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ON INTEGRAL REPRESENTATION OF THE SOLUTIONS OF A MODEL $\vec{2b}$-PARABOLIC BOUNDARY VALUE PROBLEM

A general boundary value problem for Eidelman type $\overrightarrow{2b}$-parabolic system of equation without minor terms in the equations and boundary conditions, and with constant coefficients in the group of major terms is considered in the region $\{(t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | t \in (0, T], x_j \in \mathbb{R}, j \in \{1, \ldots, n-1\}, x_n > 0\}, T > 0, n \geq 2$. It is assumed that the boundary conditions are connected with the system of equations by the complementing condition, which is analogous to the Lopatynsky complementing condition. Integral representations of the solutions for such a problem are derived. The kernels of the integrals from this representation form the Green's matrix of the problem. It is revealed that, in general, not all the elements of the Green's matrix are ordinary functions. Some of them contain terms that are linear combinations of Dirac delta functions and their derivatives. This occurs in cases when the boundary conditions include derivatives with respect to the variables t and x_n of orders that are equal or greater than the highest orders of derivatives with respect to these variables in the equations of the system. The obtained results are important, in particular, for the establishing of the correct solvability and integral representation of solutions for more general $\overrightarrow{2b}$-parabolic boundary value problems.

Key words and phrases: Eidelman type $\overrightarrow{2b}$-parabolic system of equations, boundary value problem, integral representation of solutions, Green's matrix.

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INTRODUCTION

Nowadays, the general theory of boundary value problems for systems of equations that are parabolic in the sense of I. G. Petrovsky and for more general systems parabolic in the sense of V. A. Solonnikov is well known (see, for example, [2, 3, 5]). The parabolic boundary problems are determined by the parabolicity condition of the system of equations and the complementing condition for boundary differential expressions. We note that the conditions for the parabolicity of a problem are specified only by the groups of the major in the parabolic sense terms of the system of equations and the boundary conditions.

The theorems of the correct solvability in Hölder and Sobolev–Slobodetskii spaces for parabolic boundary value problems, in the framework of their general theory, (Schauder's theory and L_p-theory) are proved. It turned out that the a priori estimates of the solutions established in this case are necessary and sufficient conditions for the parabolicity of the problem.

An important step in the construction of the theory of parabolic boundary value problems is a detailed study of the so-called model problems, namely the problems in a half-spaces with

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respect to the spatial variables in which systems of equations and boundary conditions contain only the major terms in the parabolic sense, and their coefficients are constants.

If we consider the, so-called, $\overrightarrow{2b}$-parabolic systems defined in [2] by S. D. Eidelman, then the orders of such systems are vectorial and the group of their major terms includes the derivatives of different the highest orders with respect to different spatial variables, since spatial variables are not equal. Therefore, it is perhaps impossible to construct a general theory of boundary value problems for such systems, analogous to the above theory for the I. G. Petrovsky systems and for V. A. Solonnikov systems, in which all spatial variables are equal. But for S. D. Eidelman systems, one can construct a theory of model boundary-value problems in a half-spaces in which one of the spatial variables varies is in the interval $(0,\infty)$, and all the others are in the interval $(-\infty,\infty)$.

In the works of the authors [4,6,7], for a parabolic in the sense of S. D. Eidelman system of the first-order equations with respect to the time variable a model boundary-value problem in a half-space is considered in which only the last spatial variable varies in the interval $(0,\infty)$. For such a problem, the complementary condition is formulated. The problem is correctly posed when the boundary conditions satisfy this complementary condition. Thus, the definition of a model $\overrightarrow{2b}$-parabolic boundary-value problem (P problem) is given. For P problem the Poisson kernel and the homogeneous Green's matrix were constructed, their accurate estimates and the estimates of their derivatives were obtained, the divergent representation was received. Using these results, a theorem of the correct solvability of P problem in anisotropic Hölder spaces is proved. In this article we obtain the integral representation of solutions of the P problem and investigate the structures of the kernels of the integrals from the representation. These kernels form the Green's matrix of P problem.

1 P PROBLEM FORMULATION, ITS HOMOGENEOUS GREEN'S MATRIX AND POISSON KERNELS

We will use the following notation: $n, N, b_1,\dots,b_n$ are given natural numbers; $\overrightarrow{2b} := (2b_1,\dots,2b_n)$; s is the least common multiple of numbers $b_1,\dots,b_n$; $m_j := s/b_j$, $j \in \{1,\dots,n\}$; $\mathbb{Z}^n_+$ is the set of all n-dimensional multi-indices $k := (k_1,\dots,k_n)$; $\|k\| := \sum\limits_{j=1}^{n} m_j k_j$, if $k \in \mathbb{Z}^n_+$; $\|\bar{k}\| := 2sk_0 + \|k\|$, if $\bar{k} := (k_0, k)$, where $k_0 \in \mathbb{Z}^1_+, k \in \mathbb{Z}^n_+$; $x := (x_1,\dots,x_n) \in \mathbb{R}^n$, $x' := (x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1}$; $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n | x_n > 0\}$, $\Pi^+_T := \{(t,x) \in \mathbb{R}^{n+1} | t \in (0,T], x \in \mathbb{R}^n_+\}$, $\Pi'_T := \{(t,x') | t \in (0,T], x' \in \mathbb{R}^{n-1}\}$, where T is given positive number; $\partial^k_x := \partial^{k_1}_{x_1}\dots\partial^{k_n}_{x_n}$, $\partial^{\bar{k}}_{t,x} := \partial^{k_0}_t \partial^k_x$, if $\bar{k} = (k_0,k), k_0 \in \mathbb{Z}^1_+, k \in \mathbb{Z}^n_+, t \in \mathbb{R}^1$ i $x \in \mathbb{R}^n$. Here, as usual, $\mathbb{R}^n$ is the n-dimensional real Euclidean space, and $\partial^l_y := \frac{\partial^l}{\partial y^l}$, if l is a natural number and $y \in \mathbb{R}^1$.

In the region Π^+_T we will consider a boundary value problem:

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n})u(t,x) := (I_N\partial_t - \sum_{\|k\|=2s} a_k \partial^k_x)u(t,x) = f(t,x), \quad (t,x) \in \Pi^+_T, \tag{1}$$

$$B^0_j(\partial_t, \partial_{x'}, \partial_{x_n})u(t,x)|_{x_n=0} := \sum_{\|\bar{k}\|=r_j} b_{j\bar{k}} \partial^{\bar{k}}_{t,x} u(t,x)|_{x_n=0} = g_j(t,x'), (t,x') \in \Pi'_T, \quad j \in \{1,\dots,m\}, \tag{2}$$

$$u(t,x)|_{t=0} = \varphi(x), \quad x \in \mathbb{R}^n_+, \tag{3}$$

where u, f and φ are matrix columns of height N; a_k and $b_{j\bar{k}}$ are constant matrices of size $N \times N$ and $1 \times N$ respectively; I_N is a unit matrix of order N; $g_1, \dots, g_m$ are scalar functions; $r_1, \dots, r_m$ are non-negative integers.

We assume that the system of equations (1) is parabolic according to Eidelman [1]. The number of boundary conditions $m = b_n N$ and these boundary conditions satisfy the complementing condition from [6]. The problem (1)–(3) that satisfies these conditions, we will call a model $\overrightarrow{2b}$-parabolic boundary value problem or P problem.

For P problem, we will give the definitions and the results of the studying of the homogeneous Green's matrix and Poisson kernels from [4, 6] that are necessary for further investigation.

According to [2, 3], we define a homogeneous Green's matrix and Poisson kernels of P problem as a matrix $G_0(t, x, \xi), t \in \mathbb{R}^1 \backslash \{0\}, \{x, \xi\} \subset \mathbb{R}^n$ of the size $N \times N$ and matrices $G_j(t, x), t \in \mathbb{R}^1 \backslash \{0\}, x \in \mathbb{R}^n$ of the size $N \times 1$ that are the solutions of the following problems:

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_0(t, x, \xi) = I_N \delta(t, x - \xi),$$

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_0(t, x, \xi)|_{x_n=0} = 0, \quad j \in \{1, \dots, m\},$$

$$G_0(t, x, \xi) = 0 \quad \text{as } t < 0,$$

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_j(t, x) = 0,$$

$$B_l^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_j(t, x)|_{x_n=0} = \delta_{lj} \delta(t, x'), \quad l \in \{1, \dots, m\},$$

$$G_j(t, x) = 0 \quad \text{as } t < 0, \quad j \in \{1, \dots, m\}$$

in spaces of generalized functions, where δ_{lj} is Kronecker symbol, $\delta(t, x - \xi)$ and $\delta(t, x')$ is Dirac delta functions with supports in points $t = 0, x = \xi$ and $t = 0, x' = 0$ respectively. Wherein $G_0(t, x, \xi)|_{t=0+} = I_N \delta(x - \xi)$.

From these definitions it follows that for an arbitrary smooth and finite functions $f, g_1, \dots, g_m$ and φ the solution of P problem (1)–(3) is represented in the form

$$u(t, x) = (\mathcal{G}_0 f + \sum_{j=1}^{m} \mathcal{G}_j g_j + \mathcal{G}_{m+1} \varphi)(t, x), \quad (t, x) \in \Pi_T^+,$$

where

$$(\mathcal{G}_0 f)(t, x) := \int_0^t d\tau \int_{\mathbb{R}^n_+} G_0(t - \tau, x, \xi) f(\tau, \xi) d\xi, \tag{4}$$

$$(\mathcal{G}_j g_j)(t, x) := \int_0^t d\tau \int_{\mathbb{R}^{n-1}} G_j(t - \tau, x - \xi') g_j(\tau, \xi') d\xi', \quad j \in \{1, \dots, m\}, \tag{5}$$

$$(\mathcal{G}_{m+1} \varphi)(t, x) := \int_{\mathbb{R}^n_+} G_0(t, x, \xi) \varphi(\xi) d\xi. \tag{6}$$

The existence of matrices $G_j, j \in \{0, 1, \dots, m\}$ and the correctness for their divergent representations

$$G_j = L^r(\partial_t, \partial_{x'}) G_j^{(r)}, \quad j \in \{0, 1, \dots, m\}, \tag{7}$$

where

$$L(\partial_t, \partial_{x'}) := \partial_t + a \sum_{j=1}^{n-1} (-1)^{b_j} \partial_{x_j}^{2b_j}, a > 0$$

and r is any non-negative number were proved in [4,6] and for $G_j^{(r)}$ the following estimates are fulfilled

$$|\partial_{t,x}^{\bar{k}} \partial_{\xi}^{l} G_0^{(r)}(t,x,\xi)| \le C_{\bar{k}l} t^{-M+r-(\|\bar{k}\|+\|l\|)/(2s)} E_c(t, x-\xi),\\ t > 0, \{x,\xi\} \subset \mathbb{R}_+^n, \bar{k} \in \mathbb{Z}_+^{n+1}, l \in \mathbb{Z}_+^n; \tag{8}$$

$$|\partial_{t,x}^{\bar{k}} G_j^{(r)}(t,x)| \le C_{\bar{k}} t^{-M'+r-1+(r_j-\|\bar{k}\|)/(2s)} E_c(t,x),\\ t > 0, x \in \mathbb{R}_+^n, \bar{k} \in \mathbb{Z}_+^{n+1}, j \in \{1,\dots,m\}. \tag{9}$$

In the estimates (8) and (9)

$$M := \textstyle\sum_{j=1}^{n} m_j/(2s),\ M' := \sum_{j=1}^{n-1} m_j/(2s),\\ E_c(t,x) := \exp\{-c\textstyle\sum_{j=1}^{n} t^{-1/(2b_j-1)}|x_j|^{2b_j/(2b_j-1)}\},$$

$C_{\bar{k}l}, C_{\bar{k}}$ and c are some positive constants.

2 REPRESENTATION OF SOLUTION FOR P PROBLEM WITH HOMOGENEOUS INITIAL CONDITION

Suppose that in the problem (1)–(3) f and $g_j, j \in \{1,\dots,m\}$ are sufficiently smooth functions such that they together with their derivatives are bounded and equal to zero as $t = 0$ and $\varphi = 0$. Let us find a formula for the solutions of P problem with these right-hand sides, namely for the following problem with zero initial condition:

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n}) u(t,x) = f(t,x), \quad (t,x) \in \Pi_T^+, \tag{10}$$

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) u(t,x)|_{x_n=0} = g_j(t,x'), \quad (t,x') \in \Pi_T', \quad j \in \{1,\dots,m\}, \tag{11}$$

$$u(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}_+^n. \tag{12}$$

Consider the function

$$u_0(t,x) := (\mathcal{G}_0 f)(t,x), \quad (t,x) \in \Pi_T^+. \tag{13}$$

Based on the definition (4) of the operator $\mathcal{G}_0$ and the properties of the matrix G_0, the function u_0 is a solution of system (10) that satisfies the condition (12). In addition, if the order r_j of the differential expression $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$ is less than $2s$, then

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) u_0(t,x)|_{x_n=0}\\ = \int_0^t d\tau \int_{\mathbb{R}_+^n} B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) G_0(t-\tau, x, \xi)|_{x_n=0} f(\tau,\xi) d\xi = 0, \quad (t,x') \in \Pi_T'.$$

In the case when $r_j \ge 2s$, it is impossible to apply the operation $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$ and pass to the limit as $x_n \to 0$ under the sign of the integral. In this case, we proceed as follows.

Consider $B_j^0(p, i\sigma', i\tau)$ and $A^0(p, i\sigma', i\tau)$, where i is the imaginary unit, as matrix polynomials of τ with fixed values of the parameters p and σ'. Based on the $\overrightarrow{2b}$-parabolicity of system (10), the determinant of the matrix, which is the coefficient at τ^{2b_n} in $A^0(p, i\sigma', i\tau)$, is nonzero (see Remark 1 in [4]). Therefore, there exist such matrix polynomials $C_j(p, i\sigma', i\tau)$ and $B_j'(p, i\sigma', i\tau)$ that their degrees on τ do not exceed $r_j - 2b_n$ and $2b_n - 1$ respectively, and they fulfill the equality

$$B_j^0(p, i\sigma', i\tau) = C_j(p, i\sigma', i\tau)A^0(p, i\sigma', i\tau) + B_j'(p, i\sigma', i\tau).$$

Turning to differential expressions, we obtain the equality

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) = C_j(\partial_t, \partial_{x'}, \partial_{x_n})A^0(\partial_t, \partial_{x'}, \partial_{x_n}) + B_j'(\partial_t, \partial_{x'}, \partial_{x_n}), \tag{14}$$

where C_j and B_j' are expressions containing differentiations on x_n of order not higher than $r_j - 2b_n$ and $2b_n - 1$, respectively.

For function (13), on the basis of equality (14) and the fact that $A^0(\partial_t, \partial_{x'}, \partial_{x_n})u_0 = f$, we now get

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0} = C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0} + B_j'(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0}.$$

Using representation (7) and estimates (8) for G_0 and integrating by parts, for a sufficiently large r, we obtain

$$B_j'(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0} = \int_0^t d\tau \int_{\mathbb{R}^n_+} B_j'(\partial_t, \partial_{x'}, \partial_{x_n})G_0^{(r)}(t-\tau, x, \xi)|_{x_n=0}L^r(\partial_\tau, \partial_{\xi'})f(\tau, \xi)d\xi.$$

Based on (14) and on the fact that $A^0(\partial_t, \partial_{x'}, \partial_{x_n})G_0^{(r)} = 0$, we replace B_j' by B_j^0 in the last integral. If we represent this integral as the limit of the integral over $\{\xi \in \mathbb{R}^n | \xi_n \geq \varepsilon\}, \varepsilon > 0$, as $\varepsilon \to 0$, and then integrate by parts of the expression $L^r(\partial_\tau, \partial_{\xi'})$ and use it to $G_0^{(r)}$, then we get that it is equal to zero. So,

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})u_0|_{x_n=0} = C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0}.$$

Note that $C_j = 0$ if the highest order of derivatives with respect to x_n in B_j^0 is less than $2b_n$.

Thus, the function (13) is a solution to the problem (10)–(12), in which g_j is replaced by $C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0}, j \in \{1, \ldots, m\}$. Moreover, if the function f is finite in Π_T^+, then

$$C_j(\partial_t, \partial_{x'}, \partial_{x_n})f|_{x_n=0} = 0, \quad j \in \{1, \ldots, m\}.$$

If, in the conditions (11), the functions $g_j, j \in \{1, \ldots, m\}$, such as indicated at the beginning of this section, then using for G_j the representation (7) and the estimates (9) just as in [5], we prove that the function

$$u_1(t, x) := \sum_{j=1}^m \int_0^t d\tau \int_{\mathbb{R}^{n-1}} G_j(t-\tau, x-\xi')g_j(\tau, \xi')d\xi', \quad (t, x) \in \Pi_T^+,$$

is the solution to the problem

$$A^0(\partial_t, \partial_{x'}, \partial_{x_n})u_1(t, x) = 0, \quad (t, x) \in \Pi_T^+,$$

$$B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) u_1(t,x)|_{x_n=0} = g_j(t,x'), \quad (t,x') \in \Pi'_T, \quad j \in \{1,\dots,m\},$$
$$u_1(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}^n_+.$$

Therefore, for the functions f and $g_j, j \in \{1,\dots,m\}$ indicated at the beginning of Section 2, the solution of problem (10)–(12) is determined by the formula

$$u(t,x) = (\mathcal{G}_0 f)(t,x) + \sum_{j=1}^{m} (\mathcal{G}_j(g_j - C_j(\partial_t, \partial_{x'}, \partial_{x_n}) f|_{x_n=0}))(t,x), \quad (t,x) \in \Pi^+_T. \tag{15}$$

3 THE GENERAL CASE OF P PROBLEM

Suppose now that in the problem (1)–(3) functions f, $g_j, j \in \{1,\dots,m\}$, and φ are sufficiently smooth in $\bar{\Pi}^+_T$, $\bar{\Pi}'_T$ and $\bar{\mathbb{R}}^n_+$ that are the closures of Π^+_T, Π'_T and $\mathbb{R}^n_+$, respectively and they together with their derivatives, are bounded and satisfy the corresponding matched conditions as $t = 0$ and $x_n = 0$. Then, from the results of the paper [7], it follows that there exists a unique smooth solution u of the general P problem, defined in $\bar{\Pi}^+_T$ and bounded with all its derivatives. Now we were find the integral representation of this solution u.

Let us choose the infinitely differentiable function $\zeta(t), t \in \mathbb{R}^1$, that is equal to 1 for $t \geq 1$ and is equal to 0 for $t \leq 1/2$, and the function $v_h(t,x) := \zeta_h(t) u(t,x), (t,x) \in \bar{\Pi}^+_T$, where $\zeta_h(t) := \zeta(t/h)$, h is a sufficiently small positive number. Obviously, v_h has the same smoothness properties as the function u, and it is a solution to the problem

$$\begin{aligned} &A^0(\partial_t, \partial_{x'}, \partial_{x_n}) v_h(t,x) = F_{0h}(t,x), \quad (t,x) \in \Pi^+_T, \\ &B_j^0(\partial_t, \partial_{x'}, \partial_{x_n}) v_h(t,x)|_{x_n=0} = F_{jh}(t,x'), \quad (t,x') \in \Pi'_T, \quad j \in \{1,\dots,m\}, \\ &v_h(t,x)|_{t=0} = 0, \quad x \in \mathbb{R}^n_+, \end{aligned} \tag{16}$$

where

$$F_{0h}(t,x) := \zeta_h(t) f(t,x) + \zeta_h^{(1)}(t) u(t,x),$$

$$F_{jh}(t,x') := \zeta_h(t) g_j(t,x') + \sum_{\|\bar{k}\|=r_j} \sum_{\nu=0}^{k_0} C_{k_0}^{\nu} b_{j\bar{k}} \zeta_h^{(\nu)}(t) \partial_t^{k_0-\nu} u(t,x)|_{x_n=0}, \quad j \in \{1,\dots,m\}.$$

Here and further $C_{k_0}^{\nu} := \frac{k_0!}{\nu!(k_0-\nu)!}, \zeta_h^{(\nu)}(t) := \frac{d^\nu \zeta_h(t)}{dt^\nu}$.

Since the problem (16) is a problem with zero initial condition, then according to the result of Section 2, the representation of its solution could be written in the form (15), i.e.

$$v_h(t,x) = (\mathcal{G}_0 F_{0h})(t,x) + \sum_{j=1}^{m} (\mathcal{G}_j(F_{jh} - C_j(\partial_t, \partial_{x'}, \partial_{x_n}) F_{0h}|_{x_n=0}))(t,x), \quad (t,x) \in \Pi^+_T. \tag{17}$$

Assuming that the point (t,x) is fixed from Π^+_T and $h \in (0,t)$, we pass to the limit as $h \to 0$ in (17). At the same time, we obtain $u(t,x)$ in the left-hand side. Further we find the limit of the right side.

We have

$$\mathcal{G}_0(\zeta_h f)(t,x) = (\mathcal{G}_0 f)(t,x) + \int_0^h d\tau \int_{\mathbb{R}^n_+} G_0(t-\tau, x, \xi)(\zeta_h(\tau) - 1) f(\tau, \xi) d\xi \xrightarrow[h \to 0]{} (\mathcal{G}_0 f)(t,x). \tag{18}$$

Taking into account the properties of the function ζ_h and integrating by parts, we obtain

$$
\begin{aligned}
(\mathcal{G}_0(\zeta_h^{(1)}u))(t,x) &= \int_{h/2}^{h} d\tau \int_{\mathbb{R}^n_+} G_0(t-\tau,x,\xi)\zeta_h^{(1)}(\tau)u(\tau,\xi)d\xi \\
&= \int_{\mathbb{R}^n_+} G_0(t-h,x,\xi)u(h,\xi)d\xi \\
&\quad - \int_{h/2}^{h} d\tau \int_{\mathbb{R}^n_+} \partial_\tau(G_0(t-\tau,x,\xi)u(\tau,\xi))\zeta_h(\tau)d\xi \xrightarrow[h\to 0]{} (\mathcal{G}_{m+1}\varphi)(t,x).
\end{aligned}
\tag{19}
$$

Similarly we have

$$
\begin{aligned}
&(\mathcal{G}_jF_{jh})(t,x) = (\mathcal{G}_j(\zeta_h g_j))(t,x) + \sum_{\|\bar{k}\|=r_j} b_{j\bar{k}}\Big\{(\mathcal{G}_j(\partial^{\bar{k}}_{t,x}u|_{x_n=0}\zeta_h))(t,x) \\
&+ \sum_{\nu=1}^{k_0}(-1)^{\nu-1}C^\nu_{k_0}\Big[\int_{\mathbb{R}^{n-1}} \partial_\tau^{\nu-1}(G_j(t-\tau,x-\xi')\partial_\tau^{k_0-\nu}\partial_\xi^k u(\tau,\xi)|_{\xi_n=0})d\xi'\zeta_h(\tau)|_{\tau=h/2}^{\tau=h} \\
&- \int_{h/2}^{h} d\tau \int_{\mathbb{R}^{n-1}} \partial_\tau^{\nu}(G_j(t-\tau,x-\xi')\partial_\tau^{k_0-\nu}\partial_\xi^k u(\tau,\xi)|_{\xi_n=0})\zeta_h(\tau)d\xi'\Big]\Big\} \xrightarrow[h\to 0]{} (\mathcal{G}_jg_j)(t,x) \\
&+ \sum_{\substack{\|\bar{k}\|=r_j \\ (k_0>0)}} b_{j\bar{k}} \sum_{\nu=1}^{k_0}(-1)^{\nu-1}C^\nu_{k_0}\int_{\mathbb{R}^{n-1}} \partial_\tau^{\nu-1}(G_j(t-\tau,x-\xi')\partial_\tau^{k_0-\nu}\partial_\xi^k u(\tau,\xi)|_{\xi_n=0})d\xi'|_{\tau=0}.
\end{aligned}
\tag{20}
$$

Now consider $\mathcal{G}_j(C_j(\partial_t,\partial_{x'},\partial_{x_n})F_{0h}|_{x_n=0})$. Using the record

$$C_j(\partial_t,\partial_{x'},\partial_{x_n}) = \sum_{\|\bar{k}\|\le\le r_j-2s} c_{j\bar{k}}\partial^{\bar{k}}_{t,x},$$

as above, using integration by parts, we obtain

$$
\begin{aligned}
&(\mathcal{G}_j(C_j(\partial_t,\partial_{x'},\partial_{x_n})F_{0h}|_{x_n=0}))(t,x) \\
&= \sum_{\|\bar{k}\|\le r_j-2s} c_{j\bar{k}}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} G_j(t-\tau,x-\xi')\times\partial_\tau^{k_0}(\zeta_h(\tau)\partial_\xi^k f(\tau,\xi)+\zeta_h^{(1)}(\tau)\partial_\xi^k u(\tau,\xi))|_{\xi_n=0}d\xi' \\
&= \sum_{\|\bar{k}\|\le r_j-2s} c_{j\bar{k}}\int_0^t d\tau \int_{\mathbb{R}^{n-1}} \partial_t^{k_0}G_j(t-\tau,x-\xi')(\zeta_h(\tau)\partial_\xi^k f(\tau,\xi)|_{\xi_n=0}+\zeta_h^{(1)}(\tau)\partial_\xi^k u(\tau,\xi)|_{\xi_n=0})d\xi'.
\end{aligned}
$$

The remaining terms are zero due to the properties of the functions ζ_h and G_j. Integrating by parts again and passing to the limit as $h \to 0$, we get

$$
\begin{aligned}
(\mathcal{G}_j(C_j(\partial_t,\partial_{x'},\partial_{x_n})F_{0h}|_{x_n=0}))(t,x) &\xrightarrow[h\to 0]{} \sum_{\|\bar{k}\|\le r_j-2s} c_{j\bar{k}}\Big(\int_0^t d\tau\int_{\mathbb{R}^{n-1}} \partial_t^{k_0}G_j(t-\tau,x-\xi')\partial_\xi^k f(\tau,\xi)|_{\xi_n=0}d\xi' \\
&\quad + \int_{\mathbb{R}^{n-1}} \partial_t^{k_0}G_j(t,x-\xi')\partial_\xi^k\varphi(\xi)|_{\xi_n=0}d\xi'\Big).
\end{aligned}
\tag{21}
$$

From (17)–(21) it follows the formula

$$u(t,x) = (\mathcal{G}_0 f + \sum_{j=1}^{m} \mathcal{G}_j g_j + \mathcal{G}_{m+1}\varphi)(t,x)$$
$$+ \int_0^t d\tau \int_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} R_k(t-\tau, x-\xi')\partial_\xi^k f(\tau,\xi)|_{\xi_n=0} d\xi' + \int_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} R_k(t, x-\xi')\partial_\xi^k \varphi(\xi)|_{\xi_n=0} d\xi'$$
$$+ \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{m} \sum_{\substack{\|\bar{k}\|=r_j \\ (k_0>0)}} \sum_{\nu=1}^{k_0} (-1)^{\nu-1} C_{k_0}^{\nu} b_{j\bar{k}} \partial_\tau^{\nu-1} (G_j(t-\tau, x-\xi') \partial_\tau^{k_0-\nu} \partial_\xi^k u(\tau,\xi)|_{\xi_n=0}) d\xi'|_{\tau=0},\ (t,x) \in \Pi_T^+, \tag{22}$$

where

$$R_k(t,x) := \sum_{j=1}^{m} R_{jk}(t,x), \tag{23}$$

$$R_{jk}(t,x) := \begin{cases} \sum\limits_{k_0 \le (r_j - \|k\| - 2s)/(2s)} c_{j\bar{k}} \partial_t^{k_0} G_j(t,x), & \text{if } \|k\| \le r_j - 2s, \\ 0, & \text{if } r_j - 2s < \|k\| \le r_0, \end{cases} \tag{24}$$

$$r_0 := \max(0, r_1 - 2s, \dots, r_m - 2s), \tag{25}$$

moreover $R_k = 0$, if the highest order of derivatives with respect to x_n in $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$, $j \in \{1, \dots, m\}$, n_0 is less than $2b_n$.

All terms of the right-hand side of (22), except for the last, include only the right-hand sides of the problem (1)–(3). We transform the last term (denote it by D) in such way that it also will include only the right-hand sides of the problem (1)–(3). Note that the term D is absent if the expressions $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$, $j \in \{1, \dots, m\}$, do not include differentiation with respect to t.

Using the Leibniz formula and changing the order of summation, we get

$$D = \int_{\mathbb{R}^{n-1}} \sum_{j=1}^{m} \sum_{\mu=0}^{p_j-1} \sum_{\substack{k_0= \\ =\mu+1}}^{p_j} \sum_{\substack{\|k\|=r_j- \\ -2sk_0}} N_{k_0\mu} b_{j\bar{k}} \partial_\tau^{k_0-\mu-1} G_j(t-\tau, x-\xi') \partial_\tau^{\mu} \partial_\xi^k u(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi', \tag{26}$$

where p_j is the highest order of derivatives with respect to t in the expression $B_j^0(\partial_t, \partial_{x'}, \partial_{x_n})$, and

$$N_{k_0\mu} := \sum_{\nu=k_0-\mu}^{k_0} (-1)^{\nu-1} C_{k_0}^{\nu} C_{\nu-1}^{k_0-\mu-1}.$$

We will write the formula (26) in the form

$$D = \int_{\mathbb{R}^{n-1}} \sum_{\mu=0}^{p_0-1} \sum_{\|k\|\le r_0-2s\mu} Q_{\mu k}(t-\tau, x-\xi') \partial_\tau^{\mu} \partial_\xi^k u(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi', \tag{27}$$

where $p_0 := \max(p_1, \dots, p_m)$,

$$Q_{\mu k}(t,x) := \sum_{j=1}^{m} Q_{j\mu k}(t,x), \tag{28}$$

$$Q_{j\mu k}(t,x) :=$$
$$= \begin{cases} \sum\limits_{\substack{\mu+1\le k_0\le \\ \le (r_j-\|k\|-2s)/(2s)}} N_{k_0\mu} b_{j\bar{k}} \partial_t^{k_0-\mu-1} G_j(t,x), \text{if } 0 \le \mu \le p_j - 1,\, 2s\mu + \|k\| \le r_j - 2s, \\ 0, \qquad\qquad \text{if } p_j \le \mu \le p_0 - 1 \text{ or } r_j - 2s < 2s\mu + \|k\| \le r_0. \end{cases}$$

Using the system (1) and the condition (3) for $\mu > 0$ we obtain the representation

$$\partial_\tau^\mu \partial_\xi^k u(\tau,\xi)|_{\tau=0} = \sum_{\|\nu\|=2s\mu+\|k\|} A_{\mu k\nu} \partial_\xi^\nu \varphi(\xi) + \sum_{\substack{\|\bar{\nu}\|=2s(\mu-1)+\|k\| \\ (\nu_0\le\mu-1)}} B_{\mu k\bar{\nu}} \partial_{\tau,\xi}^{\bar{\nu}} f(\tau,\xi)|_{\tau=0}, \tag{29}$$

where $A_{\mu k\nu}$ and $B_{\mu k\bar{\nu}}$ are constant matrices of the size $N \times N$, wich compiled with coefficients a_k, $\|k\| = 2s$, from the system (1). Substituting expression (29) into (27), changing the order of summation and using the notation

$$V_\nu(t,x) := \sum_{\substack{2s\mu+\|k\|=\|\nu\| \\ (\mu\le p_0-1)}} Q_{\mu k}(t,x) A_{\mu k\nu},\ W_{\bar{\nu}}(t,x) := \sum_{\substack{2s\mu+\|k\|=\|\bar{\nu}\|+2s \\ (\mu\ge\nu_0+1)}} Q_{\mu k}(t,x) B_{\mu k\bar{\nu}}, \tag{30}$$

we get the following expression for D:

$$\begin{aligned} D = &\int\limits_{\mathbb{R}^{n-1}} \sum_{\|\nu\|\le r_0} V_\nu(t, x-\xi') \partial_\xi^\nu \varphi(\xi)|_{\xi_n=0} d\xi' \\ &+ \int\limits_{\mathbb{R}^{n-1}} \sum_{\substack{\|\bar{\nu}\|\le r_0-2s \\ (\nu_0\le p_0-2)}} W_{\bar{\nu}}(t-\tau, x-\xi') \partial_{\tau,\xi}^{\bar{\nu}} f(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi'. \end{aligned} \tag{31}$$

Therefore, from formulas (22) and (31) it follows the following representation of the solution of the general P problem:

$$\begin{aligned} u(t,x) =& (\mathcal{G}_0 f + \sum_{j=1}^m \mathcal{G}_j g_j + \mathcal{G}_{m+1}\varphi)(t,x) \\ &+ \int_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} R_k(t-\tau, x-\xi') \partial_\xi^k f(\tau,\xi)|_{\xi_n=0} d\xi' \\ &+ \int\limits_{\mathbb{R}^{n-1}} \sum_{\substack{\|\bar{k}\|\le r_0-2s \\ (k_0\le p_0-2)}} W_{\bar{k}}(t-\tau, x-\xi') \partial_{\tau,\xi}^{\bar{k}} f(\tau,\xi)|_{\tau=0,\xi_n=0} d\xi' \\ &+ \int\limits_{\mathbb{R}^{n-1}} \sum_{\|k\|\le r_0} \big(R_k(t, x-\xi') + V_k(t, x-\xi')\big) \partial_\xi^k \varphi(\xi)|_{\xi_n=0} d\xi' \\ &=: I_1 + I_2 + I_3 + I_4, \quad (t,x) \in \Pi_T^+. \end{aligned} \tag{32}$$

Now, we rewrite this representation in another form. To do this, first we transform the addend I_3 from formula (32). Using the formula

$$W_{\bar{k}} \partial_\tau^{k_0} f = \sum_{l=0}^{k_0} (-1)^{k_0-l} C_{k_0}^l \partial_\tau^l (\partial_\tau^{k_0-l} W_{\bar{k}} f),$$

we get

$$
\begin{aligned}
I_3 &= \int\limits_{\mathbb{R}^{n-1}} \sum_{k_0=0}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} \sum_{l=0}^{k_0} (-1)^{k_0-l} C_{k_0}^l \partial_\tau^l (\partial_\tau^{k_0-l} W_{\bar{k}}(t-\tau, x-\xi') \partial_\xi^k f(\tau,\xi)|_{\xi_n=0})|_{\tau=0} d\xi' \\
&= \int\limits_{\mathbb{R}^{n-1}} \sum_{l=0}^{p_0-2} (-1)^l \partial_\tau^l \Bigg[\sum_{k_0=l}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} (-1)^{k_0} C_{k_0}^l \partial_\tau^{k_0-l} W_{\bar{k}}(t-\tau, x-\xi') \partial_\xi^k f(\tau,\xi)|_{\xi_n=0} \Bigg]\Bigg|_{\tau=0} d\xi' \\
&= \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} \sum_{l=0}^{p_0-2} \delta^{(l)}(\tau) \sum_{k_0=l}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} (-1)^{k_0} C_{k_0}^l \partial_\tau^{k_0-l} W_{\bar{k}}(t-\tau, x-\xi') \partial_\xi^k f(\tau,\xi)|_{\xi_n=0}|_{\tau=0} d\xi' \\
&= \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n_+} G_0''(t,x;\tau,\xi) f(\tau,\xi) d\xi,
\end{aligned}
$$

where

$$
G_0''(t,x;\tau,\xi) := \sum_{l=0}^{p_0-2} \delta^{(l)}(\tau) \sum_{k_0=l}^{p_0-2} \sum_{\substack{\|k\|\le r_0-\\ -2s(k_0+1)}} (-1)^{k_0+|k|} C_{k_0}^l \partial_\tau^{k_0-l} \partial_{\xi'}^{k'} W_{\bar{k}}(t-\tau, x-\xi') \delta^{(k_n)}(\xi_n), \quad (33)
$$

where $|k| := k_1 + \cdots + k_n$, $\delta^{(l)}(\tau)$ and $\delta^{(k_n)}(\xi_n)$ are the derivatives of delta functions concentrated at points $\tau = 0$ and $\xi_n = 0$ respectively.

Similarly transforming the addends I_2 and I_4 from (32) and taking into account the definitions (4)–(6), we write the representation (32) in the form

$$
\begin{aligned}
u(t,x) = \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n_+} \tilde{G}_0(t,x;\tau,\xi) f(\tau,\xi) d\xi + \sum_{j=1}^m \int\limits_0^t d\tau \int\limits_{\mathbb{R}^{n-1}} G_j(t-\tau, x-\xi') g_j(\tau,\xi') d\xi' \\
+ \int\limits_{\mathbb{R}^n_+} \tilde{G}_{m+1}(t,x,\xi) \varphi(\xi) d\xi, \quad (t,x) \in \Pi_T^+,
\end{aligned} \quad (34)
$$

where

$$
\begin{aligned}
\tilde{G}_0(t,x;\tau,\xi) &:= G_0(t-\tau,x,\xi) + G_0'(t-\tau,x,\xi) + G_0''(t,x;\tau,\xi), \\
\tilde{G}_{m+1}(t,x,\xi) &:= G_0(t,x,\xi) + G_0'(t,x,\xi) + +G_{m+1}'(t,x,\xi).
\end{aligned} \quad (35)
$$

Here

$$
\begin{aligned}
G_0'(t,x,\xi) &:= \sum_{\|k\|\le r_0} (-1)^{|k|} \partial_{\xi'}^{k'} R_k(t, x-\xi') \delta^{(k_n)}(\xi_n), \\
G_{m+1}'(t,x,\xi) &:= \sum_{\|k\|\le r_0} (-1)^{|k|} \partial_{\xi'}^{k'} V_k(t, x-\xi') \delta^{(k_n)}(\xi_n),
\end{aligned} \quad (36)
$$

and G_0'' is defined by the formula (33).

As a corollary we can get the following theorem from the results obtained above and from Theorem 2 [7] about the correct solvability of the P problem in Hölder spaces.

Theorem 1. *Any solution to the P problem* (1)–(3)*, that belongs to the Hölder space* $H^{2s+l+\lambda}(\bar{\Pi}_T^+, \mathbb{C}_{N1})$*, where* l *is an integer, such that* $l \geq r_0$ *and* $\lambda \in (0,1)$*, is represented in the form* (34)*. The kernels of this representation are defined by formulas* (33), (35) *and* (36)*. In these formulas* R_k*,* V_k *and* $W_{\bar{k}}$ *are defined by equalities* (23)–(25), (28) *and* (30)*. In all of these formulas,* G_0 *is a homogeneous Green's matrix, and* $G_j, j \in \{1, \ldots, m\}$*, are Poisson kernels of problem* (1)–(3)*. Moreover,* $G_0' = G_0'' = G_{m+1}' = 0$ *if* $2sp_0 + m_n n_0 < 2s$*, where* p_0 *and* n_0 *are the highest orders of derivatives with respect to* t *and* x_n *in boundary conditions* (2) *accordingly, and* $m_n = s/b_n$*.*

Definition 1. *The matrix composed of the elements of the matrices* $\tilde{G}_0, G_1, \ldots, G_m$ *and* $\tilde{G}_{m+1}$ *is called the Green's matrix of the problem* (1)–(3)*.*

So, the article describes the structure of the Green's matrix of the problem (1)–(3).

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В області $\{(t, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | t \in (0, T], x_j \in \mathbb{R}, j \in \{1, \ldots, n-1\}, x_n > 0\}, T > 0, n \geq 2$, розглядається загальна крайова задача для $\overrightarrow{2b}$-параболічної за Ейдельманом системи рівнянь, в якій у рівняннях і крайових умовах відсутні молодші члени, а коефіцієнти групи старших членів сталі. Припускається, що крайові умови пов'язані з системою рівнянь умовою доповняльності, яка є аналогом умови доповняльності Лопатинського. Для розв'язків такої задачі виведено інтегральне зображення. Ядра інтегралів з цього зображення утворюють матрицю Ґріна задачі. Виявлено, що, взагалі кажучи, не всі елементи матриці Ґріна є звичайними функціями. Деякі з них містять доданки, які є лінійними комбінаціями дельта-функцій Дірака та їх похідних. Це виникає у випадках, коли в крайові умови входять похідні за змінними t і x_n порядків, рівних або більших за найвищі порядки похідних за цими змінними в рівняннях системи. Отримані результати є важливими, зокрема, для встановлення коректної розв'язності та інтегрального зображення розв'язків загальніших $\overrightarrow{2b}$-параболічних крайових задач.

Ключові слова і фрази: $\vec{2b}$-параболічна за Ейдельманом система рівнянь, крайова задача, інтегральне зображення розв'язків, матриця Ґріна.