ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2018, **10** (2), 408–411
doi:10.15330/cmp.10.2.408-411

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2018, Т.10, №2, С.408–411

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THE CONVERGENCE CLASSES FOR ANALYTIC FUNCTIONS IN THE REINHARDT DOMAINS

Let L^0 be the class of positive increasing on $[1,+\infty)$ functions l such that $l((1+o(1))x) = (1+o(1))l(x)$ $(x \to +\infty)$. We assume that α is a concave function such that $\alpha(e^x) \in L^0$ and function $\beta \in L^0$ such that $\int_1^{+\infty} \alpha(x)/\beta(x)dx < +\infty$. In the article it is proved the following theorem: If $f(z) = \sum_{\|n\|=0}^{+\infty} a_n z_n$, $z \in \mathbb{C}^p$, is analytic function in the bounded Reinhard domain $G \subset \mathbb{C}^p$, then the condition $\int\limits_{R_0}^{1} \frac{\alpha(\ln^+ M_G(R,f))}{(1-R)^2\beta(1/(1-R))} d\,R < +\infty$, $M_G(R,f) = \sup\{|F(Rz)| \colon z \in G\}$, yields that

$$\sum_{k=0}^{+\infty} (\alpha(k) - \alpha(k-1))\beta_1\left(k/\ln^+|A_k|\right) < +\infty, \quad \beta_1(x) = \int\limits_x^{+\infty} \frac{dt}{\beta(t)}, \quad A_k = \max\{|a_n| \colon \|n\| = k\}.$$

Key words and phrases: analytic function, Reinhardt domain, convergence class.

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1 INTRODUCTION

We denote by $\mathcal{A}^p(G)$, $p \in \mathbb{N}$, the class of analytic functions f in $G \subset \mathbb{C}^p$, represented by power series of the form

$$f(z) = f(z_1, \ldots, z_p) = \sum_{\|n\|=0}^{+\infty} a_n z^n, \quad z = (z_1, \ldots, z_p), \tag{1}$$

with the domain of convergence G, where $z^n = z_1^{n_1} \ldots z_p^{n_p}$, $n = (n_1, \ldots, n_p) \in \mathbb{Z}_+^p$, $\|n\| = \sum_{j=1}^p n_j$; $\mathcal{E}^p := \mathcal{A}^p(\mathbb{C}^p)$ is the class of entire functions in several variables (i.e., analytic functions in $\mathbb{C}^p$). From the one hand, it is well-known that every analytic function f in the complete Reinhardt domain G with center at $z = 0$ can be represented in G by the series of form (1). On the other hand, the domain of convergence of each series of form (1) is the logarithmically-convex complete Reinhardt domain with center $z = 0$.

We say that a domain $G \subset \mathbb{C}^p$ is the complete Reinhardt domain if:

a) $z = (z_1, \ldots, z_p) \in G \Longrightarrow (\forall R = (R_1, \ldots, R_p) \in [0,1]^p) \colon Rz = (R_1 z_1, \ldots, R_p z_p) \in G$ (a complete domain);

b) $(z_1, \ldots, z_p) \in G \Longrightarrow (\forall (\theta_1, \ldots, \theta_p) \in \mathbb{R}^p) : (z_1 e^{i\theta_1}, \ldots, z_p e^{i\theta_p}) \in G$ (a multiple-circular domain).

УДК 517.5
2010 *Mathematics Subject Classification:* 32A05, 32A10, 32A07.

The Reinhardt domain G is called logarithmically-convex if the image of the set $G^* = \{z \in G\colon z_1 \cdot \ldots \cdot z_p \neq 0\}$ under the mapping $Ln\colon z \to Ln(z) = (\ln|z_1|, \ldots, \ln|z_p|)$ is a convex set in the space $\mathbb{R}^p$.

In one complex variable ($p=1$), a logarithmically-convex Reinhardt domain is a disc.

The following complete Reinhardt domains ($p \geq 2$) are considered most frequently:

$$C_p(R) := \{z \in \mathbb{C}^p\colon |z_1| < R_1, \ldots, |z_p| < R_p\}, \quad R = (R_1, \ldots, R_p) \in (0, +\infty)^p, \quad \text{(polydisk)},$$
$$\mathbb{B}_p(r) := \{z \in \mathbb{C}^p\colon |z| := \sqrt{|z_1|^2 + \ldots + |z_p|^2} < r\} \quad \text{(ball)},$$
$$\Pi_p(r) := \{z \in \mathbb{C}^p\colon |z_1| + \ldots + |z_p| < r\}, \quad r > 0.$$

Remark 1. $C_p(R) \subset G$ *for every* $w = (w_1, \ldots, w_p) \in G$ *and* $R = (|w_1|, \ldots, |w_p|)$. *In particular,* $C_p(rw) \subset G_r$ *for every* $w = (w_1, \ldots, w_p) \in G$.

The domains $C_p(re_1)$, $e_1 = (1, \ldots, 1) \in \mathbb{R}^p$, $\mathbb{B}_p(r)$, $\Pi_p(r)$ $(r > 0)$ are the logarithmically-convex complete Reinhardt domains. But, for example, the complete Reinhardt domain

$$G = \{z = (z_1, z_p)\colon |z_1| < 1, |z_2| < 2\} \cup \{z = (z_1, z_p)\colon |z_1| < 2, |z_2| < 1\}$$

is not logarithmically-convex.

For a domain G and any $R \in (0,1)$ we denote $G_R = R \cdot G := \{Rz\colon z \in G\}$, and for a function $f \in \mathcal{A}^p(G)$ of the form (1) set

$$M_G(R, f) = \max\{|f(z)| : z \in \overline{G}_R\}, \quad \mu_G(R, f) = \max\{|a_n z^n|\colon z \in \overline{G}_R,\ n \in \mathbb{Z}_+^p\},$$
$$d_G(n) = \max\{|z^n|\colon z \in G\}.$$

Note, that $d_G(n) = 1$ in the case $G = C_p(e_1)$.

Let us denote by L the class of positive increasing on $[0, +\infty)$ functions, and by L^0 the class of functions $\alpha \in L$ such that $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ $(x \to +\infty)$.

For $\alpha \in L$ and $\beta \in L$ we consider the following convergence classes of integrals (in one variable definition see in [1])

$$\int\limits_{R_0}^{1} \frac{\alpha(\ln^+ M_G(R, f))}{(1-R)^2 \beta(1/(1-R))} dR < +\infty, \tag{2}$$
$$\int\limits_{R_0}^{1} \frac{\alpha(\ln^+ \mu_G(R, f))}{(1-R)^2 \beta(1/(1-R))} dR < +\infty. \tag{3}$$

By $\mathcal{E}^p_{\alpha\beta}$ and $\underline{\mathcal{E}}^p_{\alpha\beta}$ we denote the classes of entire functions $f \in \mathcal{E}^p$ for which conditions (2) and (3) are fulfilled, respectively.

We prove the following theorem.

Theorem 1. *Let* α *be a concave function on* $[x_0, +\infty)$, $\alpha(e^x) \in L^0$, *and a function* $\beta \in L^0$ *satisfies the conditions* $x\beta'(x)/\beta(x) - 2 \geq h > 0$ *on* $[x_0, +\infty)$ *and* $\int_{x_0}^{+\infty} \frac{\alpha(x)}{\beta(x)} dx < +\infty$. *In order that the function* $f \in \mathcal{E}^p(G)$ *of form* (1) *belongs to the class* $\mathcal{E}^p_{\alpha\beta}(G)$, *it is necessary that*

$$\sum_{k=0}^{+\infty} (\alpha(k) - \alpha(k-1)) \beta_1 \left(\frac{k}{\ln^+ |A_k|} \right) < +\infty, \quad \beta_1(x) = \int\limits_{x}^{+\infty} \frac{dt}{\beta(t)}, \tag{4}$$

where $A_k := \max\{|a_n| : \|n\| = k\}$.

2 THE PROOF OF THE MAIN RESULT

Proposition 1. *For each function $f \in \mathcal{E}^p(G)$ and any functions $\alpha, \beta \in L_2$ we have the following implication $f \in \mathcal{E}^p_{\alpha\beta}(G) \Longrightarrow F \in \underline{\mathcal{E}}^p_{\alpha\beta}$.*

The statement of this propositions follows from Proposition 2.

Proposition 2. *For every function $f \in \mathcal{E}^p(G)$ and for any $r \in (0,1)$*

$$\mu_G(r,F) = \max\{B_k r^k\colon k \geq 0\} \leq M_G(r,f) \leq c\frac{1}{(1-r)^{p+1}}\mu_G\Big(\frac{1+r}{2}, F\Big),$$

where $c = c(p) < +\infty$.

Lemma 1 ([2]). *Let $r \in (0,1)$, $\tau \in \mathbb{C}^p$, $k \geq 0$,*

$$B_k = \max\{|a_n| d_G(n)\colon \|n\| = k\}, \quad P_k(\tau) = \sum_{\|n\|=k} a_n \tau^n, \quad F_1(r) = \sum_{k=0}^{+\infty} M_G(1, P_k) r^k.$$

Then

$$B_k \leq M_G(1,P_k) \leq B_k(k+1)^p, \quad \mu_{F_1}(r) = \max\{M_G(1,P_k)r^k\colon\ k \geq 0\} \leq M_G(r,f) \leq F_1(r).$$

Proof of Proposition 2. By Lemma 1,

$$\begin{aligned}\mu_G(r,F) &= \max\{\big|a_n z^n\big|\colon z \in \overline{G}_r, n \in \mathbb{Z}^p_+\} = \max\{\big|a_n\big| \max\{\big|z^n\big|\colon z \in \overline{G}_r, \}\colon n \in Z^p_+\} \\ &= \max\{\big|a_n\big| d_G(n) r^k\colon n \in \mathbb{Z}^p_+, \|n\| = k \geq 0\} \\ &= \max\Big\{\max\big\{|a_n| d_G(n)\colon n \in \mathbb{Z}^p_+, \|n\| = k\big\} r^k\colon k \geq 0\Big\} = \max\{B_k r^k\colon k \geq 0\} \\ &\leq \max\{M_G(1,P_k) r^k\colon\ k \geq 0\} = \mu_{F_1}(r) \leq M_G(r,f).\end{aligned}$$

On the other hand,

$$\begin{aligned}M_G(r,f) &\leq \sum_{k=0}^{+\infty}\sum_{\|n\|=k} |a_n| \max\{\big|z^n\big|\colon z \in \overline{G}_r\} = \sum_{k=0}^{+\infty}\Big(\frac{2r}{1+r}\Big)^k \sum_{\|n\|=k} |a_n| \max\{\big|z^n\big|\colon z \in \overline{G}_{\frac{1+r}{2}}\} \\ &\leq \mu_G\Big(\frac{1+r}{2}, F\Big)\sum_{k=0}^{+\infty}\Big(\frac{2r}{1+r}\Big)^k (k+1)^p \leq c\frac{1}{(1-r)^p}\mu_G\Big(\frac{1+r}{2}, F\Big),\ c = c(p) < +\infty.\end{aligned}$$

□

The proof of Theorem 1. Let

$$F_2(R) = \textstyle\sum_{k=0}^{+\infty} B_k R^k, \quad F_3(R) = \sum_{k=0}^{+\infty} A_k R^k, \quad R \in (0,1).$$

From Remark 1 it follows

$$\begin{aligned}A_k R^k &= \max\{|a_n|\colon \|n\| = k\} R^k = \max\{|a_n| \max\{|z|^n\colon z \in \overline{C}_p(e_1)\}\colon \|n\| = k\} R^k \\ &= \max\{|a_n| \max\{|z|^n\colon z \in \overline{C}_p(Re_1)\}\colon \|n\| = k\} \max\{|a_n| \max\{|z|^n\colon z \in \overline{G}_R\}\colon \|n\| = k\} \\ &= \max\{|a_n| \max\{|z|^n\colon z \in \overline{G}\}\colon \|n\| = k\} R^k = \max\{|a_n| d_G(n)\colon \|n\| = k\} R^k = B_k R^k.\end{aligned}$$

Therefore, $\mu_{F_3}(R) \leq \mu_{F_2}(R) = \mu_G(R,F)$, $R \in (0,1)$.

Hence, by Proposition 2

$$f \in \mathcal{E}^p_{\alpha\beta}(G) \Longrightarrow \int_{R_0}^{1} \frac{\alpha(\ln^+ \mu_{F_3}(R))}{(1-R)^2 \beta(1/(1-R))} d\,R < +\infty.$$

Thus, from Theorem 2 in [1] it follows that for the function F_3 condition (4) holds. □

In the case $\alpha(x) \equiv x$, $\beta(x) = e^{px}$, $p > 0$, we obtain the following converse statement to Theorem 1.

Theorem 2. *Let* $f \in \mathcal{E}^p(G)$ *of form* (1) *with* $G = C_p(e_1)$, $A_k = \max\{|a_n|\colon \|n\|\} = k \geq 0$. *If* $A_k/A_{k+1} \nearrow 1$ *as* $k_0 \leq k \uparrow +\infty$ *and*

$$\sum_{k=1}^{+\infty} \Big(\frac{\ln^+ A_k}{k}\Big)^2 \exp\Big\{-\frac{pk}{\ln^+ A_k}\Big\} < +\infty,$$

then

$$\int\limits_{R_0}^{1} \frac{\ln^+ M_G(R,F)}{(1-R)^2 \exp\{p/(1-R)\}} dR < +\infty.$$

From Lemma 1 we obtain the following statement (see also proof of Proposition 2).

Lemma 2. *For* $R \in (0,1)$

$$\mu_{F_2}(R) \leq \mu_{F_1}(R) \leq c(p)\frac{1}{(1-R)^p}\mu_{F_2}\Big(\frac{1+R}{2}\Big).$$

Then $A_k = B_k$. The statement of Theorem 2 follows from Theorem 6 in [1] in a similar way as in the proof of Theorem 1 we use Theorem 2 from [1].

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Received 23.10.2018

Сало Т.М., Тарновецька О.Ю. *Класи збіжності для аналітичних функцій в областях Рейнгарда* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 408–411.

Нехай L^0 — клас додатних неспадних на $[1,+\infty)$ функцій l таких, що $l((1+o(1))x) = (1+o(1))l(x)$ $(x \to +\infty)$. Припустимо, що α — вгнута функція така, що $\alpha(e^x) \in L^0$, а функція $\beta \in L^0$ така, що $\int_1^{+\infty} \alpha(x)/\beta(x)dx < +\infty$. У статті доведено теорему: якщо $f(z) = \sum_{\|n\|=0}^{+\infty} a_n z_n$, $z \in \mathbb{C}^p$, — аналітична в обмеженій області Рейнгарда $G \subset \mathbb{C}^p$ функція, то з того, що виконується умова $\int\limits_{R_0}^{1} \frac{\alpha(\ln^+ M_G(R,f))}{(1-R)^2\beta(1/(1-R))} d\,R < +\infty$, $M_G(R,f) = \sup\{|F(Rz)|\colon z \in G\}$, випливає, що

$$\sum_{k=0}^{+\infty}(\alpha(k) - \alpha(k-1))\beta_1\left(k/\ln^+|A_k|\right) < +\infty, \quad \beta_1(x) = \int\limits_x^{+\infty}\frac{dt}{\beta(t)}, \quad A_k = \max\{|a_n|\colon \|n\| = k\}.$$

Ключові слова і фрази: аналітична функція, область Рейнгарда, клас збіжності.