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# ON COMPARISON OF THE PRINCIPLES OF EQUIVALENT UTILITY AND ITS APPLICATIONS

An insurance premium principle is a way of assigning to every risk, represented by a non-negative bounded random variable on a given probability space, a non-negative real number. Such a number is interpreted as a premium for the insuring risk. In this paper the implicitly defined principle of equivalent utility is investigated. Using the properties of the quasideviation means, we characterize a comparison in the class of principles of equivalent utility under Rank-Dependent Utility, one of the important behavioral models of decision making under risk. Then we apply this result to establish characterizations of equality and positive homogeneity of the principle. Some further applications are discussed as well.

*Key words and phrases:* insurance premium, quasideviation mean, comparison, equality, positive homogeneity, risk loading.

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## 1 INTRODUCTION

Assume that $\mathcal{X}_+$ is a family of risks, represented by non-negative bounded random variables on a non-atomic probability space $(\Omega, \mathcal{F}, P)$. An insurance premium principle is a way of assigning to every $X \in \mathcal{X}_+$ a non-negative real number $H(X)$. The number $H(X)$ is interpreted as a premium for insuring $X$. There are many methods of defining the principles. In what follows we deal with the principle of equivalent utility. The principle, postulating a fairness in terms of utility, has been introduced in [2]. Under the Expected Utility model the premium for a risk $X \in \mathcal{X}_+$ is defined through the equation

$$E[u(w + H_{(w,u)}(X) - X)] = u(w), \tag{1}$$

where $w \in [0, \infty)$ is an initial wealth level and $u : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function such that $u(0)$=0. In general, (1) has no explicit solution. However, in some cases the premium can be expressed in an explicit way. In particular, if $u$ is linear, then

$$H_{(w,u)}(X) = E[X] \quad \text{for} \quad X \in \mathcal{X}_+,$$

i.e. the principle of equivalent utility becomes the net premium principle. If $u(x) = a(1 - e^{-cx})$ for $x \in \mathbb{R}$, with some $a, c > 0$, then from (1) we deduce that the principle of equivalent utility reduces to the exponential principle

$$H_{(w,u)}(X) = \frac{1}{c} \ln E[e^{cX}] \quad \text{for} \quad X \in \mathcal{X}_+.$$

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Note that in both cases the premium for a given risk does not depend on an initial wealth level. Some properties of the principle of equivalent utility defined by (1) can be found e.g in [1,2,6,13].

In this paper we deal with the principle of equivalent utility under Rank-Dependent Utility, one of the behavioral models of decision making under risk. In this setting the principle has been introduced and investigated in [7]. In order to define it, recall that if $g : [0,1] \to [0,1]$ is a probability distortion function, that is a non-decreasing function such that $g(0) = 0$ and $g(1) = 1$ then, for any bounded random variable $X$ on $(\Omega, \mathcal{F}, P)$, the Choquet integral with respect to $g$ is given by

$$E_g[X] = \int_{-\infty}^{0} (g(P(X > t)) - 1)\, dt + \int_{0}^{\infty} g(P(X > t))\, dt. \tag{2}$$

The premium for a risk $X \in \mathcal{X}_+$ under the Rank-Dependent Utility model is defined as a solution of the equation

$$E_g[u(w + H_{(w,u,g)}(X) - X)] = u(w). \tag{3}$$

It is known (cf. [4, Remark 1]) that if $g$ is a continuous probability distortion function and $u : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function with $u(0) = 0$ then, for every $X \in \mathcal{X}_+$, the number $H_{(w,u,g)}(X)$ is uniquely determined by (3). Some properties of the premium defined by (3) have been investigated in [7] under the assumption that $g$ is convex and $u$ is concave and differentiable.

The main result of this paper provides a characterization of a comparison in the class of the principles of equivalent utility. Applying this result we establish characterizations of further natural properties of the principle, namely equality and positive homogeneity. Some results concerning the risk loading property of the principle of equivalent utility are presented as well.

It turns out that an effective tool for dealing with this issue is a notion of a quasideviation mean. Therefore, in the next section we present a definition of the mean and a result concerning a comparison of quasideviation means.

## 2 QUASIDEVIATION MEANS

The notion of the quasideviation mean has been introduced in [10]. In order to recall the notion, assume that $I \subseteq \mathbb{R}$ is an open interval. A function $D : I^2 \to \mathbb{R}$ is called a quasideviation if it satisfies the following three conditions:

(i) $D(x,x) = 0$ for $x \in I$ and $(x-y)D(x,y) > 0$ for $x, y \in I$ with $x \neq y$;

(ii) for every $x \in I$, the function $I \ni t \to D(x,t)$ is continuous;

(iii) for every $x, y \in I$, with $x < y$, the function $(x,y) \ni t \to \frac{D(y,t)}{D(x,t)}$ is strictly increasing.

Let

$$\Delta_n := [0,\infty)^n \setminus \{\overline{0}\}.$$

In [10] it has been proved that, if $D : I^2 \to \mathbb{R}$ is a quasideviation, then for every $n \in \mathbb{N}$, $\overline{x} = (x_1, ..., x_n) \in I^n$ and $\overline{\lambda} = (\lambda_1, ..., \lambda_n) \in \Delta_n$, equation

$$\sum_{i=1}^{n} \lambda_i D(x_i, t) = 0 \tag{4}$$

has a unique solution $t_0$. Moreover

$$\min\{x_i : i \in \{1,..,n\}\} \leq t_0 \leq \max\{x_i : i \in \{1,..,n\}\}.$$

Thus, equation (4) defines a mean, called a $D$-quasideviation mean of $\overline{x}$ weighted by $\overline{\lambda}$. Following [10], we denote the mean by $\tilde{\mathfrak{M}}_D(\overline{x};\overline{\lambda})$. Several properties of quasideviation means have been proved in [11]. In our considerations we will need the following result, which is a particular case of [11, Theorem 7].

**Theorem 1.** *Assume that $I \subseteq \mathbb{R}$ is an open interval and $D_1, D_2 : I^2 \to \mathbb{R}$ are quasideviations. Then the following statements are equivalent:*

$(i)$ $\tilde{\mathfrak{M}}_{D_1}((x_1,x_2);(\lambda,1-\lambda)) \leq \tilde{\mathfrak{M}}_{D_2}((x_1,x_2);(\lambda,1-\lambda))$ *for* $x_1, x_2 \in I$, $\lambda \in [0,1]$;

$(ii)$ *there exists a function* $A : I \to (0,\infty)$ *such that*

$$D_1(x,y) \leq A(y)D_2(x,y) \quad \text{for} \quad x,y \in I.$$

## 3 Preliminary remarks

**Remark 1.** *Let $g$ be a probability distortion function. It is known (cf. [5, Proposition 5.1]) that the Choquet integral is monotone and positively homogeneous. Furthermore, for every bounded random variable $X$ on $(\Omega,\Sigma,P)$, we get*

$$E_g[X+c] = E_g[X] + c \quad \text{for} \quad c \in \mathbb{R} \tag{5}$$

*and*

$$E_g[-X] = -E_{\bar{g}}[X], \tag{6}$$

*where $\bar{g} : [0,1] \to [0,1]$, given by*

$$\bar{g}(p) = 1 - g(1-p) \quad \text{for} \quad p \in [0,1], \tag{7}$$

*is the probability distortion function conjugated to $g$.*

**Remark 2.** *Note that if $g$ is the identity on $[0,1]$ then $E_g[X] = E[X]$ for every bounded random variable $X$ on $(\Omega,\Sigma,P)$. Therefore, applying [5, Proposition 5.2 (iii)], we conclude that:*

- *if $g(p) \leq p$ for $p \in [0,1]$ then $E_g[X] \leq E[X]$ for every bounded random variable $X$ on $(\Omega,\Sigma,P)$;*
- *if $g(p) \geq p$ for $p \in [0,1]$ then $E_g[X] \geq E[X]$ for every bounded random variable $X$ on $(\Omega,\Sigma,P)$.*

**Remark 3.** *Let $g$ be a continuous probability distortion function. Since the Choquet integral is monotone, for every $X \in \mathcal{X}_+$, the function*

$$\mathbb{R} \ni t \to E_g[u(w+t-X)] - u(w)$$

*is nondecreasing. Furthermore, $H_{(w,u,g)}(X)$ is its unique zero.*

**Remark 4.** *In view of* (3) *the premium for a given risk depends only on a probability distribution of the risk. Thus, we identify the risks with their probability distributions. Note also (cf. e.g. [12, Lemma 2.7.1]) that, as the probability space* $(\Omega, \Sigma, P)$ *is non-atomic, for every* $x, y \in \mathbb{R}$*, with* $x < y$*, and every* $p \in (0,1)$*, there exists a random variable* $X$ *on* $(\Omega, \Sigma, P)$ *such that* $P(X = x) = p$ *and* $P(X = y) = 1 - p$*. Denote any such a random variable by* $\langle x, y; 1-p, p\rangle$*. Furthermore, let* $\mathcal{X}^{(2)}$ *be a family of all such random variables and*

$$\mathcal{X}^{(2)}_{+} := \{\langle x, y; 1-p, p\rangle \in \mathcal{X}^{(2)} : x \geq 0\}.$$

**Remark 5.** *If* $X = \langle x_1, x_2; 1-p, p\rangle \in \mathcal{X}^{(2)}$ *then, in view of* (2)*, we get (cf. [8])*

$$E_g[X] = (1 - g(p))x_1 + g(p)x_2.$$

**Remark 6.** *Assume that* $w \in [0, \infty)$*,* $g$ *is a continuous probability distortion function and* $u : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing continuous function such that* $u(0) = 0$*. Then, taking* $X = \langle x, y; p, 1-p\rangle \in \mathcal{X}^{(2)}_{+}$*, we obtain*

$$u(w + H_{(w,u,g)}(X) - X) = \langle (u(w + H_{(w,u,g)}(X) - y), u(w + H_{(w,u,g)}(X) - x)); 1-p, p\rangle.$$

*Therefore, applying Remark 5, from* (3) *we derive that* $H_{(w,u,g)}(X)$ *is a unique solution of the equation*

$$(1 - g(p))(u(w + H_{(w,u,g)}(X) - y) + g(p)u(w + H_{(w,u,g)}(X) - x) = u(w). \tag{8}$$

## 4 RESULTS

The following theorem is the main result of this paper.

**Theorem 2.** *Let* $w_1, w_2 \in [0, \infty)$*. Assume that* $g$ *is a continuous probability distortion function and* $u, v : \mathbb{R} \to \mathbb{R}$ *are strictly increasing continuous functions such that* $u(0) = v(0) = 0$*. Then the following statements are pairwise equivalent:*

$(i)$

$$H_{(w_1,v,g)}(X) \leq H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}^{(2)}_{+}; \tag{9}$$

$(ii)$

$$H_{(w_1,v,g)}(X) \leq H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}_{+}; \tag{10}$$

$(iii)$ *there exists* $c \in (0, \infty)$ *such that*

$$u(x) \leq cv(x + w_1 - w_2) + u(w_2) - cv(w_1) \quad \textit{for} \quad x \in \mathbb{R}. \tag{11}$$

*Proof.* Let $D_1, D_2 : (0, \infty)^2 \to \mathbb{R}$ be given by

$$D_1(x, y) = v(w_1) - v(w_1 + y - x) \quad \text{for} \quad x, y \in (0, \infty), \tag{12}$$

and

$$D_2(x, y) = u(w_2) - u(w_2 + y - x) \quad \text{for} \quad x, y \in (0, \infty), \tag{13}$$

respectively. Then, as one can easily check, $D_1$ and $D_2$ are quasideviations. Furthermore, since $g$ is continuous with $g(0) = 0$ and $g(1) = 1$, for every $\lambda \in (0,1)$ there exists (not necessarily unique) $p_\lambda \in (0,1)$ such that

$$g(p_\lambda) = \lambda. \tag{14}$$

First we show that *(i)* $\Longrightarrow$ *(iii)*. Assume that (9) holds. Let $x_1, x_2 \in (0,\infty)$ and $\lambda \in [0,1]$. We claim that

$$\tilde{\mathfrak{M}}_{D_1}((x_1,x_2);(\lambda,1-\lambda)) \leq \tilde{\mathfrak{M}}_{D_2}((x_1,x_2);(\lambda,1-\lambda)). \tag{15}$$

If $x_1 = x_2$ or $\lambda = 1$, then both sides of (15) are equal to $x_1$. Moreover, if $\lambda = 0$, then both sides of (15) are equal to $x_2$. So, assume that $\lambda \in (0,1)$ and $x_1 \neq x_2$, say $x_1 < x_2$. Let $X = \langle x_1, x_2; p_\lambda, 1-p_\lambda\rangle$, where $p_\lambda \in (0,1)$ satisfies (14). Then $X \in \mathcal{X}_+^{(2)}$ whence, taking into account (8) and (12), we get

$$\begin{aligned}
&\lambda D_1(x_1, H_{(w_1,v,g)}(X)) + (1-\lambda)D_1(x_2, H_{(w_1,v,g)}(X)) \\
&\quad= g(p_\lambda)(v(w_1) - v(w_1 + H_{(w_1,v,g)}(X) - x_1)) + (1-g(p_\lambda))(v(w_1) - v(w_1 + H_{(w_1,v,g)}(X) - x_2)) \\
&\quad= v(w_1) - ((1-g(p_\lambda))v(w_1 + H_{(w_1,v,g)}(X) - x_2) + g(p_\lambda)v(w_1 + H_{(w_1,v,g)}(X) - x_1)) = 0.
\end{aligned}$$

Thus

$$H_{(w_1,v,g)}(X) = \tilde{\mathfrak{M}}_{D_1}((x_1,x_2);(\lambda,1-\lambda)).$$

The similar arguments show that

$$H_{(w_2,u,g)}(X) = \tilde{\mathfrak{M}}_{D_2}((x_1,x_2);(\lambda,1-\lambda)).$$

Hence, in view of (9), we get (15). In this way we have proved that (15) holds for every $x_1, x_2 \in (0,\infty)$ and $\lambda \in [0,1]$. Therefore, applying Theorem 1 and making use of (12)-(13), we obtain that there exists a function $A : (0,\infty) \to (0,\infty)$ such that

$$v(w_1) - v(w_1 + y - x) \leq A(y)(u(w_2) - u(w_2 + y - x)) \quad \text{for} \quad x, y \in (0,\infty).$$

Since $u$ and $v$ are strictly increasing with $u(0) = v(0) = 0$, replacing in the last inequality $x$ by $y - x$, we get

$$v(w_1) - v(w_1 + x) \leq A(y)(u(w_2) - u(w_2 + x)) \quad \text{for} \quad x \in \mathbb{R},\ y \in (\max\{0,x\},\infty).$$

Thus

$$\frac{u(w_2) - u(w_2 + x)}{v(w_1) - v(w_1 + x)} \leq \frac{1}{A(y)} \quad \text{for} \quad x \in (0,\infty),\ y > x \tag{16}$$

and

$$\frac{u(w_2) - u(w_2 + x)}{v(w_1) - v(w_1 + x)} \geq \frac{1}{A(y)} \quad \text{for} \quad x \in (-\infty,0],\ y \in (0,\infty). \tag{17}$$

Hence, taking

$$c := \sup\left\{\frac{1}{A(y)} : y \in (0,\infty)\right\},$$

we conclude that $0 < c < \infty$. Moreover, it follows from (16) that the inequality

$$u(w_2 + x) \leq cv(w_1 + x) + u(w_2) - cv(w_1) \tag{18}$$

holds for all $x \in (0,\infty)$. Furthermore, taking in (17) the supremum over all $y \in (0,\infty)$, we obtain

$$c = \sup\left\{\frac{1}{A(y)} : y \in (0,\infty)\right\} \leq \frac{u(w_2)-u(w_2+x)}{v(w_1)-v(w_1+x)} \quad \text{for} \quad x \in (-\infty,0],$$

which implies (18) for $x \in (-\infty,0]$. Therefore, (18) holds for all $x \in \mathbb{R}$. Replacing in (18) $x$ by $x-w_2$, we obtain (11). So, $(i) \Rightarrow (iii)$.

Now, assume that (11) is satisfied. Then, as the Choquet integral is monotone and positively homogeneous, in view of (3) and (5), for every $X \in \mathcal{X}_+$, we have

$$E_g[u(w_2+H_{(w_1,v,g)}(X)-X)]-u(w_2) \leq c(E[v(w_1+H_{(w_1,v,g)}(X)-X)]-v(w_1)) = 0.$$

Moreover, according to Remark 3, for every $X \in \mathcal{X}_+$, the function

$$\mathbb{R} \ni t \to E_g[u(w_2+t-X)]-u(w_2)$$

is nondecreasing and $H_{(w_2,u,g)}(X)$ is its unique zero. Hence, (10) is valid. In this way we have proved that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

From Theorem 2 we derive the following characterization of equality in the class of principles of equivalent utility under the Rank-Dependent Utility model.

**Corollary 1.** *Let $w_1, w_2 \in [0,\infty)$. Assume that $g$ is a continuous probability distortion function and $u, v : \mathbb{R} \to \mathbb{R}$ are strictly increasing continuous functions such that $u(0) = v(0) = 0$. Then the following statements are pairwise equivalent:*

$(i)$

$$H_{(w_1,v,g)}(X) = H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}_+^{(2)}; \tag{19}$$

$(ii)$

$$H_{(w_1,v,g)}(X) = H_{(w_2,u,g)}(X) \quad \textit{for} \quad X \in \mathcal{X}_+;$$

$(iii)$ *there exists $c \in (0,\infty)$ such that*

$$u(x) = cv(x+w_1-w_2)+u(w_2)-cv(w_1) \quad \textit{for} \quad x \in \mathbb{R}. \tag{20}$$

*Proof.* Assume that (19) holds. Then, according to Theorem 2, there exist $c, \tilde{c} \in (0,\infty)$ such that (11) is valid and

$$v(x) \leq \tilde{c}u(x+w_2-w_1)+v(w_1)-\tilde{c}u(w_2) \quad \text{for} \quad x \in \mathbb{R}.$$

Hence

$$u(x)-u(w_2) \leq c(v(x+w_1-w_2)-v(w_1)) \leq \tilde{c}c(u(x)-u(w_2)) \quad \text{for} \quad x \in \mathbb{R}.$$

Therefore, since $v$ is strictly increasing, we get $c\tilde{c} = 1$ and so (20) is valid. This proves that $(i) \Rightarrow (iii)$.

If (20) holds then, replacing $x$ by $x+w_2-w_1$, we get

$$v(x) = \frac{1}{c}u(x+w_2-w_1)+v(w_1)-\frac{1}{c}u(w_2) \quad \text{for} \quad x \in \mathbb{R}. \tag{21}$$

Taking into account (20) and (21), from Theorem 2 we derive (19). Thus $(iii) \Rightarrow (ii)$. Obviously, we have also $(ii) \Rightarrow (i)$. □

Applying Corollary 1 we are going to characterize the positive homogeneity of the principle of equivalent utility. Recall that the principle $H_{(w,u,g)}$ is positively homogeneous provided, for every $X \in \mathcal{X}_+$ and $\lambda \in (0, \infty)$, it holds

$$H_{(w,u,g)}(\lambda X) = \lambda H_{(w,u,g)}(X). \tag{22}$$

If (22) holds for every $X \in \mathcal{X}_+^{(2)}$ and $\lambda \in (0, \infty)$, then the principle $H_{(w,u,g)}$ is said to be positively homogeneous on $\mathcal{X}_+^{(2)}$. The positive homogeneity of $H_{(w,u,g)}$ in the case $w = 0$ has been characterized in [7]. It is proved there that if $g$ is convex and $u$ is concave and differentiable then $H_{(0,u,g)}$ is positively homogeneous if and only if $u$ is linear.

**Theorem 3.** *Assume that $w \in [0, \infty)$, $g$ is a continuous probability distortion function and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u(0) = 0$. Then the following statements are pairwise equivalent:*

$(i)$ *$H_{(w,u,g)}$ is positively homogeneous on $\mathcal{X}_+^{(2)}$;*

$(ii)$ *$H_{(w,u,g)}$ is positively homogeneous;*

$(iii)$ *there exist $a, b, r \in (0, \infty)$ and $\gamma \in \mathbb{R}$ such that*

$$u(x) = \begin{cases} -a(w-x)^r + \gamma & \text{for } x \in (-\infty, w], \\ b(x-w)^r + \gamma & \text{for } x \in (w, \infty). \end{cases} \tag{23}$$

*Proof.* Assume that $(i)$ holds. For every $t \in (0, \infty)$, define $u_t : \mathbb{R} \to \mathbb{R}$ as follows

$$u_t(x) = u(w + tx) - u(w) \quad \text{for} \quad x \in \mathbb{R}. \tag{24}$$

Then, taking into account (3) and (5), for every $X \in \mathcal{X}_+^{(2)}$ and $t \in (0, \infty)$, we get

$$\begin{aligned} E_g[u_t(H_{(w,u,g)}(X) - X)] &= E_g[u(w + tH_{(w,u,g)}(X) - tX)] - u(w) \\ &= E_g[u(w + H_{(w,u,g)}(tX) - tX)] - u(w) = 0 = u_t(0) = E_g[u_t(H_{(0,u_t,g)}(X) - X)]. \end{aligned}$$

Therefore,

$$H_{(0,u_t,g)}(X) = H_{(w,u,g)}(X) \quad \text{for} \quad X \in \mathcal{X}_+^{(2)},\ t \in (0, \infty)$$

and so, applying Corollary 1 with $w_1 = 0$, $w_2 = w$ and $v = u_t$, we conclude that for every $t \in (0, \infty)$ there exists $c(t) \in (0, \infty)$ such that

$$u(x) = c(t)u_t(x - w) + u(w) \quad \text{for} \quad x \in \mathbb{R}.$$

Hence, replacing $x$ by $x + w$, in view of (24), we get

$$u_t(x) = \frac{1}{c(t)} u_1(x) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0, \infty).$$

Moreover, it follows from (24) that

$$u_t(x) = u_1(tx) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0, \infty).$$

Thus, we have

$$u_1(tx) = \frac{1}{c(t)} u_1(x) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0, \infty). \tag{25}$$

Since $u_1(x) > 0$ for $x \in (0, \infty)$, applying (25) with $x = 1$, we obtain

$$c(t) = \frac{u_1(1)}{u_1(t)} \quad \text{for} \quad t \in (0, \infty).$$

Hence (25) becomes

$$\bar{u}(tx) = \bar{u}(t)\bar{u}(x) \quad \text{for} \quad x \in \mathbb{R},\ t \in (0, \infty), \tag{26}$$

where $\bar{u} : \mathbb{R} \to \mathbb{R}$ is given by

$$\bar{u}(x) = \frac{u_1(x)}{u_1(1)} \quad \text{for} \quad x \in \mathbb{R}. \tag{27}$$

Note that as $u$ is strictly increasing and continuous, so is $\bar{u}$. Moreover, it follows from (26) that

$$\bar{u}(tx) = \bar{u}(t)\bar{u}(x) \quad \text{for} \quad x, t \in (0, \infty).$$

Thus, according to [9, Theorem 13.3.8], there exist $\beta, r \in (0, \infty)$ such that

$$\bar{u}(x) = \beta x^r \quad \text{for} \quad x \in (0, \infty).$$

Furthermore, replacing in (26) $x$ and $t$ by $-1$ and $-x$, respectively, we get

$$\bar{u}(x) = \bar{u}(-1)\bar{u}(-x) \quad \text{for} \quad x \in (-\infty, 0).$$

Therefore, as $u(0) = 0$ and, in view of (24),

$$u(x) = u_1(x - w) + u(w) \quad \text{for} \quad x \in \mathbb{R},$$

taking into account (27), we obtain (23) with $a := -\beta u_1(-1) > 0$, $b := \beta u_1(1) > 0$ and $\gamma := u(w)$. In this way we have proved that $(i) \Rightarrow (iii)$.

If $u$ is of the form (23) with some $a, b, r \in (0, \infty)$ and $\gamma \in \mathbb{R}$ then, for every $x \in \mathbb{R}$ and $\lambda \in (0, \infty)$, we have

$$u(w + \lambda x) = \lambda^r u(w + x) + (1 - \lambda^r)\gamma = \lambda^r u(w + x) + (1 - \lambda^r)u(w).$$

Thus, as the Choquet integral is positively homogeneous, in view of (3) and (5), for every $X \in \mathcal{X}_+$ and $\lambda \in (0, \infty)$, we obtain

$$\begin{aligned} E_g[u(w + \lambda H_{(w,u,g)}(X) - \lambda X)] &= \lambda^r E_g[u(w + H_{(w,u,g)}(X) - X)] + (1 - \lambda^r)u(w) \\ &= \lambda^r u(w) + (1 - \lambda^r)u(w) = u(w) = E_g[u(w + H_{(w,u,g)}(\lambda X) - \lambda X)]. \end{aligned}$$

Hence

$$H_{(w,u,g)}(\lambda X) = \lambda H_{(w,u,g)}(X) \quad \text{for} \quad X \in \mathcal{X}_+,\ \lambda \in (0, \infty).$$

This means that $H_{(w,u,g)}$ is positively homogeneous and shows that $(iii) \Rightarrow (ii)$.

The implication $(ii) \Rightarrow (i)$ is obvious. □

**Corollary 2.** *Assume that $w \in [0, \infty)$, $g$ is a continuous probability distortion function and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing continuous function with $u(0) = 0$. Then the following statements are pairwise equivalent:*

*(i)*
$$H_{(w,u,g)}(X) \geq E_{\bar{g}}[X] \quad \textit{for} \quad X \in \mathcal{X}_+^{(2)};$$

*(ii)*
$$H_{(w,u,g)}(X) \geq E_{\bar{g}}[X] \quad \textit{for} \quad X \in \mathcal{X}_+; \tag{28}$$

*(iii) there exists* $c \in (0, \infty)$ *such that*
$$u(x) \leq c(x - w) + u(w) \quad \textit{for} \quad x \in \mathbb{R}. \tag{29}$$

*Proof.* Let $v$ be the identity on $\mathbb{R}$. Then, taking into account (3) and (5)-(6), for every $X \in \mathcal{X}_+$, we get
$$\begin{aligned} w = v(w) =& E_g[v(w + H_{(w,v,g)}(X) - X)] \\ =& E_g[w + H_{(w,v,g)}(X) - X] = w + H_{(w,v,g)}(X) - E_{\bar{g}}[X] \end{aligned}$$
which implies that
$$H_{(w,v,g)}(X) = E_{\bar{g}}[X] \quad \text{for} \quad X \in \mathcal{X}_+.$$
Therefore, applying Theorem 2, we get the assertion. □

The next result concerns the risk loading property of the principle of equivalent utility under the Rank-Dependent Utility model. Let us recall that the principle $H_{(w,u,g)}$ has the risk loading property, provided
$$H_{(w,u,g)}(X) \geq E[X] \quad \text{for} \quad X \in \mathcal{X}_+. \tag{30}$$

**Corollary 3.** *Assume that* $w \in [0, \infty)$ *and* $u : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing continuous function with* $u(0) = 0$. *Let* $g$ *be a continuous probability distortion function such that*
$$g(p) \geq p \quad \textit{for} \quad p \in [0, 1]. \tag{31}$$
*If the premium principle* $H_{(w,u,g)}$ *has the risk loading property, then there exists* $c \in (0, \infty)$ *such that* (29) *holds.*

*Proof.* It follows from (7) and (31) that $\bar{g}(p) \leq p$ for $p \in [0, 1]$. Therefore, if $H_{(w,u,g)}$ has the risk loading property then, applying Remark 2, we get (28). Hence, according to Corollary 2, (29) is valid with some $c \in (0, \infty)$. □

**Remark 7.** *Suppose that* $g(p) \leq p$ *for* $p \in [0, 1]$. *Then* $\bar{g}(p) \geq p$ *for* $p \in [0, 1]$ *and so, according to Remark 2, we have*
$$E[X] \leq E_{\bar{g}}[X] \quad \textit{for} \quad X \in \mathcal{X}_+.$$
*Hence, if* (29) *is valid, then using a monotonicity of the Choquet integral, in view of* (3) *and* (5)-(6), *for every* $X \in \mathcal{X}_+$, *we get*
$$E_g[u(w + E[X] - X)] \leq E_g[u(w + E_{\bar{g}}[X] - X)] \leq c(E_g[E_{\bar{g}}[X] - X]) + u(w) = u(w).$$
*Therefore, applying Remark 3, we conclude that* (30) *holds, that is* $H_{(w,u,g)}$ *has the risk loading property.*

We complete the paper with a result which is a direct consequence of Corollary 3 and Remark 7. In fact, it is a slight generalization of [3, Theorem 7].

**Corollary 4.** *Assume that* $w \in [0, \infty)$, $g$ *is the identity on* $[0, 1]$ *and* $u : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing continuous function with* $u(0) = 0$. *Then the premium principle* $H_{(w,u,g)}$ *has the risk loading property if and only if there exists* $c \in (0, \infty)$ *such that* (29) *holds.*

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Принцип страхової винагороди є способом поставити у відповідність кожному ризику, зображеному за допомогою невід'ємної обмеженої випадкової величини на заданому ймовірнісному просторі, деяке дійсне невід'ємне число. Таке число можна інтерпретувати як винагороду за страховий ризик. У цій статті досліджено неявно заданий принцип еквівалентної корисності. Використовуючи властивості середнього квазівідхилення ми характеризуємо порівняння в класі принципів еквівалентної корисності за ранг-залежною корисністю, однією з важливих поведінкових моделей прийняття рішення в умовах ризику. Ми використовуємо цей результат для встановлення рівності і додатної однорідності цього принципу. Також висвітлено деякі інші застосування.

*Ключові слова і фрази:* страхова винагорода, середнє квазівідхилення, порівняння, рівність, додатна однорідність, ризик.