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THE NONLOCAL BOUNDARY VALUE PROBLEM WITH PERTURBATIONS OF MIXED BOUNDARY CONDITIONS FOR AN ELLIPTIC EQUATION WITH CONSTANT COEFFICIENTS. I

In this article we investigate a problem with nonlocal boundary conditions which are multipoint perturbations of mixed boundary conditions in the unit square G using the Fourier method.

The properties of a generalized transformation operator $R : L_2(G) \to L_2(G)$ that reflects normalized eigenfunctions of the operator L_0 of the problem with mixed boundary conditions in the eigenfunctions of the operator L for nonlocal problem with perturbations, are studied. We construct a system $V(L)$ of eigenfunctions of operator L. Also, we define conditions under which the system $V(L)$ is total and minimal in the space $L_2(G)$, and conditions under which it is a Riesz basis in the space $L_2(G)$. In the case if $V(L)$ is a Riesz basis in $L_2(G)$, we obtain sufficient conditions under which nonlocal problem has a unique solution in form of Fourier series by system $V(L)$.

Key words and phrases: differential equation with partial derivatives, eigenfunctions, Riesz basis.

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1 Introduction

The fundamentals of the theory of linear differential equations in partial derivatives with constant coefficients were established by L. Ehrenpreis, L. Hermander, V. Malgrange, I. Petrovsky.

Boundary value problems in bounded domains for certain classes of differential equations with constant coefficients have been studied in [1–13]. This paper is a continuation of the investigations that were begun in [3–6].

For our investigation we will use the following notations. Let $G := \{x := (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1,\ x_2 < 1\}$, D_1, D_2 are the operators of differentiation by the variables x_1, x_2 respectively; $H_0 := L_2(0,1)$, $H_1 := L_2(G)$; $H_2 := W_2^{2n}(G)$ be a Sobolev space with a scalar product and norm respectively

$$(u, v; H_2) := (u, v; H_1) + (D_1^{2n}u, D_1^{2n}v; H_1) + (D_2^{2n}u, D_2^{2n}v; H_1), \quad \|u; H_2\| := \sqrt{(u, u; H_2)};$$

$$W := \{v \in C[0,1] :\ v^{(s)} \in C[0,1],\ s = 1, \dots, 2n-1,\ v^{(2n)} \in H_0\};$$

$$H_{0,s} := \{u(t) \in H_0 :\ u(t) \equiv (-1)^s u(1-t)\}, \quad s \in \{0,1\};$$

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$W_r := W \cap H_{0,r}$, $r = 0, 1$; and $[H_0]$ be a set of linear continuous operators on the space H_0. Let us consider the boundary value problem

$$L(-D_1^2, -D_2^2)u := \sum_{j=0}^{n} a_j D_1^{2j} D_2^{2n-2j} u = f(x), \ x \in G, \tag{1}$$

$$\ell_{s,1}u := D_1^{2s-2}u|_{x_1=0} + D_1^{2s-2}u|_{x_1=1} + \ell_{s,1}^0 u = 0, \tag{2}$$

$$\ell_{n+s,1}u := D_1^{2s-2}u|_{x_1=0} - D_1^{2s-2}u|_{x_1=1} = 0, \tag{3}$$

$$\ell_{s,2}u := D_2^{2s-2}u|_{x_1=0} + D_2^{2s-2}u|_{x_1=1} = 0, \tag{4}$$

$$\ell_{n+s,2}u := D_2^{2s-1}u|_{x_2=0} + D_2^{2s-1}u|_{x_2=1} = 0, \ s = 1, \dots, n, \tag{5}$$

where

$$\ell_{s,1}^0 u := \sum_{q=0}^{k_{s,1}} \sum_{r=0}^{n_1} b_{s,q,r} D_1^q u|_{x_1=x_{1,r}}, \ s = 1, \dots, n, \tag{6}$$

$$0 = x_{1,1} < x_{1,2} < \cdots < x_{1,n_1} \le 1, \ a_j, \ b_{s,q,r} \in \mathbb{R},$$

$$q = 0, 1, \dots, k_{s,1}, \ k_{s,1} < 2n, \ r = 0, 1, \dots, n_1, \ s = 1, \dots, n, \ j = 0, 1, \dots, n.$$

Let $L : H_1 \to H_1$ be the operator of the problem (1)–(6) and

$$Lu := L(-D_1^2, -D_2^2)u, \ u \in D(L),$$

$$D(L) := \{u \in H_2 : \ell_{s,j}u = 0, \ s = 1, \dots, 2n, \ j = 1, 2\}.$$

Definition. *The function $y \in D(L)$, that satisfies equality $\|L(-D_1^2, -D_2^2)y - f; H_1\| = 0$, is called a solution of problem* (1)–(6).

Let us consider the following assumptions and theorems, that are necessary for further investigation.

1. Assumption P_1: $b_{s,q,r} = -(-1)^q b_{s,q,n_1-r}$, $x_{1,r} = 1 - x_{1,n_1-r}$, $r = 0, 1, \dots, n_1$, $s = 1, \dots, n$.
2. Assumption P_2: $k_{s,1} \le 2s - 2$, $s = 1, \dots, n$.
3. Assumption P_3: for any real numbers μ_1, μ_2 the positive number $C_1(L)$ exists, that the inequality $C_1(L)|\mu|^{2n} \le |L(\mu_1, \mu_2)|$, $\mu := (\mu_1, \mu_2)$, $|\mu|^2 := |\mu_1|^2 + |\mu_2|^2$, holds.

Theorem 1. *Let Assumption P_1 holds. Then, for an arbitrary $a_q \in \mathbb{R}$, $q = 0, 1, \dots, n$, $b_{s,q,r} \in \mathbb{R}$, the operator L has a set of eigenvalues*

$$\sigma := \{\lambda_{k,m} := L(\mu_{1,k}, \mu_{2,m}), \ \mu_{1,k} = \pi^2 k^2, \ \mu_{2,m} = \pi^2(2m-1)^2, \ k \in \mathbb{N}, \ m \in \mathbb{N}\}, \tag{7}$$

and the system $V(L)$ of eigenfunctions, which is complete and minimal in the space H_1.

Theorem 2. *Let Assumptions P_1–P_3 hold. Then, the operator L has the system $V(L)$ of eigenfunctions, which is the Riesz basis of the space H_1.*

Theorem 3. *Let Assumptions P_1–P_3 hold. Then, for arbitrary function $f \in H_1$ the unique solution of problem* (1)–(6) *exists.*

Let A_0 be the operator of boundary problem in the space H_0 :

$$-z^{(2)}(t) = g(t),\ t \in (0,1),\ z(0) = z(1) = 0;$$

$$A_0 z := -z^{(2)}(t),\ z(t) \in D(A_0),\ D(A_0) := \{z \in W_2^2(0,1) : z(0) = z(1) = 0\};$$

$$T_1 := \{\tau_{1,s,k}(t) \in H_0 :\ \tau_{1,s,k}(t) := \sqrt{2}\sin \rho_{s,k} t,\ \rho_{s,k} = \pi(2k + s - 1),\ k \in \mathbb{N},\ s = 0,1\};$$

$$T_{1,s} := \{\tau_{1,s,k}(t) \in H_{0,s},\ k \in \mathbb{N}\},\ \ s = 0,1;$$

$$\sigma(A_0) := \{\mu_{1,k} = \pi^2 k^2,\ k \in \mathbb{N}\}.$$

Lemma 1. *The operator A_0 has the point spectrum $\sigma(A_0)$ and system of eigenfunctions T_1.*

Proof. A direct substitution proves that the elements of system T_1 are the eigenfunctions of operator A_0, which correspond to the eigenvalues $\sigma(A_0)$.

Taking into account that the subsystem of eigenfunctions $T_{1,s}$ of the operator A_0 is an orthonormal basis of spaces $H_{0,s}$, $s = 0,1$, we obtain the statement of the lemma. □

Let $\Theta = \{\theta_k\}_{k=1}^{\infty}$ be any sequence of real numbers. We consider the operator $A_\Theta : H_0 \to H_0$, which has a set of eigenvalues $\sigma(A_0)$, and the system of eigenfunctions

$$V(A_\Theta) := \{v_{s,k}(t, A_\Theta) \in H_0 : v_{0,k}(t, A_\Theta) := \tau_{1,0,k}(t),$$
$$v_{1,k}(t, A_\Theta) := \tau_{1,1,k}(t) + \theta_k \sqrt{2}\cos 2k\pi t,\ k \in \mathbb{N}\}.$$

Lemma 2. *For an arbitrary sequence Θ the system of functions $V(A_\Theta)$ is complete and minimal in the space H_0. The system of functions $V(A_\Theta)$ is the Riesz basis of this space if and only if the sequence Θ is bounded.*

Proof. Suppose that the system $V(A_\Theta)$ is not complete in the space H_0.

Let us suppose that there exist functions $f = f_0 + f_1 \in H_0$, and $f_s \in H_{0,s}$, $s = 0,1$, for which the conditions of orthogonality hold:

$$(f, v_{s,k}(t, A_\Theta); H_0) = 0,\ s = 0,1,\ k \in \mathbb{N}.$$

Taking into account, that the system of functions $\tau_{1,0,q}(t) = v_{0,q}(t, A_\Theta)$, $q \in \mathbb{N}$, is an orthonormal basis of the space $H_{0,0}$ with respect to the condition of orthogonality, we obtain $f_0 = 0$. Thus $f = f_1 \in H_{0,1}$.

According to the condition of orthogonality we have the relation

$$(f, v_{1,k}(t, A_\Theta); H_0) = (f, \tau_{1,1,k}(t); H_0) = 0,\ k \in \mathbb{N}.$$

Taking into account the totality of the system of functions $V_1(L_0) = T_{1,1}$ in the space $H_{0,1}$, we have $f = f_1 \equiv 0$. Thus the system $V(A_\Theta)$ is total (complete) in the space H_0. Therefore, the operator A_Θ is defined on a dense set of the space H_0.

In the space H_0 let us define the operators

$$R(A_\Theta) := E + S(A_\Theta),\ \ S(A_\Theta)\tau_{1,0,q}(t) := 0,\ S(A_\Theta)\tau_{1,1,q}(t) := \theta_q \sqrt{2}\cos 2q\pi t \in H_{0,0},\ q \in \mathbb{N}.$$

According to equality $S^2(A_\Theta) = 0$ we get the relation $R^{-1}(A_\Theta) = E - S(A_\Theta)$. Therefore, the system of functions $V(A_\Theta)$ is minimal in the space H_0. Let us prove the second part of the lemma.

Necessity. We choose any bounded sequence Θ and show that $S(A_\Theta) : H_0 \to H_0$ is a bounded operator.

Let us expand an arbitrary function $h \in H_0$ into Fourier series

$$h = \sum_{k=1}^{\infty} \sum_{j=0}^{1} h_{j,k} \tau_{1,j,k}(t).$$

Consider $S(A_\Theta)h = \sum\limits_{k=1}^{\infty} \theta_m h_{1,k} \sqrt{2} \cos 2k\pi t.$

Taking into account that the system of functions $\{1, \cos 2k\pi t,\ k \in \mathbb{N}\}$ is an orthonormal basis of $H_{0,0}$ and using Cauchy's inequality, we obtain

$$\|S(A_\Theta)h; H_0\|^2 \le C_1 \|h; H_0\|^2,\ C_1 = \max |\theta_k|^2.$$

Thus $S(A_\Theta) \in [H_0]$.

Taking into account the relation $R^{-1}(A_\Theta) = E - S(A_\Theta),$ we obtain an estimate

$$\|R^{-1}(A_\Theta); [H_0]\|^2 \le C_2,\ C_2 = 2 + 2C_1.$$

Thus the system $V(A_\Theta)$ is the Riesz basis by definition.

Sufficiency. Let $V(A_\Theta)$ be the Riesz basis in the space H_0. Therefore, this system is almost normalized. Thus, for any positive numbers $C_3 \le C_4$ the next inequality holds:

$$C_3 \le \|v_{s,m}(t, A_\Theta); H_0\| \le C_4 < \infty,\ m \in \mathbb{N}.$$

Taking into account the equalities

$$\|v_{0,k}(t, A_\Theta); H_0\| = 1,\ \|v_{1,m}(t, A_\Theta); H_0\| = 1 + |\theta_m|,\ k = 0, 1, \ldots,\ m \in \mathbb{N},$$

we obtain the proof of sufficiency. □

Let B_0 be the operator of spectral problem

$$-z^{(2)}(t) = \mu z(t),\ \mu \in \mathbb{C},$$
$$\ell_1 z := z(0) + z(1) = 0,$$
$$\ell_2 z := z^{(1)}(0) + z^{(1)}(1) = 0,$$
$$B_0 z := -z^{(2)}(t),\ z(t) \in D(B_0),\ D(B_0) := \{z \in W_2^2(0,1) : \ell_s z = 0,\ s = 1, 2\},$$
$$T_2 := \{\tau_{2,r,m}(t) \in H_0 : \tau_{2,0,m}(t) := \sqrt{2} \sin \pi(2m-1)t,\ \tau_{2,1,m}(t) := \sqrt{2} \cos \pi(2m-1)t,\ m \in \mathbb{N}\},$$
$$\sigma(B_0) := \{\mu_{2,m} = \pi^2(2m-1)^2,\ m \in \mathbb{N}\}.$$

Lemma 3. *The operator B_0 has the point spectrum $\sigma(B_0)$ and system of eigenfunctions T_2.*

Proof. After performing a direct substitution we obtain that

$$\tau_{2,r,m}(t) \in D(B_0),\ -\tau_{2,r,m}^{(2)}(t) = \mu_{2,m} \tau_{2,r,m}(t),\ r = 0, 1,\ m \in \mathbb{N}.$$

Thus operator L_0 has the system of eigenfunctions $V(L_0)$, which corresponds to the set of eigenvalues σ. □

For the equation (1) we consider the boundary conditions $\ell_{0,s,j}u = 0$, $s = 1, \ldots, 2n$, $j = 1, 2$, which are the partial case of boundary conditions (2)–(6) for $\ell^1_{s,1}u = 0$, $s = 1, \ldots, n$.

Let $L_0 : H_1 \to H_1$ be the operator of the obtained problem

$$L_0 u := L\left(-D_1^2, -D_2^2\right) u,\ u \in D(L_0),\ D\left(L_0\right) := \{u \in H_2 : \ell_{0,s,j}u = 0,\ s = 1, \ldots, 2n,\ j = 1, 2\},$$

and

$$V(L_0) := \{v_{r,s,k,m}(x, L_0) \in H_1 : v_{r,s,k,m}(x, L_0) := \tau_{1,s,k}(x_1)\tau_{2,r,m}(x_2),\ r,\ s \in \{0, 1\},\ \ m,\ k \in \mathbb{N}\}$$

be the orthonormal basis of the space H_1.

Considering the ratio $L_0 = (-1)^n \sum\limits_{s=0}^{n} A_0^s B_0^{n-s}$, we obtain the following statement.

Lemma 4. *The operator L_0 has eigenvalues* (7) *and the system of eigenfunctions $V(L_0)$.*

2 THE NON SELF-AJOINT PROBLEM FOR A DIFFERENTIAL EQUATION OF EVEN ORDER

For any fixed $p \in \{1, \ldots, n\}$ we consider the problem

$$L(-D_1^2, -D_2^2)u := \sum_{s=0}^{n} a_s D_1^{2s} D_2^{2n-2s} u(x) = \lambda u(x),\ x \in G,\ \lambda \in \mathbb{C}, \tag{8}$$

$$\ell_{1,s,1}u := D_1^{2s-2}u|_{x_1=0} + D_1^{2s-2}u|_{x_1=1} = 0,\ s \neq p,\ s = 1, \ldots, n, \tag{9}$$

$$\ell_{1,p,1}u := D_1^{2p-2}u|_{x_1=0} + D_1^{2p-2}u|_{x_1=1} + \ell^0_{p,1}u = 0, \tag{10}$$

$$\ell_{1,n+s,1}u := D_1^{2s-2}u|_{x_1=0} - D_1^{2s-2}u|_{x_1=1} = 0,\ s \neq p,\ s = 1, \ldots, n, \tag{11}$$

$$\ell_{1,s,2}u := D_2^{2s-2}u|_{x_2=0} + D_2^{2s-2}u|_{x_2=1} = 0,\ s = 1, \ldots, n, \tag{12}$$

$$\ell_{1,n+s,2}u := D_2^{2s-1}u|_{x_2=0} + D_2^{2s-1}u|_{x_2=1} = 0,\ s = 1, \ldots, n. \tag{13}$$

Let $L_{1,p}$ be the operator of the problem (8)–(13):

$$L_{1,p}u := L(-D_1^2, -D_2^2)u,\ u \in D\left(L_{1,p}\right),$$
$$D\left(L_{1,p}\right) := \left\{u \in H_2 : \ell_{1,r,j}u = 0,\ r = 1, \ldots, 2n,\ j = 1, 2\right\},$$

and $V\left(L_{1,p}\right)$ be the system of eigenfunctions of the operator $L_{1,p}$.

For any fixed $m \in \mathbb{N}$ let's consider the solutions of problem (8)–(13) in the form of product

$$u(x) := z\left(x_1\right) \tau_{2,s,m}\left(x_2\right),\ s \in \{0, 1\}.$$

To determine the unknown function $z(x_1)$, we obtain the problem for eigenvalues

$$\sum_{q=0}^{n} a_q (-1)^{n-s} \mu_{2,m}^{n-q} z^{(2q)}(x_1) = \lambda z(x_1),\ \ x_1 \in (0, 1),\ \ \lambda \in \mathbb{C}, \tag{14}$$

$$l^1_{s,1}z := z^{(2s-2)}(0) + z^{(2s-2)}(1) = 0,\ s \neq p,\ s = 1, \ldots, n, \tag{15}$$

$$l^1_{p,1}z := z^{(2p-2)}(0) + z^{(2p-2)}(1) + l^0_{p,1}z = 0, \tag{16}$$

$$l^1_{n+s,1}z := z^{(2s-2)}(0) - z^{(2s-2)}(1) = 0,\ s = 1, \dots, n, \tag{17}$$

where

$$l^0_{p,1}z := \sum_{q=0}^{k_{p,1}} \sum_{r=0}^{n_1} b_{p,q,r} z^{(q)}(x_{1,r}),\ p = 1, \dots, n. \tag{18}$$

Let $L_{1,p,m}$ be the operator of problem (14)–(18):

$$L_{1,p,m}z := \sum_{s=0}^{n} a_s(-1)^{m-s}\mu_{2,m}^{n-s} z^{(2s)}, \quad z \in D\left(L_{1,p,m}\right),$$

$$D\left(L_{1,p,m}\right) := \left\{z \in W : l^1_{j,1}z = 0,\ j = 1, \dots, 2n\right\}.$$

Lemma 5. *Let Assumption P_1 holds. Therefore, for any $a_q \in \mathbb{R}$, $b_{p,q,r} \in \mathbb{R}$, $q = 0, 1, \dots, k_{p,1}$, $r = 0, 1, \dots, n_1$, m, $p \in \mathbb{N}$, the operator $L_{1,p,m}$ has the set of eigenvalues $\sigma_m := \{\lambda_{k,m} \in \sigma$, $k \in \mathbb{N}\}$, and the system of eigenfunctions $V\left(L_{1,p,m}\right)$, which is complete and minimal in the space H_0.*

Proof. The solutions $\omega_{r,m}(\lambda)$, $r = 1, \dots, n$, of equation $\sum\limits_{s=0}^{n} a_s(-1)^{n-s}\mu_{2,m}^{n-s}\omega^{2s} = \lambda$, which is characteristic for equations (14), we choose to fulfill the conditions

$$\operatorname{Re}\omega_{n,m}(\lambda) \le \operatorname{Re}\omega_{n-1,m}(\lambda) \le \dots \le \operatorname{Re}\omega_{1,m}(\lambda) \le 0.$$

Let us determine the functions

$$z_{q,m}(x_1, \lambda) := \frac{1}{2}(\exp\omega_{q,m}(\lambda)\,x_1 + \exp\omega_{q,m}(\lambda)\,(1 - x_1)) \in H_{0,0},\ q = 1, \dots, n,$$

$$z_{n+q,m}(x_1, \lambda) := \frac{1}{2}(\exp\omega_{q,m}(\lambda)\,x_1 - \exp\omega_{q,m}(\lambda)\,(1 - x_1)) \in H_{0,1},\ q = 1, \dots, n,$$

$$z_m(x_1) = \sum_{j=1}^{2n} c_j z_{j,m}(x_1, \lambda),\ c_j \in \mathbb{R}. \tag{19}$$

Substituting expression (19) into boundary conditions (15)–(17), we obtain an equation for determining of eigenvalues for operator $L_{1,p,m}$:

$$\Delta_m(\lambda) = \det(l^1_{q,1}z_{j,m}(x, \lambda))_{j,q=1}^{2n} = 0.$$

According to the relations $z_{rn+q,m}(x_1, \lambda) \in H_{0,r}$, $l^1_{q+sn} \in W^*_s$, s, $r \in \{0, 1\}$, $l^0_{p,1} \in W^*_1$, we obtain

$$l^1_{n+q,1}z_{j,m}(x_1, \lambda) = 0,\ l^1_{q,1}z_{n+j,m}(x_1, \lambda) = 0,\ j,\ q = 1, \dots, n,$$

$$\Delta_m(\lambda) = \Delta_{0,m}(\lambda)\Delta_{1,m}(\lambda),$$

$$\Delta_m(\lambda) = \prod_{q=1}^{n}(1 - e^{2\omega_{q,m}(\lambda)}) \prod_{1 \le j < q \le n} (\omega_{j,m}(\lambda) - \omega_{q,m}(\lambda))^2 = 0. \tag{20}$$

Let $\omega_{r,k,m}$ be roots of the equation (20) for $\lambda = \lambda_{k,m}$, which are selected so that $\omega_{1,k,m} = \imath\pi k$, $\operatorname{Re}\omega_{n,k,m} \le \operatorname{Re}\omega_{n-1,k,m} \le \dots \le \operatorname{Re}\omega_{1,k,m} \le 0$, $k \in \mathbb{N}$. Substituting expression (19) in boundary conditions (15)–(17), we can find the eigenfunctions of the operator $L_{1,p,m}$:

$$v_{0,k}\left(x_1, L_{1,p,m}\right) = \sqrt{2}\sin\rho_{0,k}x_1,\ \rho_{0,k} = \pi(2k - 1),\ k \in \mathbb{N}. \tag{21}$$

Let us define the system of functions

$$z_{1,1,k,m}(x_1) = \sqrt{2}\cos\rho_{1,k}x_1,\ \rho_{1,k} = 2k\pi,\ k \in \mathbb{N}, \tag{22}$$

$$z_{1,q,k,m}(x_1) := \frac{1}{2}(1 + \exp\omega_{q,k,m})^{-1}(\exp\omega_{q,k,m}x_1 + \exp\omega_{q,k,m}(1 - x_1)),\ k \in \mathbb{N}, \tag{23}$$

and a square matrix of order n, elements of which we define by the following rule: pth row is defined by functions (22), (23), and elements of other rows is defined by numbers

$$\vartheta_{q,r,k,m} := \rho_{1,k}^{2-2r} l_{1,r,1} z_{1,q,k,m},\ v_{q,r,k,m} = \rho_{1,k}^{2-2r}\omega_{q,k,m}^{2r-2},\ q = 2,3,\ldots,n,\ r \neq p,\ r = 1,\ldots,n.$$

$$\vartheta_{1,r,k,m} = 2\sqrt{2},\ r \neq p,\ = 2,3,\ldots,n,\ k \in \mathbb{N}.$$

Determinant of the given matrix is denoted by $z_{2,p,k,m}(x_1),\ k \in \mathbb{N}$.

Remark 1. *For any fixed $m \in \mathbb{N}$ and $k \to \infty$, we obtain the relation*

$$\delta_{1,k,m} := \omega_{1,k,m}\rho_{1,k}^{-1} = \imath,$$

$$\delta_{q,k,m} := \rho_{1,k}^{-1}\omega_{q,k,m} = \varepsilon_q\left(1 + O\left(k^{-1}\right)\right),$$

where ε_q are the solutions of equation $(-1)^n(\varepsilon)^{2n} = 1,\ \varepsilon_1 = \imath,\ \operatorname{Im}\varepsilon_q < 0,\ q = 2,3,\ldots,n$.

Substituting function $z_{2,p,k,m}(x_1)$ in boundary conditions (14)–(17), we obtain the equalities

$$\ell_{1,s,1}z_{2,p,k,m} = 0,\ s \neq p,\ l_{1,p,1}z_{2,p,k,m} := c_{p,k,m},\ s = 1,\ldots,2n,\ k \in \mathbb{N},$$

where $c_{p,k,m} = \sqrt{2}2\rho_{1,k}^{2p-2}W_{k,m},\ W_{k,m} = W\left(\delta^2_{1,k,m},\ldots,\delta^2_{n,k,m}\right)$ is Vandermonde determinant of order n, which is constructed by numbers $\delta^2_{q,k,m},\ q = 1,\ldots,n$.

Remark 2. *For arbitrary $m \in \mathbb{N}$ and $k \to \infty$ the number sequence $\{W_{k,m}\}_{k=1}^{\infty}$ converges to Vandermonde determinant $W\left(\varepsilon_1^2,\ \varepsilon_2^2,\ldots,\varepsilon_n^2\right)$, which is constructed by numbers $\varepsilon_1^2,\ldots,\varepsilon_n^2$.*

Therefore, $\vartheta_{q,r,k,m} = \varepsilon_q^{2r-2}(1 + O(\frac{1}{k})),\ k \to \infty,\ q = 1,\ldots,n$.

Thus, the positive numbers $C_5,\ C_6$ exist such that the following inequality holds:

$$0 < C_5 \leq \left|c_{p,k,m}\right|\rho_{1,k}^{2-2p} \leq C_6 < \infty,\ k \in \mathbb{N}.$$

Let us choose the functions

$$z_{3,p,k,m}(x_1) := W_{k,m}^{-1}z_{2,p,k,m}(x_1),\ k \in \mathbb{N}. \tag{24}$$

Taking into account equalities (24), we obtain the relations

$$\ell^1_{1,s}z_{3,p,k,m} = 0,\ s \neq p,\ \ell^1_{1,p}z_{3,p,k,m}(x_1) = 2\sqrt{2}\rho_{1,k}^{2p-2},\ s = 1,\ldots,n. \tag{25}$$

Let $\Delta_{j,s,k,m} := \det(\vartheta_{q,r,k,m})_{q,r=\overline{1,n}}^{q\neq j,r\neq s}$. Consider the functions $y_{p,k,m}(x_1) := \Delta^{-1}_{1,1,k,m}z_{3,p,k,m}(x_1)$,

$$y_{p,k,m}(x_1) = z_{1,1,k,m}(x_1) + \sum_{j=2}^{n}\gamma_{j,p,k,m}z_{1,j,k,m}(x_1),\ k \in \mathbb{N}, \tag{26}$$

where $\gamma_{j,p,k,m} = \Delta^{-1}_{1,p,k,m}\Delta_{j,p,k,m},\ j = 2,3,\ldots,n$.

From formulas (24)–(26) we obtain

$$y_{p,k,m}(x_1) = c_{1,p,k,m} z_{2,p,k,m}(x_1),$$

where

$$c_{1,p,k,m} = W_{k,m}^{-1} \Delta_{1,p,k,m},\ C_7 < c_{1,p,k,m} < C_8 < \infty.$$

Therefore,

$$l_{1,p}^1 y_{p,k,m}(x_1) = c_{1,p,k,m} 2\sqrt{2} \rho_{1,k}^{2p-2},\ l_{1,s}^1 y_{p,k,m}(x_1) = 0,\ s \neq p,\ s = 1, \ldots, n.$$

The eigenfunctions $v_{1,k}\left(x_1, L_{1,p,m}\right)$ of the operator $L_{1,p,m}$ we define by the equality

$$v_{1,k}(x_1, L_{1,p,m}) := \tau_{1,1,k}(x_1) + \eta_{p,k,m} y_{p,k,m}(x_1),\ k \in \mathbb{N}. \tag{27}$$

To determine the unknown parameters $\eta_{p,k,m}$, we substitute the expression (27) in the boundary conditions (16), (17).

Taking into account (24), we obtain

$$\eta_{p,k,m} = (-1)^p \sqrt{8^{-1}} c_{1,p,k,m}^{-1} \rho_{1,k}^{2-2p} l_{1,p}^1 \tau_{1,1,k},\ k \in \mathbb{N}. \tag{28}$$

Thus, the operator $L_{1,p,m}$ has the system $V\left(L_{1,p,m}\right)$ of eigenfunctions (21), (24), (28).

The completeness of the system of functions $V\left(L_{1,p,m}\right)$ in the space H_0 is proved from the opposite, like in the proof of the Lemma 2.

Let us consider the operators

$$R(L_{1,p,m}),\ S(L_{1,p,m}):\ H_0 \to H_0,\ \ R(L_{1,p,m}) = E + S(L_{1,p,m}),$$

$$R(L_{1,p,m})\tau_{1,0,k}(x_1) := \tau_{1,0,k}(x_1),\ \ R(L_{1,p,m})\tau_{1,1,k}(x_1) := v_{1,k}(x_1, L_{1,p,m}),\ k \in \mathbb{N}.$$

From the definition of operator $S(L_{1,p,m})$ we obtain $S(L_{1,p,m}) : H_{0,0} \to 0,\ S(L_{1,p,m}) : H_{0,1} \to H_{0,0},\ S^2(L_{1,p,m}) = 0,\ R^{-1}(L_{1,p,m}) = E - S(L_{1,p,m})$. Therefore, the system of functions $V\left(L_{1,p,m}\right)$ is minimal in the space H_0. Lemma 5 is proved. □

Let $\theta_k = \eta_{p,k,m}$, then $A_{p,m} := A_\Theta,\ k,\ m \in \mathbb{N},\ p \in \{1, \ldots, n\}$.

Lemma 6. *If $\{\eta_{p,k,m}\}_{k=1}^{\infty}$ is a bounded sequence, then the system of functions $V(L_{1,p,m})$ is the Riesz basis in the space H_0.*

Proof. Taking into account the definition of the function $y_{p,k,m}(x_1)$ and the choice of numbers $\omega_{q,k,m},\ q = 1, \ldots, n$, we can conclude: if $\theta_k = \eta_{p,k,m},\ k \in \mathbb{N},\ p \in \{1, \ldots, n\}$, is a bounded sequence, then the systems of functions $V(L_{1,p,m}),\ V\left(A_{p,m}\right)$ are quadratically approximate for every $m \in \mathbb{N},\ p \in \{1, \ldots, n\}$.

Therefore, taking into account the Lemma 5 and the theorem N.K. Bari [10], we obtain the statement of Lemma 6. □

Let us choose an arbitrary sequence of real numbers $\Theta = \{\theta_k\}_{k=1}^{\infty}$, and define the operator $A_{\Theta,p,m} : H_0 \to H_0$, which has the set of eigenvalues $\sigma_{1,m} = \{\lambda_{k,m} \in \sigma,\ k \in \mathbb{N}\}$ and the system $V(A_{\Theta,p,m}) := \{v_{s,k,m}(x_1, A_{\Theta,p,m}) \in H_0 :\ \ s = 0, 1,\ k \in \mathbb{N}\}$ of eigenfunctions

$$v_{0,k,m}(x_1, A_{\Theta,p,m}) := \tau_{1,0,k}(x_1),\ v_{1,k,m}(x_1, A_{\Theta,p,m}) := \tau_{1,1,k}(x_1) + \theta_k y_{p,k,m}(x_1),\ k \in \mathbb{N}. \tag{29}$$

Consider the operators

$$R(A_{\Theta,p,m}) := E + S(A_{\Theta,p,m}),$$
$$S(A_{\Theta,p,m})\tau_{1,0,k}(x_1) := 0,$$
$$S(A_{\Theta,p,m})\tau_{1,1,k}(x_1) := \theta_k y_{p,k,m}(x_1),\ k \in \mathbb{N}.$$

Let $\Gamma_{1,p}(L_{0,m})$ be the set of operators, which have purely point spectrum $\sigma_{1,m}$ and the system of eigenfunctions (29).

We define on $\Gamma_{1,p}(L_{0,m})$ the commutative multiplication operation

$$R(A_{\Theta_1,p,m})R(A_{\Theta_2,p,m}) = E + S(A_{\Theta_1,p,m}) + S(A_{\Theta_2,p,m}) = R(A_{\Theta_2,p,m})R(A_{\Theta_1,p,m}),$$

$$A_{\Theta_2,p,m},\ A_{\Theta_1,p,m} \in \Gamma_{1,p}(L_0),$$

and inverse operator $R^{-1}(A_{\Theta,p,m}) = E - S(A_{\Theta,p,m}),\ A_{\Theta,p,m} \in \Gamma_1(L_{0,m})$.

Lemma 7. *For any real numbers $\theta_q \in \mathbb{R},\ q \in \mathbb{N},$ the system of functions $V(A_{\Theta,p,m})$ is complete and minimal in the space H_0. The system of functions $V(A_{\Theta,p,m})$ is the Riesz basis in H_0 if and only if the sequence Θ is bounded.*

Proof. The lemma can be proved by the schema of proof the Lemma 2. □

We define by the formulas

$$v_{s,r,k,m}\left(x, L_{1,p}\right) := v_{s,k}\left(x_1, L_{1,p,m}\right)\tau_{2,r,m}\left(x_2\right),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}, \tag{30}$$

the eigenfunctions of operator $L_{1,p}$.

Lemma 8. *Suppose that the Assumption P_1 holds. Then, for arbitrary $a_s \in \mathbb{R}$, $b_{p,q,r} \in \mathbb{R}$, the operator $L_{1,p}$ has the point spectrum σ, and the system of eigenfunctions $V\left(L_{1,p}\right) := \{v_{s,r,k,m}\left(x, L_{1,p}\right),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}\}$, which is complete and minimal in H_1.*

If the Assumptions P_1–P_3 hold, then the system of functions $V\left(L_{1,p}\right)$ is the Riesz basis in the space H_1.

Proof. Substituting functions (30) into the equations (8)–(13) makes sure that the numbers $\lambda_{k,m} \in \sigma$ are eigenvalues, if $k,\ m \in \mathbb{N}$.

In the space H_1 we define the operator $R\left(L_{1,p}\right) := E + S\left(L_{1,p}\right)$, which maps the system of functions $V\left(L_0\right)$ into $V\left(L_{1,p}\right)$.

The operator $R\left(L_{1,p}\right)$ has the form

$$R\left(L_{1,p}\right) := \sum_{r,m} R(L_{1,p,m}) \times \pi_{2,r,m},$$

where $\pi_{2,r,m}$ is the orthoprojector into the one-dimensional proper subspace in H_0, which corresponds to eigenfunction $\tau_{2,r,m}(x_2)$ of operator B_0.

We consider the operator $A_p : H_1 \to H_1,$ which has purely point spectrum $\sigma(A_p) := \{\lambda_{k,m} \in \mathbb{R} : \lambda_{k,m} = \mu_{1,k} + \mu_{2,m},\ k,\ m \in \mathbb{N}\}$ and the system of eigenfunctions

$$V(A_p) := \{v_{s,r,k,m}\left(x_1, x_2, A_p\right) := v_{s,k}\left(x_1, A_{p,m}\right)\tau_{2,r,m}\left(x_2\right),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}\}.$$

Let $R(A_p) := \sum_{r,m} R(A_{p,m}) \times \pi_{2,r,m}$.

According to the Lemma 5, for an arbitrary $m \in \mathbb{N}$ the system of functions $W\left(L_{1,p,m}\right)$ exists, and it is biorthogonal to the system $V\left(L_{1,p,m}\right)$.

Therefore, we can define the elements of system $W\left(L_{1,p}\right)$, which is biorthogonal to system $V\left(L_{1,p}\right)$ in the space H_1 :

$$w_{s,r,k,m}(x_1,x_2,L_{1,p}) = w_{s,k}(x_1,L_{1,p,m})\tau_{2,r,m}(x_2),\ s,\ r \in \{0,1\},\ k,\ m \in \mathbb{N}.$$

Thus, the system $V\left(L_{1,p}\right)$ is complete and minimal in H_1.

Therefore, when the Assumptions P_2 and P_3 hold, then we obtain the inequality $|\eta_{p,k,m}| \leq C_9 < \infty$, for arbitrary $m,\ k \in \mathbb{N}$. Taking into account the estimates $\|R(A_p);[H_1]\|^2 \leq C_{10}$, we obtain the statement: eigenfunctions (30) of operator A_p are almost normalized, and system $V\left(A_p\right)$ is the Riesz basis of the space H_1.

We consider the operator $R(L_{1,p}) = E + S(L_{1,p}) = (E+Q)(E+S(A_p))$. Then the operator $Q_p := S(L_{1,p}) - S(A_p)$ is completely continuous, because the systems of functions $V(L_{1,p,m})$, $V(A_{p,m})$ are quadratically approximate and the operator $Q_{p,m} := S(L_{1,p,m}) - S(A_{p,m})$ is idempotent: $Q_{p,m}^2 = 0$.

According to the definition of function $v_{s,r,k,m}(x,L_0)$, we obtain

$$\|Q_p v_{s,r,k,m}(x,L_0);H_1\| = O(m+k)^{-3},\ m,\ k \to \infty.$$

Then, for an arbitrary $h = \sum_{s,r,m,k} h_{s,r,k,m} v_{s,r,k,m}(x,L_0) \in H_1$, from Cauchy's inequality we can get the inequality

$$\|Q_p h;H_1\|^2 = \|\sum_{s,r,k,m} h_{s,r,k,m} Q_p v_{s,r,k,m}(x,L_0);H_1\|^2 \leq C_{11}\|h;H_1\|^2.$$

Thus $\|Q_p;[H_1]\|^2 < \infty$, $(L_{1,p}) = Q_p + R(A_{1,p}) \in [H_1]$, $R(L_{1,p})^{-1} = (E - S(A_p)(E-Q) \in [H_1]$. □

Proof. Proof of the Theorem 1. Let $R(L) := \prod_{p=1}^{n} R(L_{1,p})$. The eigenfunctions of operator L we can define in the form

$$v_{s,r,k,m}(x,L) := R(L)v_{s,r,k,m}(x,L_0),\ r,\ s \in \{0,1\},\ k,\ m \in \mathbb{N}.$$

Taking into account, that operators $R(L_{1,p})$ are elements of the group $\Gamma_{1,p}(L_0)$, we obtain

$$R(L) = E + S(L),\ R^{-1}(L) = E - S(L),\ S(L) := \sum_{p=1}^{n} S(L_{1,p}).$$

Therefore, the system of eigenfunctions $V(L)$ is complete and minimal in H_1. □

Proof. Proof of the Theorem 2. Let the Assumptions P_1–P_3 hold, then the system of eigenfunctions $V\left(L_{1,p}\right)$ is the Riesz basis in the space H_1, and $R(L) = \prod_{p=1}^{n} R(L_{1,p}) \in [H_1]$. Therefore, taking into account the theorem N.K. Bari [10], we obtain the statement of the theorem. □

Let us define the elements of system $W(L)$, which is biorthogonal to system $V(L)$ in the space H_1:

$$w_{s,r,k,m}(x,L) := R(L)\tau_{1,s,k}(x_1)\tau_{2,r,m}(x_2),\ \ s,r \in \{0,1\},\ \ k,\ m \in \mathbb{N}.$$

Remark 3. *The positive numbers $C_1(L)$, $C_2(L)$ exist, such that for function*

$$f(x) = \sum_{s,r,k,m} f_{s,r,k,m} v_{s,r,k,m}(x_1, x_2, L),\ f_{s,r,k,m} := (f, w_{s,r,k,m}(x_1, x_2, L);\ H_1)$$

the following inequality holds

$$C_2(L)\|f; H_1\|^2 \leq \sum_{s,r,k,m} |f_{s,r,k,m}|^2,\ C_3(L)\|f; H_1\|^2. \tag{31}$$

Proof. Proof of the Theorem 3. We will use a solution of the problem (1)–(6) in the form of series

$$u(x) = \sum_{s,r,k,m} u_{s,r,k,m} v_{s,r,k,m}(x_1, x_2, L). \tag{32}$$

If we substitute series (31), (32) into equation (1), we obtain

$$u_{s,r,k,m} = \lambda_{k,m}^{-1} f_{s,r,k,m}.$$

Taking into account the Assumption P_3 and inequality $\lambda_{k,m}^{-1} \leq 1$, we can get

$$\|u; H_1\|^2 \leq C_5(L)\|f; H_1\|^2,\ C_5(L) = C_2(L)^{-1}(L)C_3(L)C_1^{-2}(L),$$

$$\|D_1^{2n} u; H_1\|^2 \leq C_5(L)\|f; H_1\|^2,$$

$$\|D_2^{2n} u; H_1\|^2 \leq C_5(L)\|f; H_1\|^2.$$

Therefore, $\|u; H_2\|^2 \leq 3C_5(L)\|f; H_1\|^2$. □

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Баранецький Я.О., Каленюк П.І., Копач М.І., Соломко А.В. *Нелокальна крайова задача зі збуреннями мішаних крайових умов для еліптичного рівняння зі сталими коефіцієнтами. I* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 228–239.

У роботі в одиничному квадраті G методом Фур'є досліджується задача з нелокальними умовами, які є багатоточковими збуреннями мішаних крайових умов. Вивчено властивості узагальненого оператора перетворення $R : L_2(G) \to L_2(G)$, який відображає нормовані власні функції оператора L_0 задачі із мішаними крайовими умовами у власні функції оператора L збуреної нелокальної задачі. Побудовано систему $V(L)$ власних функцій оператора L. Визначено умови, при яких система $V(L)$ повна та мінімальна в просторі $L_2(G)$, та умови, при яких вона є базисом Рісса у просторі $L_2(G)$. У випадку, якщо система $V(L)$ є базисом Рісса в просторі $L_2(G)$, встановлено достатні мови, при яких нелокальна задача має єдиний розв'язок у вигляді ряду Фур'є за системою $V(L)$.

Ключові слова і фрази: диференціальне рівняння з частинними похідними, кореневі функції, базис Рісса.