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INVERSE PROBLEM FOR $2b$-ORDER DIFFERENTIAL EQUATION WITH A TIME-FRACTIONAL DERIVATIVE

We study the inverse problem for a differential equation of order $2b$ with the Riemann-Liouville fractional derivative of order $\beta \in (0,1)$ in time and given Schwartz type distributions in the right-hand sides of the equation and the initial condition. The problem is to find the pair of functions (u,g): a generalized solution u to the Cauchy problem for such equation and the time dependent multiplier g in the right-hand side of the equation. As an additional condition, we use an analog of the integral condition

$$(u(\cdot,t),\varphi_0(\cdot)) = F(t), \quad t \in [0,T],$$

where the symbol $(u(\cdot,t),\varphi_0(\cdot))$ stands for the value of an unknown distribution u on the given test function φ_0 for every $t \in [0,T]$, F is a given continuous function.

We prove a theorem for the existence and uniqueness of a generalized solution of the Cauchy problem, obtain its representation using the Green's vector-function. The proof of the theorem is based on the properties of conjugate Green's operators of the Cauchy problem on spaces of the Schwartz type test functions and on the structure of the Schwartz type distributions.

We establish sufficient conditions for a unique solvability of the inverse problem and find a representation of an unknown function g by means of a solution of a certain Volterra integral equation of the second kind with an integrable kernel.

Key words and phrases: distribution, fractional derivative, inverse problem, Green vector-function.

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INTRODUCTION

Different initial and boundary value problems to differential and pseudo-differential equations with distributions in the right-hand sides are sufficiently investigated (see, for example, [1–9] and references therein).

Equations with fractional derivatives [10] and inverse problems to them are appearing in different branches of science and engineering, and the range of the applicability of the generated models is increase considerable. The conditions of classical solvability of the Cauchy and boundary value problems to equations with a time fractional derivative were obtained, for example, in [11–15]. The inverse boundary value problems to a time fractional diffusion equation with different unknown functions or parameters were investigated, for example, in [16–24]. Most papers were devoted to inverse problems with an unknown right-hand sides, mainly under regular data.

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In this paper for the equation

$$u_t^{(\beta)} - A(D)u = g(t)F_0(x), \quad (x,t) \in \mathbb{R}^n \times (0,T] := Q, \tag{1}$$

with the Riemann-Liouville fractional derivative of order $\beta \in (0,1)$ we study the inverse problem

$$u(x,0) = F_1(x), \ \ x \in \mathbb{R}^n, \tag{2}$$

$$(u(\cdot,t), \varphi_0(\cdot)) = F(t), \quad t \in [0,T], \tag{3}$$

of the determination the pair (u,g) where

$$A(D)u = \sum_{|\gamma|\leq 2b} A_\gamma D^\gamma u$$

is a differential expression of order $2b$ with constants coefficients A_γ, $|\gamma| \leq 2b$ such that

$$\frac{\partial u}{\partial t} - A(D)u$$

is the parabolic differential expression [7,12], F_j $(j = 0,1)$ are given Schwartz type distributions, F is a given continuous function, the symbol $(u(\cdot,t), \varphi_0(\cdot))$ stands for the value of an unknown distribution u on the given test function φ_0 for every $t \in [0,T]$.

Note that the conditions of the existence a regular solution for such fractional Cauchy problem, even with the variable coefficients $A_\gamma = A_\gamma(x)$, $|\gamma| \leq 2b$, was obtained in [8] by M.I. Matijchuk. The inverse boundary value problems of finding a pair (u,g) for a time-fractional diffusion equations under regular given data in the right-hand sides and similar (integral) over-determination conditions were studied, for example, in [16,18]. The over-determination condition of kind (3), but with the scalar product (u,φ_0) in abstract Hilbert space, was used in [17]. The inverse problem of kind (1)–(3) with $(-\Delta)^{\gamma/2}$ $(\gamma > \beta)$ instead of $A(D)$ and distributions with compact supports in the right-hand sides was studied in [22].

1 NOTATIONS, DEFINITIONS AND AUXILIARY RESULTS

We use the following: $Q = \mathbb{R}^n \times (0,T]$, $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, $\bar{\alpha} = (\alpha_0,\alpha)$, $\alpha_j \in \mathbb{Z}_+$, $j \in \{0,1,\ldots,n\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}$, $D^\alpha v(x,t) = D_x^\alpha v(x,t) = \frac{\partial^{|\alpha|} v(x,t)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $D^{\bar{\alpha}} v(x,t) = (\frac{\partial}{\partial t})^{\alpha_0} D^\alpha v(x,t)$, $\mathcal{S}(\mathbb{R}^n)$ is the space of indefinitely differentiable functions v in $\mathbb{R}^n$ such that $x^\gamma D^\alpha v$ are bounded in $\mathbb{R}^n$ for all multi-indexes α, γ (the Schwartz space of smooth rapidly decreasing functions), $S_\gamma(\mathbb{R}^n)$ $(\gamma > 0)$ is the space of type $\mathcal{S}(\mathbb{R}^n)$ (see [2, p. 201]):

$$S_\gamma(\mathbb{R}^n) = \{v \in S(\mathbb{R}^n) : |D^\alpha v(x)| \leq C_\alpha e^{-a|x|^{\frac{1}{\gamma}}}, \ \ x \in \mathbb{R}^n, \ \ \forall\, \alpha\}$$

with some positive constants $C_\alpha = C_\alpha(v)$ and $a = a(v)$,

$$S_{\gamma,(a)}(\mathbb{R}^n) = \{v \in S(\mathbb{R}^n) : |D^\alpha v(x)| \leq C_{\alpha,\delta}(v) e^{-(a-\delta)|x|^{\frac{1}{\gamma}}}, \ \ x \in \mathbb{R}^n, \ \ \forall\, \alpha, \ \ \forall\, \delta > 0\}, \ a > 0,$$

$C^{\infty,(0)}(\bar{Q}) = \{v \in C^\infty(\overline{Q}) : (\frac{\partial}{\partial t})^k v|_{t=T} = 0, \ k \in \mathbb{Z}_+\}$, $\mathcal{S}(\bar{Q})$ $(\mathcal{S}_\gamma(\bar{Q}), S_{\gamma,(a)}(\bar{Q}))$ is the space of functions $v \in C^{\infty,(0)}(\bar{Q})$ such that $(\frac{\partial}{\partial t})^s v(\cdot,t) \in \mathcal{S}(\mathbb{R}^n)$ $(\mathcal{S}_\gamma(\mathbb{R}^n)$, $\mathcal{S}_{\gamma,(a)}(\mathbb{R}^n)$, respectively) for all $t \in [0,T]$, $s \in \mathbb{Z}_+$. By E' we denote the space of linear continuous functionals over E (the space of distributions). The symbol (f, φ) stands for the value of the distribution $f \in E'$ on the test function $\varphi \in E$,

$$S'_{\gamma,C}(\bar{Q}) = \{f \in S'_\gamma(\bar{Q}) : \ \big(f(x,\cdot), \varphi(x)\big) \in C[0,T] \quad \forall \varphi \in S_\gamma(\mathbb{R}^n)\},$$
$$S'_{\gamma,(a),C}(\bar{Q}) = \{f \in S'_{\gamma,(a)}(\bar{Q}) : \ \big(f(x,\cdot), \varphi(x)\big) \in C[0,T] \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n)\}, \ a > 0.$$

We denote by $(g \hat{*} \varphi)(x) = \big(g(\xi), \varphi(x+\xi)\big)$ the convolution of the distribution g and the test function φ, by $f*g$ the convolution of the distributions f and g: $(f*g, \varphi) = (f, g \hat{*} \varphi)$ for any test function φ, by fg the direct product of the distributions f and g: $(fg, \varphi) = (f(x), (g(t), \varphi(x,t))$ for any test function $\varphi(x,t)$, use the function

$$f_\lambda(t) = \frac{\theta(t) t^{\lambda-1}}{\Gamma(\lambda)} \text{ for } \lambda > 0 \quad \text{and} \quad f_\lambda(t) = f'_{1+\lambda}(t) \ \text{ for } \lambda \leq 0,$$

where $\Gamma(\lambda)$ is the Gamma-function, $\theta(t)$ is the Heaviside function. Note that $f_\lambda * f_\mu = f_{\lambda+\mu}$, $f_\lambda \hat{*} f_\mu = f_{\lambda+\mu}$.

The Riemann-Liouville derivative $v^{(\beta)}(t)$ of order $\beta > 0$ is defined by the formula

$$v^{(\beta)}(t) = f_{-\beta}(t) * v(t),$$

the Djrbashian-Caputo (regularized) fractional derivative of order $\beta \in (0,1)$ is defined by

$$D^\beta v(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} v'(\tau) d\tau,$$

and therefore $D^\beta v(t) = v^{(\beta)}(t) - f_{1-\beta}(t) v(0)$.

We denote

$$(Lv)(x,t) \equiv v_t^{(\beta)}(x,t) - (Av)(x,t),$$
$$(L^{reg} v)(x,t) \equiv D_t^\beta v(x,t) - (Av)(x,t),$$
$$(\widehat{L} v)(x,t) \equiv f_{-\beta}(t) \hat{*} v(x,t) - (Av)(x,t), \quad (x,t) \in Q.$$

The Green formula

$$\int_Q v(x,\tau)(\widehat{L}\psi)(x,\tau) dx d\tau = \int_Q (L^{reg} v)(x,\tau) \psi(x,\tau) dx d\tau + \int_Q v(x,0) f_{1-\beta}(\tau) \psi(x,\tau) dx d\tau,$$

$v, \psi \in \mathcal{S}(\bar{Q})$, holds (see, for example, [5]).

Definition 1. *The function* $u \in \mathcal{S}'_{\gamma,(a),C}(\bar{Q})$ *is called a solution of the Cauchy problem* (1), (2) *if the identity*

$$\int_0^T \big(u(\cdot,t), (\widehat{L}\psi)(\cdot,t)\big) dt = \int_0^T g(t) \big(F_0(\cdot), \psi(\cdot,t)\big) dt + \big(F_1(y) f_{1-\beta}(t), \psi(y,t)\big) \tag{4}$$

holds for all $\psi \in \mathcal{S}_{\gamma,(a)}(\bar{Q})$.

Definition 2. *The pair* $(u, g) \in \mathcal{S}'_{\gamma,(a),C}(\bar{Q}) \times C[0,T]$ *is called a solution of the problem* (1)–(3) *if the identity* (4) *and the condition* (3) *hold.*

It follows from (2) and (3) the compatibility condition

$$(F_1, \varphi_0) = F(0). \tag{5}$$

Definition 3. *The vector-function* $\big(G_0(x,t), G_1(x,t)\big)$ *is called a Green vector-function of the Cauchy problem* (2) *to the equation* $(Lu)(x,t) = \Phi(x,t)$, $(x,t) \in Q$, *and also of such problem to the equation*

$$(L^{reg}u)(x,t) = \Phi(x,t), \quad (x,t) \in Q, \tag{6}$$

if under rather regular Φ, F_1 *the function*

$$u(x,t) = \int_0^t d\tau \int_{\mathbb{R}^n} G_0(x-y, t-\tau)\Phi(y,\tau)dy + \int_{\mathbb{R}^n} G_1(x-y,t)F_1(y)dy, \quad (x,t) \in \overline{Q}, \tag{7}$$

is the regular solution of the problem (6), (2).

Such Green vector-function exists [8] and has the following bounds:

$$\begin{aligned} |G_0(x,t)| &\le Ct^{-\frac{\beta n}{2b}+\beta-1} e^{-c(|x|t^{-\frac{\beta}{2b}})^{\frac{2b}{2b-\beta}}} \Psi_{n-2b}(|x|t^{-\frac{\beta}{2b}}), \\ |G_1(x,t)| &\le Ct^{-\frac{\beta n}{2b}} e^{-c(|x|t^{-\frac{\beta}{2b}})^{\frac{2b}{2b-\beta}}} \Psi_{n-2b}(|x|t^{-\frac{\beta}{2b}}), \end{aligned} \tag{8}$$

where $\Psi_m(z) = \Psi_m(1)$ for $|z| > 1$ and $\Psi_m(z) = \begin{cases} 1, & m < 0, \\ 1 + |ln|z||, & m = 0, \\ |z|^{-m}, & m > 0, \end{cases}$ for $|z| < 1$.

Hereinafter $c, C, c_k, \widehat{c}_k, d_k, \widehat{d}_k, C_k, \widehat{C}_k$ $(k \in \mathbb{Z}_+)$ are positive constants. Let

$$(\widehat{\mathcal{G}}_0\varphi)(y,\tau) = \int_\tau^T dt \int_{\mathbb{R}^n} \varphi(x,t)G_0(x-y,t-\tau)dx, (y,\tau) \in \bar{Q},$$

$$(\widehat{\mathcal{G}}_1\varphi)(y) = \int_0^T dt \int_{\mathbb{R}^n} \varphi(x,t)G_1(x-y,t)dx, y \in \mathbb{R}^n.$$

$$(\widehat{G_j}\varphi)(y,t) = \int_{\mathbb{R}^n} G_j(x-y,t)\varphi(x)\, dx, \ \ (y,t) \in \bar{Q}, \ \ j = \overline{0,1}.$$

Lemma 1. *If* $a > 0$, $\gamma \ge 1 - \frac{\beta}{2b}$, $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ *then there exist numbers* $C > 0$, $a' \in (0,a]$ *such that for all* $k \in \mathbb{Z}_+$, *multi-index* κ, $|\kappa| = k$, $\delta > 0$ *the following bounds hold:*

$$\big|D_y^\kappa(\widehat{G_0}\varphi)(y,t)\big| \le c_k t^{\beta-1} e^{-(a'-\delta)|y|^{\frac{1}{\gamma}}} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha\varphi(x)| e^{(a'-\delta)|x|^{\frac{1}{\gamma}}}, (y,t) \in Q,$$

$$\big|D_y^\kappa(\widehat{G_1}\varphi)(y,t)\big| \le c_k e^{-(a'-\delta)|y|^{\frac{1}{\gamma}}} \max_{|\alpha|\le k} \sup_{x\in\mathbb{R}^n} |D^\alpha\varphi(x)| e^{(a'-\delta)|x|^{\frac{1}{\alpha}}}, (y,t) \in \bar{Q}.$$

Proof. We use the bounds (8). In the case $n > 2b$ for all multi-index α, $|\alpha| = k$, $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$

and $\delta' = \delta/a$ we have

$$\Big|\int\limits_{\mathbb{R}^n} G_0(x-y,t-\tau)D^\alpha\varphi(x)dx\Big| \le \int\limits_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} |G_0(x-y,t-\tau)|\,|D^\alpha\varphi(x)|dx$$

$$+ \int\limits_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} |G_0(x-y,t-\tau)|\,|D^\alpha\varphi(x)|dx$$

$$\le C(t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\Big[\int\limits_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} \frac{|D^\alpha\varphi(x)||x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}dx$$

$$+ \int\limits_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} e^{-c[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}|D^\alpha\varphi(x)|dx\Big]$$

$$\le C_1(t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\Big[\int\limits_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} \frac{|x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}e^{-c(1-\delta')[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx$$

$$+ \int\limits_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} e^{-c\delta'[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}e^{-c(1-\delta')[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx\Big]\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_1(t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\Big[\int\limits_{\{x\in\mathbb{R}^n:|x-y|<(t-\tau)^{\frac{\beta}{2b}}\}} \frac{|x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}e^{-c(1-\delta')[|x-y|T^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx$$

$$+ \int\limits_{\{x\in\mathbb{R}^n:|x-y|>(t-\tau)^{\frac{\beta}{2b}}\}} e^{-c\delta'[|x-y|(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}e^{-c(1-\delta')[|x-y|T^{-\frac{\beta}{2b}}]^{\frac{1}{\gamma}}}e^{-a(1-\delta')|x|^{\frac{1}{\gamma}}}dx\Big]\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}.$$

Putting $c_\gamma = 2^{1-\frac{1}{\gamma}}$ for $\gamma \in [1-\frac{\beta}{2b},1]$, $c_\gamma = 1$ for $\gamma \ge 1$, $a' = c_\gamma \min\{cT^{-\frac{\beta}{2b\gamma}}, a\}$ and using the inequality [12, p. 25] $|A|^{\frac{1}{\gamma}} + |B|^{\frac{1}{\gamma}} \ge c_\gamma|A+B|^{\frac{1}{\gamma}}$ we get

$$c\big(|x-y|T^{-\frac{\beta}{2b}}\big)^{\frac{1}{\gamma}} + a|x|^{\frac{1}{\gamma}} \ge \min\{cT^{-\frac{\beta}{2b\gamma}},a\}[|x-y|^{\frac{1}{\gamma}}+|x|^{\frac{1}{\gamma}}] \ge a'|y|^{\frac{1}{\gamma}}.$$

Then

$$\Big|\int\limits_{\mathbb{R}^n} G_0(x-y,t-\tau)D^\alpha\varphi(x)dx\Big| \le C_2\Big[\frac{1}{(t-\tau)}\int\limits_0^{(t-\tau)^{\frac{\beta}{2b}}} r^{2b-1}dr$$

$$+(t-\tau)^{-\frac{\beta n}{2b}+\beta-1}\int\limits_{t^{\frac{\beta}{2b}}}^{\infty} r^{n-1}e^{-c\delta'[r(t-\tau)^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}dr\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_3(t-\tau)^{\beta-1}\Big[1+\int\limits_1^{+\infty} z^{(1-\frac{\beta}{2b})n-1}e^{-c\delta' z}\,dz\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_4(t-\tau)^{\beta-1}e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}, \quad y\in\mathbb{R}^n,\ \ 0\le\tau<t\le T,$$

$$\Big|\int\limits_{\mathbb{R}^n} G_1(x-y,t)D^\alpha\varphi(x)dx\Big| \le C(t-\tau)^{-\frac{\beta n}{2b}}\Big[\int\limits_{\{x\in\mathbb{R}^n:|x-y|<t^{\frac{\beta}{2b}}\}} \frac{|x-y|^{2b-n}}{(t-\tau)^{\frac{\beta(2b-n)}{2b}}}dx$$

$$+\int\limits_{\{x\in\mathbb{R}^n:|x-y|>t^{\frac{\beta}{2b}}\}} e^{-c[|x-y|t^{-\frac{\beta}{2b}}]^{\frac{2b}{2b-\beta}}}dx\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta)|\xi|^{\frac{1}{\gamma}}}$$

$$\le C_5\Big[1+\int\limits_1^\infty z^{n-1-\frac{\beta n}{2b}}e^{-cz}dz\Big]e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}}$$

$$= C_6e^{-a'(1-\delta')|y|^{\frac{1}{\gamma}}}\sup_{\xi\in\mathbb{R}^n}|D^\alpha\varphi(\xi)|e^{a(1-\delta')|\xi|^{\frac{1}{\gamma}}},\quad (y,t)\in\bar{Q}$$

and similarly for $n\le 2b$. Integrating by parts we finish the proof. □

Lemma 2. *If $a>0$, $\gamma\ge 1-\frac{\beta}{2b}$, $a'=c_\gamma\min\{cT^{-\frac{\beta}{2b\gamma}},a\}$, then*

$\widehat{G}_0: S_{\gamma,(a)}(\mathbb{R}^n)\to S_{\gamma,(a')}(\mathbb{R}^n)$, $\widehat{G}_1: S_{\gamma,(a)}(\mathbb{R}^n)\to S_{\gamma,(a')}(\mathbb{R}^n)$, *for each* $t\in[0,T]$,

$\widehat{\mathcal{G}}_0: S_{\gamma,(a)}(\bar{Q})\to S_{\gamma,(a')}(\bar{Q})$, $\widehat{\mathcal{G}}_1: S_{\gamma,(a)}(\bar{Q})\to S_{\gamma,(a')}(\mathbb{R}^n)$.

Proof. It follows from Lemma 1 the correctness of the mappings for $\widehat{G}_j$, $j=0,1$. Using the property of the convolution and convolution's differentiation we finish the proof. □

Lemma 3. *For $\gamma\ge 1$, $0<aT^{\frac{\beta}{2b\gamma}}\le c$, any $\psi\in S_{\gamma,(a)}(\bar{Q})$ the following relations hold:*

$$\begin{aligned}(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau)&=\psi(y,\tau),\quad (y,\tau)\in\bar{Q},\\ (\widehat{\mathcal{G}}_1(\widehat{L}\psi))(y)&=\big(f_{1-\beta}(\tau),\psi(y,\tau)\big),\quad y\in\mathbb{R}^n.\end{aligned}\tag{9}$$

Proof. For all $\psi\in S_{\gamma,(a)}(\bar{Q})$, $(y,s)\in\bar{Q}$ and multi-index α we have

$$(f_{-\beta}\hat{*}\psi)(y,s)=f'_{1-\beta}(s)\hat{*}\psi(y,s)=-\int_0^{T-s}\frac{q^{-\beta}}{\Gamma(1-\beta)}\frac{\partial}{\partial s}\psi(y,q+s)dq,$$

$$\Big(\frac{\partial}{\partial s}\Big)^k D_y^\alpha(f_{-\beta}\hat{*}\psi)(y,s)=(-1)^k\Big(f_{1-\beta}\hat{*}\Big(\frac{\partial}{\partial s}\Big)^{k+1}D_y^\alpha\psi\Big)(y,s).$$

Therefore, $f_{-\beta}\hat{*}\psi\in S_{\gamma,(a)}(\bar{Q})$ and $\widehat{L}\psi\in S_{\gamma,(a)}(\bar{Q})$. Then it follows from Lemmas 1 and 2 that $a'=a$ and $(\widehat{\mathcal{G}}_0(\widehat{L}\psi))\in S_{\gamma,(a)}(\bar{Q})$, $(\widehat{\mathcal{G}}_1(\widehat{L}\psi))\in S_{\gamma,(a)}(\mathbb{R}^n)$.

By [8], under rather regular (in particular, compactly supported) F_0,F_1, $g\in C[0,T]$ the unique regular solution (7) with $\Phi=F_0g$ of the Cauchy problem (1), (2) exists. Substituting it in the Green formula (instead of v) we get

$$\int\limits_Q\Big(\int\limits_0^t d\tau\int\limits_{\mathbb{R}^n}G_0(x-y,t-\tau)F(y)g(\tau)dy\Big)(\hat{L}\psi)(x,t)dxdt$$

$$+\int\limits_Q\Big(\int\limits_{\mathbb{R}^n}G_1(x-y,t)F_1(y)dy\Big)(\hat{L}\psi)(x,t)dxdt$$

$$=\int\limits_Q F(x)g(t)\psi(x,t)dxdt+\int\limits_{\mathbb{R}^n}F_1(x)\big(f_{1-\beta}(t),\psi(x,t)\big)dx,$$

$$\int\limits_Q \Big(\int\limits_\tau^T dt \int\limits_{\mathbb{R}^n} G_0(x-y,t-\tau)(\hat{L}\psi)(x,t)dx\Big) F(y)g(\tau)dyd\tau$$
$$+\int\limits_{\mathbb{R}^n}\Big(\int\limits_Q G_1(x-y,t)(\hat{L}\psi)(x,t)dxdt\Big)F_1(y)dy$$
$$=\int\limits_Q \psi(y,\tau)F(y,\tau)dyd\tau+\int\limits_{\mathbb{R}^n}\big(f_{1-\beta}(t),\psi(y,t)\big)F_1(y)dy,$$

and obtain the desirable formulas (9) after an arbitrariness of F_0, F_1, g. □

Lemma 4. *For any $\varphi \in S_{\gamma,(a)}(\bar{Q})$ there exists $\psi \in S_{\gamma,(a)}(\bar{Q})$ such that*

$$(\widehat{L}\psi)(x,t)=\varphi(x,t), \qquad (x,t)\in\bar{Q}.$$

Proof. As in [21], we show that

$$\psi(y,\tau)=\int\limits_\tau^T dt\int\limits_{\mathbb{R}^n} G_0(x-y,t-\tau)\varphi(x,t)dx,\ \ (y,\tau)\in\bar{Q}$$

is the unknown function. □

2 Existence and uniqueness theorem for the Cauchy problem

Theorem 1. *Assume that $\gamma \geq 1$, $0 < aT^{\frac{\beta}{2b\gamma}} \leq c$, $F_0, F_1 \in S'_{\gamma,(a)}(\mathbb{R}^n)$, $g \in C[0,T]$. Then there exists the unique solution $u \in S'_{\gamma,(a),C}(\bar{Q})$ of the Cauchy problem (1), (2). It is defined by*

$$\Big(u(\cdot,t),\varphi(\cdot)\Big)=\int\limits_0^t g(\tau)\Big(F_0(\cdot),(\widehat{G}_0\varphi)(\cdot,t-\tau)\Big)d\tau+\Big(F_1(\cdot),(\widehat{G}_1\varphi)(\cdot,t)\Big)$$
$$\forall\varphi\in S_{\gamma,(a)}(\mathbb{R}^n),\ t\in[0,T]. \tag{10}$$

Moreover, for any $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ there exist positive constants $\widehat{d}_j = \widehat{d}_j(\varphi)$, $j = 0,1$ such that

$$\big|\big(u(\cdot,t),\varphi(\cdot)\big)\big|\leq \widehat{d}_0 t^\beta+\widehat{d}_1,\ \ t\in[0,T]. \tag{11}$$

Proof. Using Lemma 2 we get that for any $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ the right-hand side of (10) exists and belongs to $C[0,T]$. As in [22], we show that the function (10) satisfies the equality (4). For all $\psi \in S_{\gamma,(a)}(\overline{Q})$ we have

$$\big(u,\widehat{L}\psi\big)=\int\limits_0^T\Big(u(\cdot,t),(\widehat{L}\psi)(\cdot,t)\Big)dt$$
$$=\int\limits_0^T\Big(\int\limits_0^t g(\tau)\Big(F_0(y),(\widehat{G}_0(\widehat{L}\psi))(y,t,\tau)\Big)d\tau\Big)dt+\int\limits_0^T\Big(F_1(y),(\widehat{G}_1(\widehat{L}\psi))(y,t)\Big)dt$$
$$=\Big(F_0(y),\int\limits_0^T g(\tau)d\tau\int\limits_\tau^T(\widehat{G}_0(\widehat{L}\psi))(y,t,\tau)dt\Big)+\Big(F_1(y),\int\limits_0^T(\widehat{G}_1(\widehat{L}\psi))(y,t)dt\Big)$$
$$=\Big(F_0(y)\int\limits_0^T g(\tau)(\widehat{\mathcal{G}}_0(\widehat{L}\psi))(y,\tau)d\tau\Big)+\Big(F_1(y),\widehat{\mathcal{G}}_1(\widehat{L}\psi)(y)\Big).$$

Using Lemma 3 we get the identity (4). By Definition 1 the function (10) is the solution of the problem (1), (2).

To prove the performance of (11) for the function (10) we use [2, p. 211] that

$$S_{\gamma,(a)}(\mathbb{R}^n) = \{v \in C^\infty(\mathbb{R}^n) : ||v||_{k,a} = \sup_{|\alpha|\le k, x\in\mathbb{R}^n} e^{a(1-\frac{1}{k})|x|^{\frac{1}{\gamma}}} |D^\alpha v(x)| < +\infty \ \ \forall k \in \mathbb{N}, \ k \ge 2\}$$

and the sequence $v_m(x)$ converges to zero ($m \to +\infty$) in the space $S_{\gamma,(a)}(\mathbb{R}^n)$ if the sequence $D^\alpha v_m(x)$ converges to zero uniformly on an arbitrary compact $|x| \le C < +\infty$ for each multi-index α and the norms $||v_m||_{k,a}$ are limited at random $m, k \in \mathbb{N}$, $k \ge 2$. Note that

$$||v||_{k,a} \le ||v||_{k+p,a} \ \ \forall k, p \in \mathbb{N}, \ \ k \ge 2, \ \ a > 0, \ \ v \in S_{\gamma,(a)}(\mathbb{R}^n).$$

We say (see [25, p. 151]) that the distribution $F \in S'_{\gamma,(a)}(\mathbb{R}^n)$ has the order $k \in \mathbb{Z}_+$ if there exists $C > 0$ such that

$$\big|(F, \varphi)\big| \le C||\varphi||_{k,a} \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n). \tag{12}$$

A distribution from $S'_{\gamma,(a)}(\mathbb{R}^n)$ has a finite order. Indeed, the functional F satisfying (12) is continuous on $S_{\gamma,(a)}(\mathbb{R}^n)$. Conversely, if $F \in S'_{\gamma,(a)}(\mathbb{R}^n)$ and (12) is incorrect, then for each $k \in \mathbb{N}, \ k \ge 2$ there exists $\varphi_k \in S_{\gamma,(a)}(\mathbb{R}^n)$ such that $|(F, \varphi_k)| > k||\varphi_k||_{k,a}$. Then

$$|(F, \psi_k)| > 1, \ \text{ where } \ \psi_k(x) = \frac{\varphi_k(x)}{k||\varphi_k||_{k,a}}, \ \ x \in \mathbb{R}^n.$$

By definition, $||\psi_k||_{k,a} \le \frac{1}{k}$, and the sequence $\psi_k \to 0$ ($k \to \infty$) in the space $S_{\gamma,(a)}(\mathbb{R}^n)$. We get a contradiction with the previous inequality $|(F, \psi_k)| > 1$ for all $k \in \mathbb{N}, \ k \ge 2$.

So, there exist $k_j \in \mathbb{Z}_+$ and positive constants B_j such that

$$\big|(F_j, \varphi)\big| \le B_j||\varphi||_{k_j,a} \quad \forall \varphi \in S_{\gamma,(a)}(\mathbb{R}^n), \quad j = \overline{0,1}.$$

Using it and Lemma 1, for all $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$ we get

$$\begin{aligned}\big|(F_0(y), (\widehat{G_0}\varphi)(y, t-\tau))\big| &\le B_0||(\widehat{G_0}\varphi)(\cdot, t-\tau)||_{k_0,a} \\ &\le B_0 c_{k_0}(t-\tau)^{\beta-1}||\varphi||_{k_0,a} \le \widehat{c}_0(\varphi)(t-\tau)^{\beta-1}||\varphi||_{k_0,a}, \ 0 \le \tau < t \le T,\end{aligned}$$

$$\int_0^t |g(\tau)| \ \big|(F_0(y), (\widehat{G_0}\varphi)(y, t-\tau))\big| d\tau \le d_0 t^\beta ||\varphi||_{k_0,a} \le \widehat{d}_0 t^\beta, \ \text{ and similarly,}$$

$$\big|(F_1(\cdot), (\widehat{G}_1\varphi)(\cdot, t))\big| \le B_1||(\widehat{G}_1\varphi)(\cdot, t)||_{k_1,a} \le d_1||\varphi||_{k_1,a} \le \widehat{d}_1, \quad t \in [0, T].$$

Therefore, we obtain (11) with $\widehat{d_j} = d_j||\varphi||_{k,a}$, $k = \max\{k_0, k_1\}$, and see that the solution u of the Cauchy problem has the order k for each $t \in [0, T]$.

If u_1, u_2 are two solutions of the problem (1), (2) then for $u = u_1 - u_2$ from (4) we obtain

$$(u, \widehat{L}\psi) = 0 \ \ \forall \psi \in S_{\gamma,(a)}(\overline{Q}).$$

By using Lemma 4 we get $(u(\cdot, t), \varphi(\cdot)) = 0$ for all $\varphi \in S_{\gamma,(a)}(\mathbb{R}^n)$, $t \in [0, T]$. We obtain $u = 0$ in $S'_{\gamma,(a),C}(\overline{Q})$. □

3 SOLUTION OF THE INVERSE PROBLEM

We pass to the problem (1)–(3).

Theorem 2. *Assume that* $\gamma \geq 1$, $0 < aT^{\frac{\beta}{2b\gamma}} \leq c$, $F_0, F_1 \in S'_{\gamma,(a)}(\mathbb{R}^n)$, $g, F, F^{(\beta)} \in C[0,T]$, $\varphi_0 \in S_{\gamma,(a)}(\mathbb{R}^n)$, $(F_0, \varphi_0) \neq 0$ *and* (5) *holds. Then there exists the unique solution* $(u, g) \in S'_{\gamma,(a),C}(\bar{Q}) \times C[0,T]$ *of the problem* (1)–(3): u *is defined by (10) with*

$$g(t) = \left[F^{(\beta)}(t) - r(t)\right]\left[(F_0, \varphi_0)\right]^{-1}, \quad t \in [0,T], \tag{13}$$

where $r(t)$ *is the solution of the integral equation*

$$r(t) = -\int_0^t K(t,\tau) r(\tau) d\tau + v(t), \ \ t \in [0,T], \tag{14}$$

$$K(t,\tau) = \frac{\left(F_0(\cdot), (\widehat{G}_0 A\varphi_0)(\cdot, t-\tau)\right)}{(F_0, \varphi_0)}, \tag{15}$$

$$v(t) = \int_0^t K(t,\tau) F^{(\beta)}(\tau) d\tau + \left(F_1(\cdot), (\widehat{G}_1 A\varphi_0)(\cdot, t)\right), \ \ t \in [0,T]. \tag{16}$$

Proof. Let $u \in S'_{\gamma,(a),C}(\bar{Q})$ be the solution of the problem (1), (2). The equation (1) implies

$$\left(u_t^{(\beta)}(\cdot, t), \varphi_0(\cdot)\right) = \left(u(\cdot, t), A\varphi_0(\cdot)\right) + (F_0, \varphi_0) g(t).$$

By the over-determination condition (3) we get

$$F^{(\beta)}(t) = \left(u(\cdot, t), A\varphi_0(\cdot)\right) + (F_0, \varphi_0) g(t).$$

Using the assumption we find

$$g(t) = \left[F^{(\beta)}(t) - \left(u(\cdot, t), A\varphi_0(\cdot)\right)\right]\left[(F_0, \varphi_0)\right]^{-1}, \quad t \in [0,T]. \tag{17}$$

By Theorem 1 the right-hand side of (17) is the continuous function on $[0,T]$. By substituting it in (10) instead of $g(t)$ and putting $\varphi = \varphi_0$ one obtains

$$\left(u(\cdot, t), A\varphi_0(\cdot)\right) = \frac{1}{(F_0, \varphi_0)} \int_0^t \left[F^{(\beta)}(\tau) - \left(u(\cdot, \tau), A\varphi_0(\cdot)\right)\right] \left(F_0(\cdot), (\widehat{G}_0 A\varphi_0)(\cdot, t-\tau)\right) d\tau$$
$$+ \left(F_1(\cdot), (\widehat{G}_1 A\varphi_0)(\cdot, t)\right), \ \ t \in [0,T].$$

We denote

$$r(t) = \left(u(\cdot, t), A\varphi_0(\cdot)\right).$$

Then the previous equation takes the form of equation (14). As in the proof of Theorem 1 we get

$$\left|\left(F_0(\cdot), (\widehat{G}_0 A\varphi_0)(\cdot, t, \tau)\right)\right| \leq B_0 ||(\widehat{G}_0 A\varphi_0)(\cdot, t-\tau)||_{k_0}$$
$$\leq \widehat{C}_0 ||A\varphi_0(\cdot, t-\tau)||_{k_0} \leq \widehat{C}_0 \, (t-\tau)^{\beta-1} ||\varphi_0(\cdot, t-\tau)||_{k_0+2b},$$
$$\left|\left(F_1(\cdot), (\widehat{G}_1 A\varphi_0)(\cdot, t)\right)\right| \leq B_1 ||(\widehat{G}_1 A\varphi_0)(\cdot, t)||_{k_1} \leq \widehat{C}_1 ||\varphi_0(\cdot, t)||_{k_1+2b}.$$

So, the kernel (15) is integrable, the function (16) is continuous on $[0,T]$, and the second type Volterra integral equation (14) has the unique solution $r \in C[0,T]$.

Let r, g be defined by (14), (13), respectively. Then by Theorem 1 the function (10) is the solution of the Cauchy problem (1)–(2) with the known $g(t)$. Using the property

$$\lim_{t\to 0} \int_{\mathbb{R}^n} \varphi(x) G_1(x-y,t)dx = \varphi(0) \quad \forall \varphi \in S(\mathbb{R}^n)$$

and the condition (5) we get

$$(u(\cdot,0), \varphi_0(\cdot)) = \Big(F_1(\cdot), (\widehat{G}_1\varphi_0)(\cdot,0)\Big) = (F_1, \varphi_0) = F(0).$$

Show that the function (10) with $g(t)$ defined by (13) satisfies the condition (3). If $F^*(t) = \big(u(\cdot,t), \varphi_0(\cdot)\big)$ then $F^*(0) = F(0)$, and from the over-determination condition (3) we get

$$g(t) = \big[F^{*(\beta)}(t) - \big(u(\cdot,t), A\varphi_0(\cdot)\big)\big][(F_0,\varphi_0)]^{-1}, \quad t \in [0,T]. \tag{18}$$

As in the previous reasoning we obtain that the function $\big(u(\cdot,t), A\varphi_0(\cdot)\big)$ satisfies the equation (14), and by uniqueness of a solution of this equation we obtain $\big(u(\cdot,t), A\varphi_0(\cdot)\big) = r(t)$ for all $t \in [0,T]$. Then it follows from (18) and (13) that $F^{*(\beta)}(t) = F^{(\beta)}(t)$, and therefore, $F^*(t) = F(t)$, $t \in [0,T]$. So, the pair (u,g) defined by (10) and (13), with r defined by (14), is the solution of the problem (1)–(3).

If (u_1, g_1), (u_2, g_2) are two solutions of the problem (1)–(3), then for $u = u_1 - u_2$, $g = g_1 - g_2$ we obtain the problem

$$Lu(x,t) = F_0(x)g(t), \quad (x,t) \in Q,$$
$$u(x,0) = 0, \quad x \in \mathbb{R}^n,$$
$$\big(u(\cdot,t), \varphi_0(\cdot)\big) = 0, \quad t \in [0,T].$$

As before, we find

$$\big(u(\cdot,t), \varphi(\cdot)\big) = -\int_0^t r(\tau)\big(F_0(\cdot), (\widehat{G}_0\varphi)(\cdot,t-\tau)\big)d\tau \quad \forall \varphi \in S(\mathbb{R}^n),$$

$$g(t) = -\frac{r(t)}{(F_0,\varphi_0)}, \quad t \in [0,T],$$

where $r(t)$ is a solution of the second type homogeneous Volterra integral equation

$$r(t) = -\int_0^t K(t,\tau)r(\tau)d\tau, \quad t \in [0,T].$$

By uniqueness of a solution of this equation we obtain $r(t) = 0$ for all $t \in [0,T]$. Then, from the previous equalities, $g(t) = 0$ for all $t \in [0,T]$ and $u = 0$ in $S'_{\gamma,(a),C}(\bar{Q})$. □

4 CONCLUSIONS

We proved the solvability of an inverse problem of the determination a time-dependent continuous part of a source for a time fractional 2b-order equation with constant coefficients and Schwartz type distributions in the right-hand sides using the over-determination condition (3). In a such way, by using the results of [8] the obtained results extend to some case of the operator $A(x,D)$ with infinitely differentiable coefficients.

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Лопушанський А., Лопушанська Г. *Обернена задача для диференціального рівняння порядку* $2b$ *з дробовою похідною за часом* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 107–118.

Вивчаємо обернену задачу для диференціального рівняння порядку $2b$ з дробовою похідною порядку $\beta \in (0, 1)$ за часом і заданими узагальненими функціями типу Шварца у правих частинах рівняння і початкової умови. Задача полягає у знаходженні пари функцій (u, g): узагальненого розв'язку u задачі Коші для такого рівняння і залежного від часу неперервного множника g у правій частині рівняння. Як додаткову умову використовуємо аналог інтегральної умови

$$(u(\cdot, t), \varphi_0(\cdot)) = F(t), \quad t \in [0, T],$$

де $(u(\cdot, t), \varphi_0(\cdot))$ — значення шуканого узагальненого розв'язку u задачі Коші на фіксованій основній функції $\varphi_0(x)$, $x \in \mathbb{R}^n$ для кожного значення t, F — задана неперервна функція.

Доводимо теорему існування і єдиності узагальненого розв'язку задачі Коші, одержуємо його зображення за допомогою вектор-функції Ґріна. Доведення теореми ґрунтується на властивостях спряжених операторів Ґріна задачі Коші на просторах типу Шварца основних функцій і структурі узагальнених функцій типу Шварца.

Встановлюємо достатні умови однозначної розв'язності оберненої задачі і знаходимо зображення невідомої функції g через розв'язок певного інтегрального рівняння Вольтерри другого роду з інтегровним ядром.

Ключові слова і фрази: узагальнена функція, похідна дробового порядку, обернена задача, вектор-функція Ґріна.