we generalize some vector lattice properties to the category of sublinear operators i.e., positively homogenous and subadditive. The set obtained is not a Banach space but a positive convex cone. Hence, this paper deals with the extension of this set and their properties. The paper is organized as follows.

In Section 1 we recall some basic definitions and properties of Riesz spaces, we also recall the notion of sublinear operators between a vector space X and a Riesz space Y.

In Section 2 we introduce the spaces spanned by different cones of sublinear operators. In other hand we present some principal notions concerning the theory of Riesz spaces like order continuity, order ideal, and we apply these notions on these spaces.

In Section 3 we introduce the adjoint of positively homogeneous operator. We first establish the following result.

Let u be in $\mathcal{L}(X,Y)$. Then the bounded adjoint operator u^* of u can be extended to a bounded linear operator $\widetilde{u}^*$ belongs to $\mathcal{L}(H^*(Y),H^*(X))$ such that $\widetilde{u}^* = u^*$ on Y^* and $\|\widetilde{u}^*\| = \|u^*\| = \|u\|$, where $H^*(Y)$ is the space of all bounded positively homogeneous functionals on Y, Y^* is the topological dual space of Y and $\mathcal{L}(X,Y)$ is the Banach space of all bounded linear operators from X into Y. Finally we adapt the existence theorem of bounded adjoint linear operator to the category of positively homogeneous operators as follows.

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Let X, Y be two Banach spaces and $T \in \mathcal{H}(X, Y)$. Then, $T_h^* \in \mathcal{L}(H^*(Y), H^*(X))$ such that $\|T\| = \|T_h^*\|$, where T_h^* denotes the adjoint of T and $\mathcal{H}(X, Y)$ is the Banach space of all bounded positively homogeneous operators from X into Y.

1 Preliminaries

In this section, we introduce some terminology concerning Riesz spaces and Banach lattices. These spaces are well known. For more details, the interested reader can consult, for example, the references [2, 4–6]. But for our convenience, we include some recalls. We also introduce the class of positively homogeneous operators.

Let X be a real vector space. Then X is called a Riesz space (or vector lattice) if it is an ordered vector space with the additional property that the supremum of every nonempty finite subset of X exists in X. We denote the supremum of the set $\{x, y\}$ by $\sup\{x, y\}$ or $x \vee y$. Similarly, $\inf\{x, y\}$ or $x \wedge y$ denote the infimum of the set $\{x, y\}$.

Let X be a Riesz space. The subset $X^+ = \{x \in E : x \geq 0\}$ is called the positive cone of X (which is salient, i.e. $X^+ \cap (-X^+) = \{0\}$) and the elements of X^+ are called the positive elements of X.

Let X be a Riesz space, equipped with a norm. The norm in X is called a Riesz norm if

$$|x| \leq |y| \Longrightarrow \|x\| \leq \|y\|,$$

where $|x| = \sup\{x, -x\}$. Denote $x^+ = \sup\{x, 0\}$, $x^- = \sup\{-x, 0\}$. Then obviously we have $x = x^+ - x^-$ and $|x| = x^+ + x^-$. Note that this implies that for any $x \in X$, the elements x and $|x|$ have the same norm. A Riesz space X equipped with a Riesz norm, is called a normed Riesz space. If the norm is complete, X is called a Banach lattice. The convex cone X^+ is norm closed. A complete Banach lattice is a Banach lattice such that every order bounded set in X has a supremum.

By a Riesz subspace (or a vector sublattice) of a Riesz space X we mean a linear subspace E of X so that $\sup\{x, y\}$ belongs to E whenever $x, y \in E$. A vector subspace E of a Riesz space X is said to be an order ideal or simply ideal whenever $|x| \leq |y|$ and $y \in E$ imply $x \in E$.

A non-empty subset D is said to be upwards directed (respectively downwards directed) if for all $x_1, x_2 \in D$ there is $x_3 \in D$ such that $x_1 \vee x_2 \leq x_3$ (respectively $x_1 \wedge x_2 \geq x_3$), if $\sup D = x$ exists and D upwards directed (respectively $\inf D = y$ exists and D downwards directed) we shall write $D \uparrow x$ (respectively $D \downarrow y$).

Definition. *Let X be a vector space and Y be a Riesz space. An operator $T : X \longrightarrow Y$ is*

1- positively homogeneous if for all x in X and λ in $\mathbb{R}_+$ we have

$$T(\lambda x) = \lambda T(x),$$

2- subadditive if for all x, y in X we have

$$T(x + y) \leq T(x) + T(y).$$

The operator T is sublinear if it is positively homogeneous and subadditive. The operator T is said to be superlinear if T is positively homogeneous and superadditive (i.e. $T(x + y) \geq T(x) + T(y)$ for all x, y in X). We have for all x in X

$$-T(-x) \leq T(x). \tag{1}$$

We denote by $H(X,Y)$ (respectively $SL(X,Y)$) the real vector space of all positively homogeneous (the set of all sublinear) operators from X into Y, equipped with the natural order inducted by Y, i.e.

$$T \leq S \quad \text{if} \quad T(x) \leq S(x), \quad \forall\, x \in X.$$

The set $SL(X,Y)$ is a pointed convex cone of $H(X,Y)$ which is not salient.

Let T be in $SL(X,Y)$. We will denote by ∇T the subdifferential of T, which is the set of all linear operators $u : X \longrightarrow Y$ such that $u(x) \leq T(x)$ for all x in X. We know (see, for example, [1]), that ∇T is not empty if Y is a complete Banach lattice and $T(x) = \sup\{u(x) : u \in \nabla T\}$, moreover, the supremum is attained. If Y is simply a Banach lattice, then ∇T is empty in general (see [3]).

If X is a Banach space and Y is a Banach lattice, then we will denote by $\mathcal{SL}(X,Y)$ the set of all bounded (= continuous) sublinear operators from X into Y and by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y. Let T be in $SL(X,Y)$. We have (see [1]), that T is bounded if and only if u is bounded for all u in ∇T. The set $\mathcal{SL}(X,Y)$ (respectively the space $\mathcal{L}(X,Y)$) is a subset (respectively a subspace) of the space $\mathcal{H}(X,Y)$ of all homogeneous bounded operators from X into Y. The space $\mathcal{H}(X,Y)$ is normed by the standard norm

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|.$$

2 SPACES SPANNED BY SUBLINEAR OPERATORS

Let X be a vector space and Y be a Riesz space. We denote by

$$\triangle SL(X,Y) = SL(X,Y) - SL(X,Y)$$

the subspace of $H(X,Y)$ spanned by $SL(X,Y)$, i.e.

$$\triangle SL(X,Y) = \{T - S : T, S \in SL(X,Y)\}.$$

We denote by $\triangle\mathcal{SL}(X,Y)$ the subspace of all bounded operators in $\triangle SL(X,Y)$.

Proposition 1. *Let X be a vector space and Y be a Riesz space. Then $H(X,Y)$ is a Riesz space. If in addition X is a Banach space and Y is a Banach lattice, then $\mathcal{H}(X,Y)$ is also a Banach lattice.*

Proof. It is sufficent to endow the vector space $H(X,Y)$ with the partial order induced by Y. It is clear that $H(X,Y)$ is a Riesz space with respect to this order. Suppose now X be a Banach space and Y be a Banach lattice. Let $(T_n)_n \subset H(X,Y)$ be a Cauchy sequence, then $\lim\limits_{n\to+\infty} \|T_{n+p} - T_n\| = 0$ implies that $\lim\limits_{n\to+\infty} \|T_{n+p}(x) - T_n(x)\| = 0$ for all x in X.

As Y is a Banach space there is $T(x) \in Y$ such that $\lim\limits_{n\to+\infty} T_n(x) = T(x)$. Since $T_n(\alpha x) = \alpha T_n(x)$ for all α in $\mathbb{R}_+$ and all x in X we have $T(\alpha x) = \lim\limits_{n\to+\infty} T_n(\alpha x) = \lim\limits_{n\to+\infty} \alpha T_n(x) = \alpha T(x)$ for all α in $\mathbb{R}_+$ and all x in X. Thus, T is positively homogeneous. The operator T is clearly bounded and hence $\mathcal{H}(X,Y)$ is a Banach space. Let now $T, S \in \mathcal{H}(X,Y)$ such that $|T| \leq |S|$ then $\|T(x)\| \leq \|S(x)\|$ for all x in X, so $\|T\| \leq \|S\|$ and $\mathcal{H}(X,Y)$ is a Banach lattice. $\square$

Proposition 2. *Let X be a vector space and Y be a Riesz space. Then*

(a) the space $\triangle SL(X,Y)$ is a Riesz subspace of $H(X,Y)$;

(b) if X is a normed space and Y be a normed Riesz space, then $\triangle\mathcal{SL}(X,Y)$ is a normed Riesz space.

Proof. (a) The space $\triangle SL(X,Y)$, which is included in $H(X,Y)$, is partially ordered by the natural order inducted by Y. Consider T,S in $\triangle SL(X,Y)$. Then, there are T_1,T_2,S_1,S_2 in $SL(X,Y)$ such that

$$T = T_1 - T_2, S = S_1 - S_2.$$

For all x in X we define $T \vee S$ by

$$(T \vee S)(x) = T(x) \vee S(x).$$

Using for x,y,z in X the identity $x \vee y + z = (x+z) \vee (y+z)$, we obtain

$$\begin{aligned}(T \vee S)(x) &= (T_1 - T_2)(x) \vee (S_1 - S_2)(x)\\ &= (T_1 + S_2)(x) \vee (S_1 + T_2)(x) - (T_2 + S_2)(x) = \widetilde{T}(x) - \widetilde{S}(x)\end{aligned}$$

with $\widetilde{T},\widetilde{S} \in SL(X,Y)$, where

$$\widetilde{T} = (T_1 + S_2) \vee (S_1 + T_2) \quad \text{and} \quad \widetilde{S} = T_2 + S_2.$$

(b) It is clear that $\triangle\mathcal{SL}(X,Y)$ is a normed Riesz space with the norm induced by the standard norm of $\mathcal{H}(X,Y)$ on $\triangle\mathcal{SL}(X,Y)$, i.e. by the norm $\|T\|_{\triangle\mathcal{SL}(X,Y)} = \sup\limits_{\|x\|\le 1} \|T(x)\|$. □

Proposition 3. *Let X be a vector space and Y be a Dedekind complete Riesz space. Then $H(X,Y)$ is also a Dedekind complete Riesz space.*

Proof. Let $M \subset H(X,Y)$ be a nonempty subset, which is upper bounded. Then there is $S \in H(X,Y)$ such that for all $T \in M$ we have $T \le S$, that is for all $T \in M$ and all $x \in X$ we have $T(x) \le S(x)$. This implies that for all $x \in X$ the set $\{T(x) : T \in M\}$ is upper bounded by $S(x) \in Y$. Since Y is a Dedekind complete Riesz space, the supremum of $\{T(x) : T \in M\}$ exists in Y. We can put now $R(x) = \sup\{T(x) : T \in M\}$. It is clear that R is a positively homogeneous operator. □

Remark 1. *For all $T = P - Q$ in $\triangle SL(X,Y)$ there is $\varphi_T \in SL(X,Y)$ and $\overline{\varphi}_T$ super linear (i.e. $-\overline{\varphi}_T$ sublinear) such that $\overline{\varphi}_T \le T \le \varphi_T$ and $\varphi_T(-x) = \varphi_{-T}(x)$ (respectively $\overline{\varphi}_T(-x) = \overline{\varphi}_{-T}(x)$) for all x in X. It suffices to define φ_T, $\overline{\varphi}_T$ by*

$$\varphi_T(x) = P(x) + Q(-x), \quad \overline{\varphi}_T(x) = -P(-x) - Q(x)$$

and use the inequality (1).

Definition 1. *Let $T \in \triangle SL(X,Y)$ be an operator between two Riesz spaces. The operator T is said to be order bounded if T carries order bounded subsets of X to order bounded subsets of Y.*

Definition 2. *Let $T \in \triangle SL(X,Y)$ be an order bounded operator. Then T is said to be*

(1) order continuous if for any downwards directed set D in E having infimum the null element (i.e. $D \downarrow 0$) we have $\inf(|T(x)|, x \in D) = 0$ in Y;

(2) σ-order continuous if for all $x_n \downarrow 0$ in X we have in Y

$$\inf(|T(x_n)|, n \geq 0) = 0.$$

We denote by

$$\begin{aligned} \triangle SL_b(X,Y) &= \{T \in \triangle SL(X,Y), T \text{ order bounded}\}, \\ \triangle SL_{co}(X,Y) &= \{T \in \triangle SL(X,Y), T \text{ order continuous}\}. \end{aligned}$$

It should be clear that all these collections are real vector spaces under the usual pointwise algebraic operations.

Proposition 4. *The set $\triangle SL_b(X,Y)$ is a Riesz subspace of $\triangle SL(X,Y)$.*

Proof. Consider T_1, T_2 in $\triangle SL_b(X,Y)$, (α, β) in $\mathbb{R}^2$ and $\alpha \leq x \leq \beta$. Then

$$|(\alpha T_1 + \beta T_2)(x)| \leq |\alpha||T_1(x)| + |\beta||T_2(x)| \leq |\alpha| c_1 + |\beta| c_2 = c.$$

This implies that $\alpha T_1 + \beta T_2 \in \triangle SL_b(X,Y)$ and hence $T_1 \vee T_2 \in \triangle SL_b(X,Y)$ because $T_1 \vee T_2 = \frac{1}{2}(T_1 + T_2 + |T_1 - T_2|)$. Consequently, $\triangle SL_b(X,Y)$ is a Riesz subspace of the Riesz space $\triangle SL(X,Y)$. $\square$

3 THE ADJOINT OF POSITIVELY HOMOGENEOUS OPERATORS

Definition 3. *Let X, Y be two Riesz spaces. Put*

$$\triangle_r SL(X,Y) = \{T_1 - T_2 : \quad T_1, T_2 \in (SL(X,Y))^+\} \subset \triangle SL(X,Y).$$

A sublinear operator $T \in SL(X,Y)$ is said to be regular if $T \in \triangle_r SL(X,Y)$.

We denote by

$$\begin{aligned} SL_i(X,Y) &= \{T \in SL(X,Y) : T \text{ increasing}\}, \\ \triangle SL_i(X,Y) &= \{T_1 - T_2 : \quad T_1, T_2 \in SL_i(X,Y)\} \\ &= SL_i(X,Y) - SL_i(X,Y), \\ L_i(X,Y) &= \{T \in L(X,Y) : T\text{increasing}\}, \\ \triangle L_i(X,Y) &= \{T_1 - T_2 : \quad T_1, T_2 \in L_i(X,Y)\} \\ &= L_i(X,Y) - L_i(X,Y), \end{aligned}$$

and we put $X'_i = \triangle L_i(X, \mathbb{R})$, $X'_{i,s} = \triangle SL_i(X, \mathbb{R})$.

Proposition 5. *The spaces $\triangle_r SL(X,Y)$, $\triangle SL_i(X,Y)$ are Riesz subspaces of $\triangle SL(X,Y)$.*

Proof. The set $\triangle_r SL(X,Y)$ is a subspace of $\triangle SL(X,Y)$. Further, if $T_1, T_2 \in \triangle_r SL(X,Y)$, then there is $P_1, Q_1, P_2, Q_2 \in (SL(X,Y))^+$ such that $T_1 = P_1 - Q_1$ and $T_2 = P_2 - Q_2$. We have $T_1 \vee T_2 = (P_1 + Q_2) \vee (P_2 + Q_1) - (Q_1 + Q_2)$, which is in $\triangle_r SL(X,Y)$ because

$$(P_1 + Q_2) \vee (P_2 + Q_1), (Q_1 + Q_2) \in (SL(X,Y))^+.$$

The same for $\triangle SL_i(X,Y)$. □

Proposition 6. *The spaces $\triangle_r SL(X,Y), \triangle SL_i(X,Y)$ are Riesz subspaces of $\triangle SL(X,Y)$.*

Proof. The set $\triangle_r SL(X,Y)$ is a subspace of $\triangle SL(X,Y)$. Further, if $T_1, T_2 \in \triangle_r SL(X,Y)$, then there is $P_1, Q_1, P_2, Q_2 \in (SL(X,Y))^+$ such that $T_1 = P_1 - Q_1$ and $T_2 = P_2 - Q_2$. We have $T_1 \vee T_2 = (P_1 + Q_2) \vee (P_2 + Q_1) - (Q_1 + Q_2)$, which is in $\triangle_r SL(X,Y)$ because

$$(P_1 + Q_2) \vee (P_2 + Q_1), (Q_1 + Q_2) \in (SL(X,Y))^+.$$

The same for $\triangle SL_i(X,Y)$. □

Remark 2. *1) Any linear operator is a regular sublinear operator. Indeed, if $u \in L(X,Y)$, then $u = u^+ - u^-$ with $u^+(x) = 0 \vee u(x)$, $u^-(x) = 0 \vee (-u(x))$, which are positive sublinear operators.*

2) The existence of the regular sublinear operators (not linear) is assured by the fact that if $T \in SL(X,Y)$ such that $|T| \in SL(X,Y)$, then T is regular

$$T = T^+ - T^- = 2T^+ - |T| \quad (2T^+, |T| \in (SL(X,Y))^+).$$

As example, consider $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha > \beta$ and $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$T(x) = \begin{cases} \alpha x, & \text{if } x \geq 0, \\ \beta x, & \text{if } x < 0. \end{cases}$$

Then T is sublinear ($T(x) = (\alpha x) \vee (\beta x)$) and $|T|$ also because

$$|T|(x) = |T(x)| = (\alpha x) \vee (-\beta x).$$

Lemma 1 ([6, Lemma 21.3]). *Let E be an ordered vector space, and let A, B be two subsets of E such that* $\inf A = x_0,$ $\inf B = y_0$. *Then*

$$x_0 + y_0 = \inf(A + B) = \inf\{a + b \text{ such that } a \in A, b \in B\}.$$

Proposition 7. *Let X, Y be two Riesz spaces. Put*

$$SL_o(X,Y) = \{T \in SL_i(X,Y) \text{ such that } T \text{ order continuous}\},$$
$$\triangle SL_o(X,Y) = SL_o(X,Y) - SL_o(X,Y).$$

Then

(a) the set $SL_o(X,Y)$ is a convex cone;

(b) the space $\triangle SL_o(X,Y) \subset \triangle SL_{co}(X,Y)$ is an order ideal.

Proof. (a) Let $D \downarrow 0$, and $p, q \in SL_o(X,Y)$, then $(p+q)(D)$ is upwards directed such that $(p+q)(D) \downarrow 0$. Indeed, if $x_1, x_2 \in D$, then there is $x_3 \in D$ such that $x_3 \leq x_1$ and $x_3 \leq x_2$. This implies that $(p+q)(x_3) \in (p+q)(D)$. Thus

$$(p+q)(x_3) \leq (p+q)(x_1) \text{ and } (p+q)(x_3) \leq (p+q)(x_2).$$

Let h be the infimum of $(p+q)(D)$, then for all $x_1, x_2 \in D$ there is $x_3 \in D$ such that

$$h \leq (p+q)(x_3) \leq p(x_1) + q(x_2) \text{ for all } x_1, x_2 \in D.$$

We have

$$\begin{aligned} h &\leq \inf\{p(x_1) + q(x_2), x_1, x_2 \in D\} \\ &\leq \inf\{p(x_1), x_1 \in D\} + \inf\{q(x_2), x_2 \in D\ \} \\ &\leq \inf\{|p(x_1)|, x_1 \in D\} + \inf\{|q(x_2)|, x_2 \in D\} \leq 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \inf\{|(p+q)(x)|,\ x \in D\} &\leq \inf\{|p(x)| + |q(x)|,\ x \in D\} \\ &\leq \inf\{p(x) + q(x),\ x \in D\} \leq 0. \end{aligned}$$

It is clear that $\lambda p \in SL_o\,(X,Y)$ for all $\lambda \in \mathbb{R}^+$ and all $p \in SL_o(X,Y)$. Furthermore

$$\begin{aligned} \inf\{|(p \vee q)(x)|,\ x \in D\} &= \inf\{(p \vee q)(x),\ x \in D\} \\ &\leq \inf\{(p+q)(x),\ x \in D\} \leq 0. \end{aligned}$$

(b) Let $T \in \triangle SL_o(X,Y)$. Then $T = p - q$ with $p, q \in SL_o(X,Y)$. Let $D \downarrow 0$. We have

$$|p-q|(x) \leq |p(x)| + |q(x)| \leq p(x) + q(x) \text{ for all } x \in D.$$

So,

$$\inf\{|(p-q)(x)|,\ x \in D\} \leq \inf\{(p+q)(x),\ x \in D\} \leq 0.$$

Consequently, $T \in \triangle SL_{co}(X,Y)$.
Let now $D \downarrow 0$. Assume that $|T| \leq |S|$, $S \in \triangle SL_o(X,Y)$, then

$$\inf\{|T|(x),\ x \in D\} \leq \inf\{|S|(x),\ x \in D\} \leq 0.$$

This ends the proof. □

In the sequel, we extend the notion of adjoint operator on some spaces defined above. Let X be a Banach space and Y be a Banach lattice. Put

$$\begin{aligned} X' &= L(X,\mathbb{R}), \\ X^* &= \mathcal{L}(X,\mathbb{R}), \\ X'_\Delta &= \triangle SL(X,\mathbb{R}), \\ X^*_\Delta &= \triangle\mathcal{SL}(X,\mathbb{R}), \\ H'(X) &= H(X,\mathbb{R}), \\ H^*(X) &= \mathcal{H}(X,\mathbb{R}). \end{aligned}$$

We have $X' \subset X'_\Delta \subset H'(X)$ and $X^* \subset X^*_\Delta \subset H^*(X)$.

Theorem 1. *Let X, Y be two Riesz spaces and u be in $L(X, Y)$. Then there exists an $\widetilde{u}'$ in $L(H'(Y), H'(X))$ such that $\widetilde{u}' = u'$ on Y' and $\widetilde{u}'(\varphi) \leq |\varphi \circ u|$ for all $\varphi \in H'(Y)$, where u' is the adjoint operator of u.*

Proof. Let u be in $L(X, Y)$, the adjoint operator of u is defined by

$$u' : Y' \longrightarrow X' \subset H'(X)$$

such that

$$u'(\varphi) = \varphi \circ u \text{ for all } \varphi \in Y'.$$

Let now $P \in SL(H'(Y), H'(X))$ be defined by

$$P(\varphi) = |\varphi \circ u|.$$

We have

$$u'(\varphi) = \varphi \circ u \leq |\varphi \circ u| = P(\varphi) \text{ for all } \varphi \in Y'.$$

By the Hahn-Banach theorem (the analytic form), there is $\widetilde{u}' \in L(H'(Y), H'(X))$ such that $\widetilde{u}' = u'$ on Y' and

$$\widetilde{u}'(\varphi) \leq P(\varphi) \leq |\varphi \circ u|$$

for all $\varphi \in H'(Y)$ and this completes the proof. ☐

Theorem 2. *Let X, Y be two Banach spaces and u be in $\mathcal{L}(X, Y)$. Then there exists an $\widetilde{u}'$ in $\mathcal{L}(H^*(Y), H^*(X))$ such that $\widetilde{u}' = u^*$ on Y^* and $\|\widetilde{u}'\| = \|u^*\| = \|u\|$. In this case $\widetilde{u}'$ is denoted by $\widetilde{u}^*$.*

Proof. Let u be in $\mathcal{L}(X, Y)$. By Theorem 1 there is $\widetilde{u}'$ in $L(H'(Y), H'(X))$ such that $\widetilde{u}' = u^*$ on Y' and $\widetilde{u}'(\varphi) \leq |\varphi \circ u|$ for all $\varphi \in H'(Y)$. On the other hand, because $\widetilde{u}'(\varphi) \leq |\varphi \circ u|$ we obtain $|\widetilde{u}'(\varphi)| \leq |\varphi \circ u|$ and hence for all $\varphi \in H^*(Y)$

$$\|\widetilde{u}'(\varphi)\| \leq \|\varphi \circ u\| \leq \|u\|\|\varphi\|.$$

So, $\widetilde{u}' \in \mathcal{L}(H^*(Y), H^*(X))$. It remains to show that $\|\widetilde{u}'\| = \|u\|$. Since $\|\widetilde{u}'(\varphi)\| \leq \|u\|\|\varphi\|$, we conclude that $\|\widetilde{u}'\| \leq \|u\|$. For the converse inequality, we know that $\|u^*\| = \|u\|$, hence

$$\begin{aligned}
\|u\| = \|u^*\| &= \sup_{\varphi \in B_{Y^*}} \|u^*(\varphi)\| \\
&= \sup_{\varphi \in B_{Y^*}} \|\widetilde{u}'(\varphi)\| \text{ (because } \widetilde{u}'_{\diagup Y^*} = u^*) \\
&\leq \sup_{\varphi \in B_{H^*(Y)}} \|\widetilde{u}'(\varphi)\| \text{ (because } B_{Y^*} \subset B_{H^*(Y)}) \\
&= \|\widetilde{u}'\|
\end{aligned}$$

and then the theorem is proved. ☐

Now, we extend the notion of adjoint operator to positively homogeneous operators.

Definition 4. *Let X, Y be two Riesz spaces and $T \in H(X, Y)$. We define the adjoint of T by*

$$\begin{aligned}
T'_h : H'(Y) &\longrightarrow H'(X) \\
\varphi &\longmapsto T'_h(\varphi) = \varphi \circ T
\end{aligned}$$

such that $T'_h(\varphi)(x) = \varphi \circ T(x)$.

Proposition 8. *Let X, Y be two Banach spaces and $T \in \mathcal{H}(X,Y)$. Then $T'_h \in \mathcal{L}(H^*(Y), H^*(X))$ such that $\|T\| = \|T'_h\|$. In this case T'_h is denoted by T^*_h.*

Proof. Consider T in $\mathcal{H}(X,Y)$. We have for all $\varphi \in H^*(Y)$

$$\|T'_h(\varphi)\| = \|\varphi \circ T\| \leq \|\varphi\|\|T\|.$$

So, $T'_h \in \mathcal{L}(H^*(Y), H^*(X))$. To show that $\|T\| = \|T'_h\|$, we first consider the mapping $i : x \in X \longmapsto i(x) \in H^{**}(X)$ such that

$$\begin{aligned} i(x) : H^*(X) &\longrightarrow \mathbb{R}, \\ \varphi &\longmapsto (i(x), \varphi) = \langle \varphi, x \rangle. \end{aligned}$$

Then i is such that $\|i(x)\| = \|x\|$ for all $x \in X$. Indeed,

$$\begin{aligned} \|i(x)\| &= \sup_{\varphi \in B_{H^*(X)}} \|(i(x), \varphi)\| \\ &= \sup_{\varphi \in B_{H^*(X)}} \|\langle \varphi, x \rangle\| \\ &\leq \|x\|. \end{aligned}$$

Conversely

$$\begin{aligned} \|x\| &= \sup_{\xi \in B_{X^*}} \|\langle \xi, x \rangle\| \leq \sup_{\varphi \in B_{H^*(X)}} \|\langle \varphi, x \rangle\| \text{ (because } B_{X^*} \subset B_{H^*(X)}) \\ &\leq \sup_{\varphi \in B_{H^*(X)}} \|(i(x), \varphi)\| \leq \|i(x)\|. \end{aligned}$$

Finally, we have

$$\begin{aligned} \|T'_h\| = \sup_{\varphi \in B_{H^*(Y)}} \|T'_h(\varphi)\| &= \sup_{\varphi \in B_{H^*(Y)}} \|\varphi \circ T\| \\ &= \sup_{\varphi \in B_{H^*(Y)}} \big(\sup_{x \in B_X} \|\langle \varphi \circ T, x \rangle\|\big) \\ &= \sup_{\varphi \in B_{H^*(Y)}} \big(\sup_{x \in B_X} \|\langle \varphi, T(x) \rangle\|\big) \\ &= \sup_{x \in B_X} \big(\sup_{\varphi \in B_{H^*(Y)}} \|\langle \varphi, T(x) \rangle\|\big) \\ &= \sup_{x \in B_X} \big(\sup_{\varphi \in B_{H^*(Y)}} \|(i(T(x)), \varphi)\|\big) \\ &= \sup_{x \in B_X} \|i(T(x))\| \\ &= \sup_{x \in B_X} \|T(x)\| = \|T\|. \end{aligned}$$

This completes the proof. □

Definition 5. *Let X, Y be two Riesz spaces. Consider $T \in \triangle SL(X,Y)$ with $T = P - Q$. We define a linear operator on $Y'_{i,s}$ denoted T'_i by*

$$\begin{aligned} &T'_i : Y'_{i,s} \longrightarrow X'_\Delta, \\ &T_1 - T_2 \longmapsto T'_i(T_1 - T_2) = T_1 \circ P + T_2 \circ Q - (T_1 \circ Q + T_2 \circ P). \end{aligned}$$

Note that this operator is well defined. Indeed, if $S \in Y'_{i,s}$ such that $S = S_1 - S_2 = S_3 - S_4$, then

$$\begin{aligned} T'_i(S_1 - S_2) &= S_1 \circ P + S_2 \circ Q - (S_1 \circ Q + S_2 \circ P) \\ &= (S_1 - S_2) \circ P - (S_1 - S_2) \circ Q \\ &= (S_3 - S_4) \circ P - (S_3 - S_4) \circ Q = T'_i(S_3 - S_4). \end{aligned}$$

Proposition 9. *Let X, Y be two Riesz spaces, then there is $\widetilde{T}'_i$ in $L(H'(Y), H'(X))$ such that $\widetilde{T}'_i = T'_i$ on $Y'_{i,s}$.*

Proof. We define a sublinear operator $S : H'(Y) \longrightarrow H'(X)$ by

$$S(\varphi) = |\varphi \circ P| + |\varphi \circ Q|.$$

For all $\varphi = \varphi_1 - \varphi_2 \in Y'_{i,s}$ we have

$$\begin{aligned} T'_i(\varphi) = T'_i(\varphi_1 - \varphi_2) = \varphi_1 \circ P + \varphi_2 \circ Q - (\varphi_1 \circ Q + \varphi_2 \circ P) &= (\varphi_1 - \varphi_2) \circ P - (\varphi_1 - \varphi_2) \circ Q \\ \leq |(\varphi_1 - \varphi_2) \circ P| + |(\varphi_1 - \varphi_2) \circ Q| &= S(\varphi). \end{aligned}$$

The Hahn-Banach theorem implies that T'_i can be extended to a linear operator $\widetilde{T}'_i \in L(H'(Y), H'(X))$ such that $\widetilde{T}'_i(\varphi) \leq S(\varphi)$ for all $\varphi \in H'(Y)$. □

Remark 3. *If $T \in L(X, Y)$, then we have $\widetilde{T}' = T^*_h$ on Y', where T'_h denote the operator defined in Definition 4. If $T \in \triangle SL(X, Y)$, then we have $\widetilde{T}'_i = T'_h$ on Y'_i.*

Proposition 10. *Let X, Y be two Riesz spaces and T be in $(SL(X, Y))^+$. Then the following properties are satisfied.*

(1) We have $|T|'_i \leq |T'_i|$.

(2) The restriction of T'_i to $SL_i(Y, \mathbb{R})$ verifies $|T'_i| = |T|'_i$.

Proof. (1) Let $T \in (SL(X, Y))^+$ and $\varphi \in Y'_{i,s}$, then there is $\varphi_1, \varphi_2 \in SL_i(X, Y)$ such that $\varphi = \varphi_1 - \varphi_2$ and

$$\begin{aligned} |T'_i|(\varphi) = |T'(\varphi)| = |\varphi_1 \circ T - \varphi_2 \circ T| &\geq \varphi_1 \circ T - \varphi_2 \circ T \\ &\geq \varphi_1 \circ |T| - \varphi_2 \circ |T| \geq |T|'_i(\varphi). \end{aligned}$$

(2) Let $T \in (SL(X, Y))^+$ and $\varphi \in SL_i(Y, \mathbb{R})$ we have

$$\begin{aligned} |T'_i|(\varphi) &= |T'(\varphi)| = |\varphi(T)| = \varphi(T) \text{ (because } \varphi \uparrow \text{ and } T \geq 0) \\ &= \varphi(|T|) = |T|'_i(\varphi) \end{aligned}$$

and this completes the proof. □

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У цій статті досліджуються деякі класи нелінійних операторів. Нехай $SL(X, Y)$ — множина всіх сублінійних операторів між двома просторами Ріса X та Y. Це є опуклий конус в просторі $H(X, Y)$ всіх позитивно однорідних операторів. У цій статті досліджено деякі простори, породжені цим конусом, зокрема ми досліджуємо деякі властивості, які добре відомі в теорії просторів Ріса, такі як порядкова неперервність, порядкова обмеженість та ін. Насамкінець, ми пробуємо узагальнити концепцію спряженого оператора. Спочатку, використовуючи аналітичну форму теореми Гана-Банаха, ми пристосовуємо поняття спряженого оператора до категорії позитивно однорідних операторів, а потім застосовуємо його до класу операторів, породжених сублінійними операторами.

Ключові слова і фрази: простір Ріса, банахова ґратка, однорідний оператор, сублінійний оператор, порядково неперервний оператор.