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ASYMPTOTICS OF THE ENTIRE FUNCTIONS WITH v-DENSITY OF ZEROS ALONG THE LOGARITHMIC SPIRALS

Let v be the growth function such that $rv'(r)/v(r) \to 0$ as $r \to +\infty$, $l^c_\varphi = \{z = te^{i(\varphi + c\ln t)}, 1 \leqslant t < +\infty\}$ be the logarithmic spiral, f be the entire function of zero order. The asymptotics of $\ln f(re^{i(\theta+c\ln r)})$ along ordinary logarithmic spirals l^c_θ of the function f with v-density of zeros along l^c_φ outside of the C_0-set is found. The inverse statement is true just in case zeros of f are placed on the finite logarithmic spirals system $\Gamma_m = \bigcup_{j=0}^m l^c_{\theta_j}$.

Key words and phrases: entire function, density of zeros, logarithmic spiral.

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INTRODUCTION

The issues related to the study of behavior of entire functions along the logarithmic spirals were considered in [1–4, 6]. In particular, Macintyre [6] introduced the notion of an indicator along the logarithmic spiral and generalized the concept of associated function. Kennedy [3] generalized the concept of Mittag - Leffler function on the curvilinear area. Valiron-type and Valiron-Titchmarsh-type theorems for entire functions of positive order with zeros on the logarithmic spiral were proved by Balašov [2] and Kheifits [4] correspondingly. The relation between regular behavior of logarithm of modulus of entire function f of positive order along the curves of regular rotation (in particular, the logarithmic spirals) and existance of density of zeros of f along these curves was investigated in [1]. The results of [1] generalize the well-known Levin and Pfluger research of entire functions of completely regular growth (see, for example, [5, p. 118-122; p. 199]).

In this paper we study issues that similar to ones considered in [1] for entire functions of zero order.

1 SECTION WITH RESULTS

For $c \in \mathbb{R}$, $\varphi \in [-\pi;\pi)$ we denote by $l^c_\varphi(a,r) = \{z : z = te^{i(\varphi + c\ln t)}, a \leqslant t < r\}$, $l^c_\varphi(1,+\infty) = l^c_\varphi$ the logarithmic spiral, $D^c(r;\alpha,\beta) = \bigcup\limits_{\alpha\leqslant\varphi<\beta} l^c_\varphi(1,r)$ the curvilinear sector, $-\pi \leqslant \alpha < \beta < \pi$.

Let L be the set of all growth functions v such that $rv'(r)/v(r) \to 0$ as $r \to +\infty$ where growth function $v : [0;+\infty) \to \mathbb{R}_+$ is a continuously differentiable increasing to $+\infty$ function. It is clear that a set L coincides with accuracy to equivalent functions with a set of slow growing

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functions in the sense of Karamata ([7, p. 15]). For $v \in L$ we denote by $H_0(v)$ the class of entire functions f of zero order that satisfy the condition $n(r) = O(v(r))$, $r \to +\infty$, where $n(r) = n(r, 0, f)$ is counting function of zeros $(a_n)_{n=1}^{+\infty}$ of function f.

We say that zeros of the function $f \in H_0(v)$ have v-density $\Delta^c(\alpha, \beta)$ along logarithmic spirals l^c_φ if the limit

$$\lim_{r\to\infty} \frac{n^c(r;\alpha,\beta)}{v(r)} = \Delta^c(\alpha,\beta)$$

exists for all $\alpha, \beta \in \mathbb{R}$, $0 < \beta - \alpha \leqslant 2\pi$ with the exception, perhaps, of α or β belongs to some countable set $\mathcal{N}$, where $n^c(r;\alpha,\beta)$ is a number of zeros of the function f in $D^c(r;\alpha,\beta)$.

The equality $\Delta^c(\varphi) = \Delta^c(\varphi_1, \varphi)$ for a fixed $\varphi_1 \notin \mathcal{N}$ defines on the segment $[\varphi_1, \varphi_1 + 2\pi]$ a non-decreasing function $\Delta^c(\varphi)$ which we extend on $\mathbb{R}$ by the rule $\Delta^c(\varphi + 2\pi) - \Delta^c(\varphi) = \Delta^c(\varphi_1 + 2\pi) - \Delta^c(\varphi_1)$.

The logarithmic spiral l^c_θ satisfying the condition

$$\lim_{h\to 0+} \overline{\lim_{r\to+\infty}} \frac{n^c(r;\theta-h,\theta+h)}{v(r)} = 0$$

is called *ordinary* for $f \in H_0(v)$. The other logarithmic spirals are called *exceptional*. It follows from monotonicity of the function $\Delta^c(\varphi)$ that the set of exceptional logarithmic spirals is no more than countable if zeros of $f \in H_0(v)$ have v-density $\Delta^c(\alpha,\beta)$ along l^c_φ.

Denote by $\ln\left(1 - \dfrac{z}{a_n}\right)$, $a_n \in l^c_\theta$ the single-valued in the domain $D(l^c_\theta) = \mathbb{C} \setminus l^c_\theta(|a_n|, +\infty)$ branch of multi-valued function $Ln\left(1 - \dfrac{z}{a_n}\right)$ such that $\ln\left(1 - \dfrac{z}{a_n}\right)\Big|_{z=0} = 0$. Let

$$f(z) = \prod_{n=1}^{+\infty}\left(1 - \frac{z}{a_n}\right) \in H_0(v). \tag{1}$$

Then

$$\ln f(z) = \sum_{n=1}^{+\infty} \ln\left(1 - \frac{z}{a_n}\right),\ z \in \mathbb{C} \setminus \bigcup_{n=1}^{+\infty} l^c_{\varphi_i}(r_i, +\infty),$$

where r_i is the minimum module of zeros a_i of f that lie on the logarithmic spiral $l^c_{\varphi_i}$, $\varphi_i = \arg a_i \in [-\pi, \pi)$.

We call a set $E \in \mathbb{C}$ the C_0-set if it can be covered by a system of circles $\{z : |z - a_k| < r_k\}$, $k \in \mathbb{N}$ such that $\sum\limits_{|a_k|\leqslant r} r_k = o(r)$, $r \to +\infty$.

We write $\hat{h}(\theta;\psi)$ for the 2π-periodic extension of the function $h(\theta;\psi) = \theta - \psi - \pi$ from $(\psi; \psi + 2\pi)$ to $\mathbb{R}$, $-\pi \leqslant \psi < \pi$. Note $N(r) = N(r, 0, f) = \displaystyle\int_0^r \frac{n(t)}{t}\,dt$,

$$H^c_f(\theta) = \int_{\theta-2\pi}^{\theta} (\theta - \psi - \pi)\, d\Delta^c(\psi) = \int_{-\pi}^{\pi} \hat{h}(\theta;\psi) d\Delta^c(\psi). \tag{2}$$

Theorem 1. *Let $v \in L$, $f \in H_0(v)$, zeros of f have v-density $\Delta^c(\alpha,\beta)$ along l^c_φ. Then there is a C_0-set E such that the following asymptotic relation holds ($|z| = r$):*

$$\ln f(z) = (1 + ic)N(r) + iH^c_f(\theta)v(r) + o(v(r)),\ z \in l^c_\theta,\ z \notin E, \tag{3}$$

where l^c_θ is ordinary logarithmic spiral.

Let $\Gamma_m = \bigcup\limits_{j=1}^{m} l^c_{\theta_j}$, $-\pi \leqslant \theta_1 < \ldots < \theta_m < \pi$ be a finite system of logarithmic spirals, $\theta_{m+1} = \theta_1 + 2\pi$.

Theorem 2. *Let $v \in L$, $f \in H_0(v)$, zeros of f lie on Γ_m, H be a piecewise continuous on $[-\pi, \pi)$ function. If for any $\delta > 0$ the following asymptotic relation*

$$\ln f\left(re^{i(\theta + c\ln r)}\right) = (1+ic)N(r) + iH(\theta)v(r) + o(v(r)),\ r \to \infty \tag{4}$$

holds uniformly with respect to $\theta \in [-\pi, \pi) \setminus \bigcup\limits_{j=1}^{m+1} (\theta_j - \delta; \theta_j + \delta)$, then zeros of f have v-density $\Delta^c(\alpha, \beta)$ along l^c_φ.

Remark. *The condition that zeros of $f \in H_0(v)$ lie on a finite system of logarithmic spirals Γ_m is significant in Theorem 2. In the general case of zeros arrangement the statement of Theorem 2 is wrong (see [8] in case c = 0).*

2 THE PROOF OF RESULTS

At first we present the lemmas that will be used in the proof of the theorems.

Lemma 1 ([11]). *Let $\Delta > 0$, $v \in L$, $f \in H_0(v)$, zeros of f lie on the logarithmic spiral l^c_ψ, $\psi \in \mathbb{R}$,*

$$n(r) = (1 + o(1))\Delta v(r),\ r \to +\infty.$$

Then for $\theta \in \mathbb{R}\backslash\{\psi + 2\pi k : k \in \mathbb{Z}\}$ the following asymptotic relation holds:

$$\ln f\left(re^{i(\theta + c\ln r)}\right) = (1+ic)N(r) + i\Delta\hat{h}(\theta;\psi)v(r) + o(v(r)),\ r \to \infty, \tag{5}$$

moreover, relation (5) is uniform with respect to $\theta \in [\psi + \delta; \psi + 2\pi - \delta]$, $0 < \delta < 1$.

Lemma 2. *Let f has the form defined in (1), zeros of f have v-density $\Delta^c(\alpha, \beta)$ along l^c_φ, $\varepsilon > 0$ is arbitrary number. Then there exist $\delta > 0$ and a C_0-set E such that for all ordinary logarithmic spirals l^c_θ of the function f the following inequality holds:*

$$\left|\ln f(z) - \ln f^\delta(z)\right| < \varepsilon v(r),\ z \in l^c_\theta,\ z \notin E,$$

where $f^\delta(z) = \prod\limits_{n=1}^{+\infty}\left(1 - \dfrac{z}{a'_n}\right)$, $|a'_n| = |a_n|$, $|\arg a_n - \arg a'_n| < \delta$.

The proof of the Lemma 2 follows from the considerations similar to [5, p. 132-133], [1, p. 352-353] and Theorem 1 from [10].

We say that a set $F \subset \mathbb{R}_+$ is E_0-set if F is a measurable and $mes(E \cap [0,r]) = o(r)$, $r \to +\infty$.

In view of Lemmas 4 and 5 from [9], we get

Lemma 3. *Let $\theta \in [-\pi, \pi)$, $v \in L$, $f \in H_0(v)$, $\delta > 0$. Then there exists a E_0-set F such that*

$$r\int\limits_{\theta-\delta}^{\theta+\delta}\left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right| d\varphi = O(v(r))\left(\delta + \delta\ln\left(1 + \frac{1}{\delta}\right)\right),\ r \to +\infty,\ r \notin F.$$

Proof of Theorem 1. Let $\varepsilon > 0$ is given arbitrary number, function $H_f^c(\theta)$ defined by formula (2). Choose $\delta > 0$ such that the integral sum

$$S_m(\theta) = \sum_{j=0}^{m-1} \hat{h}(\theta;\psi_j)(\Delta^c(\psi_{j+1}) - \Delta^c(\psi_j)),$$

where $-\pi = \psi_0 < \psi_1 < \ldots < \psi_{m-1} < \psi_m = \pi$, $\max\limits_{0\leqslant j\leqslant m-1} |\psi_{j+1} - \psi_j| < \delta$, satisfies the inequality

$$\left| H_f^c(\theta) - S_m(\theta) \right| < \frac{\varepsilon}{3}. \tag{6}$$

Then take numbers a_k' such that $|a_k'| = |a_k|$, $a_k' \in l_{\psi_j}^c$ if $a_k \in l_\psi^c$, $\psi_j \leqslant \psi < \psi_{j+1}$ $(j = 0, 1, \ldots, m-1)$ and build the function $f^\delta(z)$. Applying Lemma 2 we obtain that there exist $\delta > 0$ and C_0-set E_1 such that for all ordinary logarithmic spirals l_θ^c of f and f^δ the following inequality holds:

$$\left| \ln f(z) - \ln f^\delta(z) \right| < \frac{\varepsilon}{3} v(r),\ z \notin E_1,\ z \in l_\theta^c. \tag{7}$$

Zeros of $f^\delta(z)$ lie on a finite system of logarithmic spirals Γ_m so $f^\delta(z)$ can be depicted as a product of m entire functions such that zeros of each function lie on a single logarithmic spiral $l_{\psi_j}^c$. From Lemma 1 (see (5)) we get that inequality

$$\left| \frac{\ln f^\delta(z) - (1+ic)N(r)}{v(r)} - iS_m(\theta) \right| < \varepsilon,\ z \in l_\theta^c$$

holds uniformly with respect to $\theta \in \mathbb{R} \setminus \bigcup_{j=1}^m (\psi_j - \delta; \psi_j + \delta)$, where $\delta > 0$ is an arbitrary number.

Further taking into account (6), (7) we obtain that for $z \notin E_1$, $z \in l_\theta^c$, $\theta \in \mathbb{R} \setminus \bigcup_{j=1}^m (\psi_j - \delta; \psi_j + \delta)$ the following inequality holds:

$$\left| \frac{\ln f(z) - (1+ic)N(r)}{v(r)} - iH_f^c(\theta) \right| < \varepsilon. \tag{8}$$

Choosing another segmentation of $[-\pi;\pi]$ by points $(\psi_j')_{j=0}^m$, $|\psi_{j+1}' - \psi_j'| < \delta$ such that intervals $(\psi_j' - \delta; \psi_j' + \delta)$ do not have the mutual points with intervals $(\psi_j - \delta; \psi_j + \delta)$, we get that (8) holds for $z \notin E_2$, $z \in l_\theta^c$, $\theta \in \mathbb{R} \setminus \bigcup_{j=1}^m (\psi_j' - \delta; \psi_j' + \delta)$, where E_2 is some C_0-set.

This yields that (3) holds for all ordinary logarithmic spirals l_θ^c of function f. So Theorem 1 is proved. □

Proof of Theorem 2. Let $v \in L$, $\Omega = \{|a_n| : n \in \mathbb{N}\}$, a_n be zeros of $f \in H_0(v)$ that lie on a finite system of logarithmic spirals $\Gamma_m = \bigcup\limits_{j=1}^m l_{\theta_j}^c$, $-\pi \leqslant \theta_1 < \ldots < \theta_m < \pi$. Set

$$\partial D^c(r;\alpha,\beta) = l_\alpha^c(1,r) \cup \Gamma(r;\alpha,\beta) \cup \left(l_\beta^c(1,r) \right)^{-1} \cup \left(\Gamma(1;\alpha,\beta)\right)^{-1},$$

where $r \notin \Omega$, $-\pi \leqslant \theta_{k_0-1} < \alpha < \theta_{k_0} < \ldots < \theta_{s_0} < \beta < \theta_{s_0+1} < \pi$,

$$\Gamma(\tau;\alpha,\beta) = \{z = \tau e^{i(\varphi + c\ln\tau)} : \alpha \leqslant \varphi \leqslant \beta\}.$$

Since $dz=(1+ic)e^{i(\varphi+c\ln t)}dt$ for $l^c_\theta(1,r)$ then with the notation

$$F(\tau,\varphi)=\tau e^{i(\varphi+c\ln\tau)}\frac{f'(\tau e^{i(\varphi+c\ln\tau)})}{f(\tau e^{i(\varphi+c\ln\tau)})}$$

using Residue theorem we have

$$\begin{aligned}2\pi i\, n^c(r;\alpha,\beta)&=\int\limits_{\partial D^c(r;\alpha,\beta)}\frac{f'(z)}{f(z)}\,dz=\left(\int\limits_{l^c_\alpha(1,r)}+\int\limits_{\Gamma(r;\alpha,\beta)}-\int\limits_{l^c_\beta(1,r)}-\int\limits_{\Gamma(1;\alpha,\beta)}\right)\frac{f'(z)}{f(z)}\,dz\\&=(1+ic)\int\limits_1^r\left(\frac{F(t,\alpha)}{t}-\frac{F(t,\beta)}{t}\right)dt+\int\limits_\alpha^\beta\left(F(r,\theta)-F(1,\theta)\right)id\theta\\&=\ln f(re^{i(\alpha+c\ln r)})-\ln f(re^{i(\beta+c\ln r)})\\&\qquad+\left(\int\limits_\alpha^{\theta_{k_0}-\delta}+\sum_{j=k_0}^{s_0-1}\int\limits_{\theta_j+\delta}^{\theta_{j+1}-\delta}+\int\limits_{\theta_{s_0}+\delta}^{\beta}+\sum_{j=k_0}^{s_0}\int\limits_{\theta_j-\delta}^{\theta_j+\delta}\right)F(r,\theta)id\theta+C,\end{aligned}\tag{9}$$

where $C=-\ln f(e^{i\alpha})+\ln f(e^{i\beta})-\int\limits_\alpha^\beta F(1,\theta)id\theta$, $0<\delta<\min\left\{\frac{\theta_{k_0}-\alpha}{2},\frac{\beta-\theta_{s_0}}{2},\frac{\theta_{j+1}-\theta_j}{2}\right\}$, $j=\overline{k_0,s_0-1}$.

Taking account of $\int\limits_{\theta_j+\delta}^{\theta_{j+1}-\delta}F(r,\theta)id\theta=\ln f(re^{i(\theta_{j+1}-\delta+c\ln r)})-\ln f(re^{i(\theta_j+\delta+c\ln r)})$, from (9) we obtain

$$\begin{aligned}2\pi i\, n^c(r;\alpha,\beta)&=\sum_{j=k_0}^{s_0}\left(\ln f(re^{i(\theta_j-\delta+c\ln r)})-\ln f(re^{i(\theta_j+\delta+c\ln r)})\right)\\&\qquad+\sum_{j=k_0}^{s_0}\int\limits_{\theta_j-\delta}^{\theta_j+\delta}F(re^{i(\theta+c\ln r)})id\theta=\sum\nolimits_1+\sum\nolimits_2.\end{aligned}\tag{10}$$

Applying (4) we get

$$\sum\nolimits_1=i\sum_{j=k_0}^{s_0}\left(H(\theta_j-\delta)-H(\theta_j+\delta)\right)v(r)+o(v(r)),\ r\to\infty.$$

In view of Lemma 3, there exist E_0-sets F_j such that $(j=\overline{k_0,s_0})$

$$\begin{aligned}\left|\int\limits_{\theta_j-\delta}^{\theta_j+\delta}F(re^{i(\theta+c\ln r)})id\theta\right|&\leqslant r\int\limits_{\theta_j-\delta}^{\theta_j+\delta}\left|\frac{f'(re^{i(\theta+c\ln r)})}{f(re^{i(\theta+c\ln r)})}\right|d\theta=r\int\limits_{\theta_j-\delta}^{\theta_j+\delta}\left|\frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\right|d\varphi\\&=O(v(r))\left(\delta+\delta\ln\left(1+\frac{1}{\delta}\right)\right),\ r\to+\infty,\ r\notin F_j.\end{aligned}$$

So,

$$\left|\sum\nolimits_2\right| \leqslant K_1(v(r))\left(\delta+\delta\ln\left(1+\frac{1}{\delta}\right)\right),\ r\to+\infty,\ r\notin F,$$

where $F=\bigcup\limits_{j=k_0}^{s_0}F_j$ is a E_0-set, K_1 is some constant.

Combining the last inequalities and (10) yields

$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{n^c(r;\alpha,\beta)}{v(r)}=\frac{1}{2\pi}\sum_{j=k_0}^{s_0}\big(H(\theta_j-\delta)-H(\theta_j+\delta)\big)+K_2\left(\delta+\delta\ln\left(1+\frac{1}{\delta}\right)\right).$$

Directing δ to 0+ gives

$$\lim_{\substack{r\to+\infty\\ r\notin E}}\frac{n^c(r;\alpha,\beta)}{v(r)}=\frac{1}{2\pi}\sum_{j=k_0}^{s_0}\big(H(\theta_j-0)-H(\theta_j+0)\big):=\Delta(\alpha,\beta).$$

Whereas F is E_0-set, then any interval $(R,(1+\eta)R)$, $\eta>0$, includes points that are not in F. Due to the monotonicity of the function $n^c(r;\alpha,\beta)$ with respect to r for $r>R_0$ we can assert that

$$\frac{n^c(r_1;\alpha,\beta)}{v(r_1)}\frac{v(r_1)}{v(r)}\leqslant\frac{n^c(r;\alpha,\beta)}{v(r)}\leqslant\frac{n^c(r_2;\alpha,\beta)}{v(r_2)}\frac{v(r_2)}{v(r)},$$

where $r(1-\eta)<r_1<r<r_2<(1+\eta)r,\ r_1,r_2\notin F.$

Since $v(r_2)\sim v(r)\sim v(r_1), r\to\infty$, the last relation yields

$$\lim_{r\to\infty}\frac{n^c(r;\alpha,\beta)}{v(r)}=\Delta(\alpha,\beta).$$

Theorem 2 is proved. □

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Нехай функція зростання v така, що $rv'(r)/v(r) \to 0$ при $r \to +\infty$, $l_\varphi^c = \{z = te^{i(\varphi + c\ln t)}, 1 \leqslant t < +\infty\}$ — логарифмічна спіраль, f — ціла функція нульового порядку. За умови існування v-щільності нулів f вздовж l_φ^c знайдено асимптотику $\ln f(re^{i(\theta + c\ln r)})$ вздовж звичайних логарифмічних спіралей l_θ^c функції f зовні C_0-множини. Показано, що обернене до цього твердження правильне лише у випадку розташування нулів f на скінченній системі логарифмічних спіралей $\Gamma_m = \bigcup_{j=0}^m l_{\theta_j}^c$.

Ключові слова і фрази: ціла функція, щільність нулів, логарифмічна спіраль.