ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 250–257
doi:10.15330/cmp.11.2.250-257

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.250–257

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LEIBNIZ ALGEBRAS: A BRIEF REVIEW OF CURRENT RESULTS

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a left Leibniz algebra if it satisfies the left Leibniz identity $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all $a,b,c \in L$. This paper is a brief review of some current results, which related to finite- dimensional and infinite-dimensional Leibniz algebras

Key words and phrases: Leibniz algebra, cyclic Leibniz algebra, ideal, subideal, contraideal, center, lower (upper) central series, finite-dimensional Leibniz algebra, nilpotent Leibniz algebra, Leibniz T-algebra, anticenter, antinilpotent Leibniz algebra.

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To Professor L.A. Kurdachenko on the occasion of his 70th birthday

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a,b,c \in L$.

Leibniz algebras appeared first in the papers of A.M. Bloh [5–7], in which he called them the *D-algebras*. However, a real interest to Leibniz algebras rose after the paper of J.-L. Loday [25] (see also [26, Section 10.6]), who rediscovered these algebras and used the term *Leibniz algebras* since it was G.W. Leibniz who discovered and proved the Leibniz rule for differentiation of functions.

Note that the Leibniz algebras have many connections with some areas of mathematics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, non- commutative geometry, and physics (see, for example, [8,12,13]).

The theory of Leibniz algebras has been developing intensively but very sporadic. On the one hand, many analogues of important results from the theory of Lie algebras were proven (see, for example, a survey [18]). On the other hand, many natural questions about the structure of Leibniz algebras are not considered. For example, we can note the situation about the structure of finite-dimensional Leibniz algebras. The first natural step in the study of all types of algebras is the description of algebras having small dimensions. Unlike the simpler cases of 1- and 2-dimensional Leibniz algebras, the structure of 3- dimensional Leibniz algebras is more complex, as well as it is more complex than the structure of 3- dimensional Lie algebras. The study of Leibniz algebras, having dimension 3, has been conducted in the papers [1,2,9,11]

УДК 512.554
2010 *Mathematics Subject Classification:* 17A32, 17A60.

for the fields of characteristic 0, moreover for the field $\mathbb{C}$ of complex numbers or algebraically closed fields of characteristic 0. In [33], Yashchuk V.S. considered the opposite situation. She described the structure of Leibniz algebras of dimension 3 over finite fields. In some cases, the structure of such algebras essentially depends on the characteristic of the field. In other words, it depends on the solvability of specific equations in the fields, and so on. It is also worth mentioning here that the description of Leibniz algebras of dimension 3 is very different from the description of Lie algebras of dimension 3, which indicates a significant difference between these types of algebras.

Note that if L is a Lie algebra, then $[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$. It follows that

$$\begin{aligned}[[a,b],c] &= -[[b,c],a] - [[c,a],b] \\ &= [a,[b,c]] + [b,[c,a]] \\ &= [a,[b,c]] - [b,[a,c]],\end{aligned}$$

which shows that every Lie algebra is a Leibniz algebra.

Conversely, suppose that $[a,a] = 0$ for all elements $a \in L$. Then for arbitrary elements $a, b \in L$ we have $0 = [a+b, a+b] = [a,a] + [a,b] + [b,a] + [b,b] = [a,b] + [b,a]$. It follows that $[a,b] = -[b,a]$. Then

$$\begin{aligned}0 &= [[a,b],c] - [a,[b,c]] + [b,[a,c]] \\ &= [[a,b],c] + [[b,c],a] - [[a,c],b] \\ &= [[a,b],c] + [[b,c],a] + [[c,a],b]\end{aligned}$$

for all $a, b, c \in L$. In other words, Lie algebras can be characterized as Leibniz algebras in which $[a,a] = 0$ for every element $a \in L$.

Recall some basic definitions.

A Leibniz algebra L is called *abelian* if $[a,b] = 0$ for all elements $a, b \in L$. Thus, an abelian Leibniz algebra is a Lie algebra.

Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by all elements $[a,b]$, where $a \in A$, $b \in B$. A subspace A of L is called a *subalgebra* of L, if $[x,y] \in A$ for every $x, y \in A$. It follows that $[A, A] \leqslant A$.

Let L be a Leibniz algebra over a field F, M be a non-empty subset of L, then $\langle M \rangle$ denote the subalgebra of L generated by M.

A subalgebra A of L is called a *left* (respectively *right*) *ideal* of L, if $[y,x] \in A$ (respectively $[x,y] \in A$) for every $x \in A$, $y \in L$. In other words, if A is a left (respectively right) ideal, then $[L, A] \leqslant A$ (respectively $[A, L] \leqslant A$).

A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal, that is $[y,x], [x,y] \in A$ for every $x \in A$, $y \in L$.

If A is an ideal of L, we can consider a *factor-algebra* L/A. It is not hard to see that this factor-algebra also is a Leibniz algebra.

Denote by $\mathbf{Leib}(L)$ the subspace, generated by the elements $[a, a]$, $a \in L$. Note that $\mathbf{Leib}(L)$ is an ideal of L, which is called the *Leibniz kernel* of algebra L.

The *left* (respectively *right*) *center* $\zeta^{left}(L)$ (respectively $\zeta^{right}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{left}(L) = \{x \in L | \ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{right}(L) = \{x \in L | \ [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. The right center is a subalgebra of L, and in general, the left and right centers are different. Moreover, they even may have different dimensions as shows an example 2.1 from [19].

The *center* $\zeta(L)$ of L is the intersection of the left and right centers, that is

$$\zeta(L) = \{x \in L | \ [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

Clearly, the center $\zeta(L)$ is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

Now we define the *upper central series*

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \ldots \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \bigcup\limits_{\mu<\lambda} \zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L. A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter.

Let L be a Leibniz algebra. Define the *lower central series*

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \gamma_\delta(L)$$

of L by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap\limits_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. For the last term $\gamma_\delta(L)$ we have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

The introduced here concepts of the upper and lower central series for Leibniz algebras are an analogous of others similar concepts, which became standard in several algebraic structures. They play an important role, for example, in Lie algebras and groups. Following this analogy, we say that a Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$.

We note that in [22] Kurdachenko L.A., Subbotin I.Ya. and Semko N.N. proved a series of results, which connected with (locally) nilpotent and hypercentral Leibniz algebras. In particular, these results are analogues of well-known group-theoretical results.

It is a well-known that in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length. The same result is also true for Leibniz algebras (see, for example, [19]).

Let L be a Leibniz algebra. Let us define the *lower derived series*

$$L = \delta_0(L) \geqslant \delta_1(L) \geqslant \ldots \delta_\alpha(L) \geqslant \delta_{\alpha+1}(L) \geqslant \ldots \delta_\nu(L)$$

of L by the following rule: $\delta_0(L) = L$, $\delta_1(L) = [L, L]$, and recursively $\delta_{\alpha+1}(L) = [\delta_\alpha(L), \delta_\alpha(L)]$ for all ordinals α and $\delta_\lambda(L) = \bigcap_{\mu<\lambda} \delta_\mu(L)$ for the limit ordinals λ. For the last term $\delta_\nu(L)$ we have $\delta_\nu(L) = [\delta_\nu(L), \delta_\nu(L)]$. If $\delta_n(L) = \langle 0 \rangle$ for some positive integer n, then we say that L is a *soluble* Leibniz algebra.

One of the first questions that naturally arises in the study of any algebraic structure is the question of the structure of its cyclic substructures. Unlike Lie algebras, associative algebras, groups, etc., cyclic Leibniz algebras are no necessarily abelian. In [10, Theorem 1.1] Chupordia V.A., Kurdachenko L.A. and Subbotin I.Ya. described the structure of such Leibniz algebras. This description made it possible to obtain a structure of the Leibniz algebras, whose proper subalgebras are Lie algebras. Such algebras are either Lie algebras, or nilpotent cyclic algebras, or they can be represented as a direct sum of an abelian ideal (from the left center of algebra) and Lie subalgebra of dimension 1 with some additional properties [10, Theorem 1.2]. As a corollary it was described Leibniz algebras whose proper subalgebras are abelian [10, Corollary 1.1]. This result implies that a description of Leibniz algebras, whose proper subalgebras are abelian, can be deduced to the case of Lie algebras, whose proper subalgebras are abelian. Such Lie algebras are either simple, or soluble. Soluble minimal non-abelian Lie algebras (even soluble minimal non-nilpotent Lie algebras) were described in [16, 30, 31]. Simple minimal non-abelian Lie algebras were studied in [14, 15], but their complete description remains an open problem.

Another natural question concerns the relationship of the subalgebras and ideals. In particular, what is a structure of Leibniz algebras, all of whose subalgebras are ideals? It is not hard to prove that a Lie algebra, all of whose subalgebras are ideals, is abelian. For groups the situation is different: there exists non-abelian groups, all of whose subgroups are normal. Such groups have been described in [3]. For Leibniz algebras the situation is quite diverse. Recall that a Leibniz algebra L is called an *extraspecial* algebra if it satisfies the following condition: $\zeta(L)$ is non-trivial and has dimension 1, and $L/\zeta(L)$ is abelian. It is important to observe that there are extraspecial Leibniz algebras in which not every subalgebra is an ideal. In [20] Kurdachenko L.A., Semko N.N. and Subbotin I.Ya. proved that if L is a Leibniz algebra over a field F, all of whose subalgebras are ideals and L is non-abelian, then $L = E \oplus Z$ where $Z \leqslant \zeta(L)$, and E is an extraspecial subalgebra such that $[a, a] \neq 0$ for every element $a \notin \zeta(E)$.

Consider now some other natural questions of the general theory of Leibniz algebras. Note that the relation "to be a subalgebra of a Leibniz algebra" is transitive. However, the relation "to be an ideal" is not transitive even for Lie algebras. Therefore it is natural to ask the question about the structure of Leibniz algebras, in which the relation "to be an ideal" is transitive. In this context, the following important type of subalgebras naturally arises. A subalgebra A of a Leibniz algebra L is called a *left* (respectively *right*) *subideal* of L, if there is a finite series of subalgebras $A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$ such that A_{j-1} is a left (respectively, right) ideal of A_j, $1 \leqslant j \leqslant n$.

Similarly, a subalgebra A of a Leibniz algebra L is called a *subideal* of L, if there is a finite series of subalgebras $A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$ such that A_{j-1} is an ideal of A_j, $1 \leqslant j \leqslant n$.

We note the following property of nilpotent Leibniz algebras (see, for example [18]): if L is a nilpotent Leibniz algebra over a field F, then every subalgebra of L is a subideal of L.

A Leibniz algebra L is called a *T-algebra*, if a relation "to be an ideal" is transitive. In other words, if A is an ideal of L and B is an ideal of A, then B is an ideal of L. It follows that in a Leibniz T-algebra every subideal is an ideal. Lie algebras, in which a relation "to be an ideal" is transitive have been studied by I. Stewart [28] and A.G. Gejn and Yu.N. Muhin [17]. In particular, soluble T-algebras and finite dimensional T-algebras over a field of characteristic 0 has been described. As in the mentioned above cases, the situation in Leibniz algebras is much more complex and diverse than it was in Lie algebras (see, for examples [18]). The description of Leibniz T-algebras has been obtained by Kurdachenko L.A., Subbotin I.Ya. and Yashchuk V.S. in the paper [24].

Consider now some new approach in Leibniz algebra theory. Two ideals are naturally associated with each subalgebra A of a Leibniz algebra L: the ideal A^L which is the intersection of all ideals including A (that is an ideal, generated by A); and the ideal $\mathbf{Core}_L(A)$ which is the sum of all ideals that are contained in A. A subalgebra A of L is called an *contraideal* of L, if $A^L = L$. From the definition it follows that the contraideals are natural antipodes to the concepts of ideals. Therefore, the study of Leibniz algebras whose subalgebras are either ideals or contraideals is very natural. The description of such Leibniz algebras was obtained by Kurdachenko L.A., Subbotin I.Ya. and Yashchuk V.S. in the paper [23]. As a corollary, the authors obtained the structure of Lie algebras, whose subalgebras are either ideals or contraideals [23].

As we noted above, the fact that $\gamma_{c+1}(L) = \langle 0 \rangle$ is equivalent to the fact that $\zeta_c(L) = L$, i.e. the lower and the upper central series in nilpotent Leibniz algebras have the same length. The next natural step is the consideration of the case, when the upper (respectively lower) central series has finite length. For this case the question about the relationships between $L/\zeta_k(L)$ and $\gamma_{k+1}(L)$ naturally appears.

If L is a Lie algebra such that $L/\zeta_k(L)$ is finite-dimensional, then $\gamma_{k+1}(L)$ is also finite-dimensional [29]. A corresponding result for groups has been obtained early by R. Baer [4]. Kurdachenko L.A., Otal J. and Pypka A.A. in the paper [19] obtained the following analog of these theorems: if L is a Leibniz algebra over a field F and $\mathbf{codim}_F(\zeta_k(L)) = d$ is finite for some positive integer k, then $\gamma_{k+1}(L)$ has finite dimension; moreover $\mathbf{dim}_F(\gamma_{k+1}(L)) \leqslant 2^{k-1}d^{k+1}$.

An important specific case here is the case when the center of a Leibniz algebra L has finite codimension. For Lie algebras the following result is well known (see, for example [32]). A corresponding result for groups was proved much earlier: if G is a group and C is a subgroup of the center $\zeta(G)$ such that G/C is finite, then the derived subgroup $[G, G]$ is finite. In this formulation, for the first time it appears in the paper of B.H. Neumann [27]. This theorem was obtained also by R. Baer [4].

For Leibniz algebras the following analog of these results was proved by Kurdachenko L.A., Otal J. and Pypka A.A. in [19]: *if L is a Leibniz algebra over a field F, $\mathbf{codim}_F(\zeta^{left}(L)) = d$ and $\mathbf{codim}_F(\zeta^{right}(L)) = r$ are finite, then $[L, L]$ has finite dimension; moreover, $\mathbf{dim}_F([L, L]) \leqslant d(d + r)$.*

In this connection, the following question appears: suppose that only $\mathbf{codim}_F(\zeta^{left}(L))$ is finite. Is $\mathbf{dim}_F([L, L])$ finite? The Example 3.1 from [19] gives a negative answer on this question. However, if L is a Leibniz algebra over a field F and $\mathbf{codim}_F(\zeta(L)) = d$ is finite, then $[L, L]$ has finite dimension; in particular, $\mathbf{dim}_F([L, L]) \leqslant d^2$ [19]. Moreover, if L is a Leibniz algebra over a field F and $\mathbf{codim}_F(\zeta(L)) = d$ is finite, then the Leibniz kernel of L has finite dimension at most $\frac{1}{2}d(d-1)$ [19].

Finally, we note that in [21] Kurdachenko L.A., Semko N.N. and Subbotin I.Ya. introduced the concepts of anticenter of Leibniz algebras and antinilpotent Leibniz algebras. Let L be a Leibniz algebra. Put

$$\alpha(L) = \{z \in L | [a, z] = -[z, a] \text{ for each element } a \in L\}.$$

This subset is called the *anticenter* of a Leibniz algebra L. Note that the anticenter is an ideal of L. Note also that we must consider the case, when $char(F) \neq 2$, because in the case when $char(F) = 2$ anticenter in general is not ideal [21].

For this concept the above authors proved some analogs of result from Leibniz algebra theory. In particular, in [21] they proved that if L is a Leibniz algebra over a field F and the anticenter of L has finite codimension d, then the Leibniz kernel of L has finite dimension at most d^2.

Starting from the anticenter, we define the *upper anticentral series*

$$\langle 0 \rangle = \alpha_0(L) \leqslant \alpha_1(L) \leqslant \ldots \alpha_\lambda(L) \leqslant \alpha_{\lambda+1}(L) \leqslant \ldots \alpha_\gamma(L) = \alpha_\infty(L)$$

of a Leibniz algebra L by the following rule: $\alpha_1(L) = \alpha(L)$ is the anticenter of L, and recursively, $\alpha_{\lambda+1}(L)/\alpha_\lambda(L) = \alpha(L/\alpha_\lambda(L))$ for all ordinals λ,and $\alpha_\mu(L) = \bigcup\limits_{\nu<\mu} \alpha_\nu(L)$ for the limit ordinals μ. By definition, each term of this series is an ideal of L. The last term $\alpha_\infty(L)$ of this series is called the *upper hyperanticenter* of L. A Leibniz algebra L is said to be *hyperanticentral* if it coincides with the upper hyperanticenter. Denote by $al(L)$ the length of upper anticentral series of L. If L is hyperanticentral and $al(L)$ is finite, then L is said to be *antinilpotent*.

If U, V the ideals of L, then we denote by (U, V) a subspace, generated by all elements $[u, v] + [v, u]$, $u \in U, v \in V$. Note that $[u, v] + [v, u] \in \zeta^{left}(L)$ and (U, V) is an ideal of L [21]. Define the *lower anticentral series* of L

$$L = \kappa_1(L) \geqslant \kappa_2(L) \geqslant \ldots \kappa_\alpha(L) \geqslant \kappa_{\alpha+1}(L) \geqslant \ldots \kappa_\delta(L)$$

by the following rule: $\kappa_1(L) = L$, $\kappa_2(L) = (L, L)$, and recursively $\kappa_{\lambda+1}(L) = (L, \kappa_\lambda(L))$ for all ordinals λ and $\kappa_\mu(L) = \bigcap\limits_{\nu<\mu} \kappa_\nu(L)$ for the limit ordinals μ. For the last term $\kappa_\delta(L)$ we have $\kappa_\delta(L) = (L, \kappa_\delta(L))$.

As we noted above in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length. For antinilpotent Leibniz algebras Kurdachenko L.A., Semko N.N. and Subbotin I.Ya. [21] proved the analog of this statement: if L is an antinilpotent Leibniz algebra, then the length of the lower anticentral series coincides with the length of the upper anticentral series; moreover, the length of these two series is the smallest among the lengths of all anticentral series of L.

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Received 11.09.2019

Чупордя В.А., Пипка О.О., Семко М.М., Ящук В.С. *Алгебри Лейбніца: короткий огляд сучасних результатів* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 250–257.

Нехай L – алгебра над полем F з двома бінарними операціями $+$ та $[\cdot,\cdot]$. Тоді L називатимемо лівою алгеброю Лейбніца, якщо вона задовольняє ліву тотожність Лейбніца $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ для всіх $a,b,c \in L$. Дана стаття є коротким оглядом деяких сучасних результатів, пов'язаних зі скінченновимірними та нескінченновимірними алгебрами Лейбніца.

Ключові слова і фрази: алгебра Лейбніца, циклічна алгебра Лейбніца, ідеал, субідеал, контраідеал, центр, верхній (нижній) центральний ряд, скінченновимірна алгебра Лейбніца, нільпотентна алгебра Лейбніца, T-алгебра Лейбніца, антицентр, антинільпотентна алгебра Лейбніца.