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ALGEBRAS GENERATED BY SPECIAL SYMMETRIC POLYNOMIALS ON ℓ_1

Let X be a weighted direct sum of infinity many copies of complex spaces $\ell_1 \bigoplus \ell_1$. We consider an algebra consisting of polynomials on X which are supersymmetric on each term $\ell_1 \bigoplus \ell_1$. Point evaluation functionals on such algebra gives us a relation of equivalence '$\sim$' on X. We investigate the quotient set $X/\sim$ and show that under some conditions, it has a real topological algebra structure.

Key words and phrases: symmetric and supersymmetric polynomials on Banach spaces, algebras of analytic functions on Banach spaces, spectra algebras of analytic functions.

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INTRODUCTION AND PRELIMINARIES

Let X be a complex Banach space and (P_α) a family of continuous complex valued polynomials on X. Often, it is interesting to consider algebras of analytic functions on X, generated by the family of polynomials (see e. g. [6, 12, 16]). If the family (P_α) does not separate points of X, then the same is true for any function, generated by (P_α). So, we have a natural relation of equivalence on X: $z \sim w$ if and only if $P_\alpha(z) = P_\alpha(w)$ for every α. If X is finite-dimensional, then from the Algebraic Geometry is well known that the quotient set $X/\sim$ is dens in an algebraic variety. The same is true for infinite-dimensional case, if the family (P_α) is finite [2]. But in the general case, the situation may be more complicated.

Let S be the group of all permutations on the set of natural numbers $\mathbb{N}$. A polynomial $P\colon \ell_1 \to \mathbb{C}$ is said to be *symmetric* if $P(\sigma(x)) = P(x)$ for every $X \in \ell_1$ and $\sigma \in S$. It is known [15] that polynomials

$$F_k(X) = \sum_{n=1}^{\infty} x_n^k, \quad k = 1, 2, \dots,$$

form an algebraic basis in the algebra of all continuous symmetric polynomials $\mathcal{P}_s(\ell_1)$. In other words, $\{F_k\}_{k=1}^{\infty}$ are algebraically independent and $\mathcal{P}_s(\ell_1)$ is the minimal unital algebra containing $\{F_k\}_{k=1}^{\infty}$. In [1] it was shown that two vectors with finite supports $x, y \in \ell_1$ are equivalent in the means $F_k(x) = F_k(y)$ for every k, if and only if $x = \sigma(y)$ for some $\sigma \in S$. Some algebraic operations on $\ell_1/\sim$ which form a semi-ring structure [4] were considered in [5, 7]. Composition operators, associated with these operations, on analytic functions were investigated in [8]. Algebras of analytic functions generated by symmetric polynomials on ℓ_p were investigated in [1, 3, 5–7, 13, 14].

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Let $X = \ell_1 \bigoplus \ell_1$. We represent each element z of X by $z = (y|x)$, $x, y \in \ell_1$. Let us consider polynomials $T_m \colon X \to \mathbb{C}$,

$$T_m(z) = F_m(x) - F_m(y) = \sum_{k=1}^{\infty} (x_k^m - y_k^m).$$

Polynomials T_m, $m \in \mathbb{N}$ are algebraically independent and form an algebraic basis on the algebra of *supersymmetric* polynomials on X. In [11] the algebra of supersymmetric polynomials was investigated and a commutative ring structure on the corresponding quotient set $X/\sim$ was described.

For a given complex Banach space E with an unconditional basis $\{e_n\}_{n=0}^{\infty}$ we denote by $\ell_1^{(E)}$ a Banach space defined by the following way. If $x \in \ell_1^{(E)}$, then

$$x = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}, \ldots), \tag{1}$$

where each $x^{(n)} = (x_1^{(n)}, \ldots, x_k^{(n)}, \ldots) \in \ell_1$ and

$$\sum_{n=0}^{\infty} \|x^{(n)}\|_{\ell_1} e_n \in E \qquad \text{with} \qquad \|x\|_{\ell_1^{(E)}} = \left\| \sum_{n=0}^{\infty} \|x^{(n)}\|_{\ell_1} e_n \right\|_E .$$

A polynomial P on $\ell_1^{(E)}$ is *separately symmetric* [10] if for every sequence of permutations on $\mathbb{N}$, $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$, $\sigma_n \in S$ we have $P(\sigma(x)) = P(\sigma_0(x^{(0)}), \ldots, \sigma_n(x^{(n)}), \ldots) = P(x)$ for all $x \in \ell_1^{(E)}$. Polynomials

$$F_m^{(j)}(x) = \sum_{k=1}^{\infty} (x_k^{(j)})^m, \quad j \in \mathbb{Z}_+, \quad m \in \mathbb{N}$$

are separately symmetric and algebraically independent.

In this paper we consider a complex Banach space X which is a weighted direct sum of infinity copies of $\ell_1 \bigoplus \ell_1$ and polynomials which are supersymmetric on each term of this sum. We show that under some assumptions, $X/\sim$ is a real locally convex algebra which contains a normed subalgebra. This is an extension of results on supersymmetric polynomials, obtained in [11]. For details about analytic mappings on Banach spaces we refer the reader to [9].

1 THE RING $\mathcal{M}^{\omega}$

Let ω be a positive number, $0 < \omega \leq 1$. We denote by $\ell_{1,\infty}^{\omega}$ a "weighted" version of the space ℓ_1^E. Namely, if $x \in \ell_{1,\infty}^{\omega}$, then

$$x = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}, \ldots), \quad x^{(n)} = (x_k^{(n)}) \in \ell_1$$

and

$$\|x\| = \|x\|_{\ell_{1,\infty}^{\omega}} = \max\left(\sum_{n=1}^{\infty} \omega^n \|x^{(n)}\|_{\ell_1}, \sup_{n,k} |x_k^{(n)}| \right).$$

We denote by Λ_1^{ω} the direct sum of two copies of $\ell_{1,\infty}^{\omega}$, $\Lambda_1^{\omega} = \ell_{1,\infty}^{\omega} \bigoplus \ell_{1,\infty}^{\omega}$. Elements of Λ_1^{ω} will be denoted by $(y|x)$, $y \in \ell_{1,\infty}^{\omega}$, $x \in \ell_{1,\infty}^{\omega}$ and $\|(y|x)\| = \|y\|_{\ell_{1,\infty}^{\omega}} + \|x\|_{\ell_{1,\infty}^{\omega}}$. In other words, any element $z \in \Lambda^\omega$ can be represented as

$$z = (y|x) = \begin{pmatrix} \dots y_k^{(0)} \dots y_1^{(0)} & | & x_1^{(0)} \dots x_k^{(0)} \dots \\ \dots & | & \dots \\ \dots y_k^{(n)} \dots y_1^{(n)} & | & x_1^{(n)} \dots x_k^{(n)} \dots \\ \dots & | & \dots \end{pmatrix}$$

or

$$z = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} x_k^{(n)} e_k^{(n)} + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} y_k^{(n)} e_k^{-(n)}, \tag{2}$$

where

$$x_k^{(n)} e_k^{(n)} = \begin{pmatrix} \dots 0 \dots 0 & | & 0 \dots 0 \dots \\ \dots & | & \dots \\ \dots 0 \dots 0 & | & 0 \dots 0 \, x_k^{(n)} \, 0 \dots \\ \dots 0 \dots 0 & | & 0 \dots 0 \dots \\ \dots & | & \dots \end{pmatrix}$$

and

$$y_k^{(n)} e_k^{-(n)} = \begin{pmatrix} \dots 0 \dots 0 & | & 0 \dots 0 \dots \\ \dots & | & \dots \\ \dots \, 0 \, y_k^{(n)} \, 0 \dots 0 & | & 0 \dots 0 \dots \\ \dots 0 \dots 0 & | & 0 \dots 0 \dots \\ \dots & | & \dots \end{pmatrix}.$$

Note that the expansion (2) is formal, that is, the series on the right is not convergent in general.

We denote by $\Lambda_1^{\omega+}$ and $\Lambda_1^{\omega-}$ subspaces $\{(0|x) \colon x \in \ell_{1,\infty}^{\omega}\}$ and $\{(y|0) \colon y \in \ell_{1,\infty}^{\omega}\}$ respectively. If $z = (y|x)$ we will use also notations $z_+ = x$ and $z_- = y$ when it will be convenient.

Let us define the following polynomials on Λ_1^ω

$$\begin{aligned} T_m^\omega(y|x) &= \sum_{n=0}^{\infty} \omega^n F_m^{(n)}(x^{(n)}) - \sum_{n=0}^{\infty} \omega^n F_m^{(n)}(y^{(n)}) \\ &= \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} (x_k^{(n)})^m - \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} (y_k^{(n)})^m, \qquad (y|x) \in \Lambda_1^\omega. \end{aligned} \tag{3}$$

Proposition 1. *For every $m \in \mathbb{N}$ the polynomial T_m^ω is continuous on Λ_1^ω and $\|T_m\| = 1$.*

Proof. Let $\|(y|x)\| \le 1$. Then $\|y\|_{\ell_1^\omega} + \|x\|_{\ell_1^\omega} \le 1$, and $|x_k^{(n)}| \le 1$ and $|y_k^{(n)}| \le 1$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Thus

$$|T_m^\omega(x)| \le \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} \left(|x_k^{(n)}|^m + |y_k^{(n)}|^m\right) \le \sum_{n=0}^{\infty} \omega^n \sum_{k=1}^{\infty} \left(|x_k^{(n)}| + |y_k^{(n)}|\right) \le \|(y|x)\|.$$

So $\|T_m\| \le 1$. Let now $(y|x)$ be such that $y = 0$, $x^{(0)} = (1, 0, 0, \dots)$, $x^{(n)} = 0$ for $n > 0$. Then $\|(y|x)\| = 1$ and $T_m(y|x) = 1$. Thus $\|T_m\| = 1$. □

Definition 1. *Let us say that a polynomial $P \colon \Lambda_1^\omega \to \mathbb{C}$ is ω-supersymmetric if it is an algebraic combination of polynomials T_m^ω, $m \in \mathbb{N}$. We denote by $\mathcal{P}_s^\omega = \mathcal{P}_s^\omega(\Lambda_1^\omega)$ the algebra of all ω-supersymmetric polynomials on Λ_1^ω.*

Theorem 1. *Let $\omega = 1/N$ for some $N \in \mathbb{N}$, $N > 1$. For every number $a \in \mathbb{R}$ there exists $z_{\{a\}} \in \Lambda_1^\omega$ such that*

$$\|z_{\{a\}}\| = \begin{cases} |a| & \text{if } |a| \geq 1 \\ 1 & \text{if } |a| < 1 \end{cases}$$

and $T_m^\omega(z_{\{a\}}) = a$ for every $m \in \mathbb{N}$.

Proof. Let $a > 0$. Then we can write

$$a = \sum_{j=0}^{\infty} \frac{a_j}{N^j}, \quad a_j \in \mathbb{N}, \tag{4}$$

that is, $a_0 = [a]$ the integer part of a and $(0.a_1a_2\ldots)_N$ is the representation of $a - [a]$ in the positional base N numeral system. Let $z_{\{a\}}$ be of the form $z_{\{a\}} = (0|x_{\{a\}})$, where

$$x_{\{a\}} = \sum_{n=0}^{\infty} x_{\{a\}}^{(n)}$$

and

$$x_{\{a\}}^{(n)} = (\underbrace{1,\ldots,1}_{a_n}, 0, 0, \ldots) = e_1^{(n)} + e_2^{(n)} + \cdots + e_{a_n}^{(n)}, \quad n = 0, 1, 2, \ldots.$$

Then for $|a| \geq 1$,

$$\|z_{\{a\}}\| = \max\left(\sum_{n=0}^{\infty} \frac{a_n}{N^n}, 1\right) = \sum_{n=0}^{\infty} \frac{a_n}{N^n} = T_m^\omega(z_{\{a\}}) = a, \quad m \in \mathbb{N}$$

and $\|z_{\{a\}}\| = 1$ for $|a| < 1$. If $a < 0$ we can consider $b = -a > 0$. By the same way, using (4) for b, we can find the vector $x_{\{b\}}$. Let us define now $z_{\{a\}} = (x_{\{b\}}|0)$. Then

$$\|z_{\{a\}}\| = \begin{cases} \mu = |a| & \text{if } |a| \geq 1, \\ 1 & \text{if } |a| < 1, \end{cases}$$

and $T_m^\omega(z_{\{a\}}) = a$ for every $m \in \mathbb{N}$. □

Let us recall that two operations on ℓ_1 "$\bullet$" and "$\diamond$" which preserve symmetric polynomials were introduced in [7] and [5]. Namely, let $x = (x_1, x_2, \ldots, x_k, \ldots)$ and $x = (y_1, y_2, \ldots, y_k, \ldots)$ are in ℓ_1, then

$$x \bullet y = (x_1, y_1, x_2, y_2, \ldots, x_k, y_k, \ldots)$$

and $x \diamond y$ is the resulting sequence of ordering the set $\{x_iy_j : i, j \in \mathbb{N}\}$ with one single index in some fixed order. It is easy to check that for every symmetric polynomial P on ℓ_1 and fixed $y \in \ell_1$, polynomials $P(x \bullet y)$ and $P(x \diamond y)$ are symmetric. In [11] these operations were extended to $\ell_1 \bigoplus \ell_1$ with preserving supersymmetric polynomials. Now we propose natural extensions of these operations to Λ_1^ω.

Definition 2. *Let $z = (z_-|z_+)$ and $r = (r_-|r_+)$ are in Λ_1^ω. We say that $h = z \bullet r$ if $h_-^{(n)} = z_-^{(n)} \bullet r_-^{(n)}$ and $h_+^{(n)} = z_+^{(n)} \bullet r_+^{(n)}$ for every $n \in \mathbb{Z}_+$. We also say that $s = z \diamond r$ if*

$$s_+^{(n)} = (z_+^{(0)} \diamond r_+^{(n)}) \bullet (z_+^{(1)} \diamond r_+^{(n-1)}) \bullet \cdots \bullet (z_+^{(n)} \diamond r_+^{(0)}) \bullet (z_-^{(0)} \diamond r_-^{(n)}) \bullet (z_-^{(1)} \diamond r_-^{(n-1)}) \bullet \cdots \bullet (z_-^{(n)} \diamond r_-^{(0)})$$

and

$$s_-^{(n)} = (z_+^{(0)} \diamond r_-^{(n)}) \bullet (z_+^{(1)} \diamond r_-^{(n-1)}) \bullet \cdots \bullet (z_+^{(n)} \diamond r_-^{(0)}) \bullet (z_-^{(0)} \diamond r_+^{(n)}) \bullet (z_-^{(1)} \diamond r_+^{(n-1)}) \bullet \cdots \bullet (z_-^{(n)} \diamond r_+^{(0)}).$$

Proposition 2. *$T_m^\omega(z \bullet r) = T_m^\omega(z) + T_m^\omega(r)$ and $T_m^\omega(z \diamond r) = T_m^\omega(z)T_m^\omega(r)$ for all $z, r \in \Lambda_1^\omega$ and $m \in \mathbb{N}$.*

Proof. The first equality directly follows from the definition of T_m^ω (3). Also, in [5] it is proved that $F_m(x \diamond y) = F_m(x)F_m(y)$, $x, y \in \ell_1$, $m \in \mathbb{N}$. So, using (3) and Definition 2, we have for $s = z \diamond r$

$$\begin{aligned}
T_m^\omega(s) = T_m^\omega(z \diamond r) &= \sum_{n=0}^\infty \omega^n F_m^{(n)}(s_+^{(n)}) - \sum_{n=0}^\infty \omega^n F_m^{(n)}(s_-^{(n)}) \\
&= \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(n)}(z_+^{(j)} \diamond r_+^{(n-j)}) + \sum_{j=0}^n F_m^{(n)}(z_-^{(j)} \diamond r_-^{(n-j)}) \right) \\
&\quad - \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(n)}(z_+^{(j)} \diamond r_-^{(n-j)}) + \sum_{j=0}^n F_m^{(n)}(z_-^{(j)} \diamond r_+^{(n-j)}) \right) \\
&= \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(j)}(z_+^{(j)}) F_m^{(n-j)}(r_+^{(n-j)}) + \sum_{j=0}^n F_m^{(j)}(z_-^{(j)}) F_m^{(n-j)}(r_-^{(n-j)}) \right) \\
&\quad - \sum_{n=0}^\infty \omega^n \left(\sum_{j=0}^n F_m^{(j)}(z_+^{(j)}) F_m^{(n-j)}(r_-^{(n-j)}) + \sum_{j=0}^n F_m^{(j)}(z_-^{(j)}) F_m^{(n-j)}(r_+^{(n-j)}) \right) \\
&= \left(\sum_{n=0}^\infty \omega^n F_m^{(n)}(z_+^{(n)}) - \sum_{n=0}^\infty \omega^n F_m^{(n)}(z_-^{(n)}) \right) \left(\sum_{n=0}^\infty \omega^n F_m^{(n)}(r_+^{(n)}) - \sum_{n=0}^\infty \omega^n F_m^{(n)}(r_-^{(n)}) \right) \\
&= T_m^\omega(z) T_m^\omega(r).
\end{aligned}$$

□

Corollary 1. *Let $P(z) \in \mathcal{P}_s^\omega$. Then, for every fixed $r \in \Lambda_1^\omega$ polynomials $P(z \bullet r)$ and $P(z \diamond r)$ are in $\mathcal{P}_s^\omega$.*

For a given $z = (y|x) \in \Lambda_1^\omega$ we denote $z^- = (x|y)$. Clearly, the map $z \mapsto z^-$ is a continuous involution in $r \in \Lambda_1^\omega$ and $T_m^\omega(z^-) = -T_m^\omega(z)$.

Let us introduce the following relation of equivalence on Λ_1^ω. We say that $z \sim r$ if and only if $T_m^\omega(z) = T_m^\omega(r)$ for every $m \in \mathbb{N}$. Let us denote by $\mathcal{M}^\omega$ the quotient set $\Lambda_1^\omega / \sim$ and by $[z]$ the class of equivalence which contains z.

Proposition 3. *The following operations $[z] + [r] := [z \bullet r]$; $[z][r] := [z \diamond r]$, $z, r \in \Lambda_1^\omega$, of addition and multiplication are well-defined on $\mathcal{M}^\omega \times \mathcal{M}^\omega$ and $(\mathcal{M}^\omega, +, \cdot)$ is a unital commutative ring.*

Proof. Let $z' \in [z]$ and $r' \in [r]$. By Proposition 2 and the definition of the equivalence we have that for every $m \in \mathbb{N}$,

$$T_m^\omega(z) + T_m^\omega(r) = T_m^\omega(z') + T_m^\omega(r') = T_m^\omega(z' \bullet r')$$

and

$$T_m^\omega(z)T_m^\omega(r) = T_m^\omega(z')T_m^\omega(r') = T_m^\omega(z' \diamond r').$$

So the operations on $\mathcal{M}^\omega$ do not depend on representatives. Let $[u] = [z]([r] + [s])$ and $[v] = [z][r] + [z][s]$. Since for every $m \in \mathbb{N}$

$$T_m^\omega(u) = T_m^\omega(z)(T_m^\omega(r) + T_m^\omega(s)) = T_m^\omega(z)T_m^\omega(r) + T_m^\omega(z)T_m^\omega(s) = T_m^\omega(v),$$

so $[u] = [v]$ and we have the distributive law. Clearly that the associativity and commutativity of the addition and multiplication can be proved by the same way. Also, $-[z] = [z^-]$ and $\mathbb{I} = [e_1^{(0)}]$ is the identity. Thus $\mathcal{M}^\omega$ is a unital commutative ring. □

For any $\lambda \in \mathbb{C}$ and $z \in \mathcal{M}^\omega$ we set $\lambda * [z] = [\lambda z]$. Since, $T_m^\omega(\lambda z) = \lambda^m T_m^\omega(z)$, the operation "$*$" is well defined on $\mathbb{C} \times \mathcal{M}^\omega$. But $(\mathcal{M}^\omega, +, *)$ is not a linear space. Indeed, if $z \in \Lambda_1^\omega$ and $z \neq 0$, then $[z] + [z] = [z \bullet z] \neq 2 * [z]$ because $T_m^\omega([z \bullet z]) = 2T_m^\omega(z)$ but $T_m^\omega(2z) = 2^m T_m^\omega(z)$.

2 OPERATORS AND SEMINORMS ON $\mathcal{M}^{1/N}$

For a given $z = (y|x) \in \Lambda_1^\omega$, we denote by $\operatorname{supp} z$ the *support* of z, that is, the following pair of sets of indexes

$$\operatorname{supp} z = (\{i \in \mathbb{N}, j \in \mathbb{Z}_+ \colon y_i^{(j)} \neq 0\}, \{k \in \mathbb{N}, n \in \mathbb{Z}_+ \colon x_k^{(n)} \neq 0\}).$$

Let us define the following maps on $\Lambda_1^{1/N}$:

$$S_k^{+(n,m)}(z) = (z - x_k^{(n)} e_k^{(n)}) \bullet (\underbrace{x_k^{(m)} e_k^{(m)} \bullet \cdots \bullet (x_k^{(m)} e_k^{(m)}}_{N^{m-n}})$$

and

$$S_k^{-(n,m)}(z) = (z - y_k^{(n)} e_k^{-(n)}) \bullet (\underbrace{y_k^{(m)} e_k^{-(m)} \bullet \cdots \bullet (y_k^{(m)} e_k^{-(m)}}_{N^{m-n}}),$$

where $m \geq n$ and $z = (y|x) \in \Lambda_1^{1/N}$ for some $N \in \mathbb{N}$, $N > 1$. Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a permutation. We denote by $S_\sigma^{+(i)}$ and $S_\sigma^{-(i)}$ linear operators on $\Lambda_1^{1/N}$ such that

$$S_\sigma^{+(i)}(e_k^{(j)}) = e_{\sigma(k)}^{(i)} \text{ if } i = j \text{ and } S_\sigma^{+(i)}(e_k^{\pm(j)}) = e_k^{\pm(j)} \text{ otherwise,}$$

and

$$S_\sigma^{-(i)}(e_k^{-(i)}) = e_{\sigma(k)}^{-(i)} \text{ if } i = j \text{ and } S_\sigma^{-(i)}(e_k^{\pm(j)}) = e_k^{\pm(j)} \text{ otherwise.}$$

Lemma 1. *For every $z = (y|x) \in \Lambda_1^{1/N}$, permutation σ on $\mathbb{N}$ and $m \geq n$ we have*

$$[z] = [S_\sigma^{+(i)}(z)] = [S_\sigma^{-(i)}(z)] = [S_k^{+(n,m)}(z)] = [S_k^{-(n,m)}(z)].$$

Proof. The proof follows from the definitions and direct calculations. □

Proposition 4. *Let $z = (y|x) \in \Lambda_1^{1/N}$ for some $N \in \mathbb{N}$, $N > 1$ and z has a finite support. If $[z] = [0]$, then there is a number $j \in \mathbb{N}$ and a composition S of a finite set of mappings $\{S_k^{\pm(n,m)}, S_\sigma^{\pm(j)}\}$ defined above such that*

$$S(z) = (y'|x') = \begin{pmatrix} \ldots 0 \ldots 0 & | & 0 \ldots 0 \ldots \\ \ldots & | & \ldots \\ \ldots 0 \ldots 0 & | & 0 \ldots 0 \ldots \\ \ldots y_k^{'(j)} \ldots y_1^{'(j)} & | & x_1^{'(j)} \ldots x_k^{'(j)} \ldots \\ \ldots 0 \ldots 0 & | & 0 \ldots 0 \ldots \\ \ldots & | & \ldots \end{pmatrix} = \sum_{k=1}^\infty x_k^{'(j)} e_k^{(j)} + \sum_{k=1}^\infty y_k^{'(j)} e_k^{-(j)} \tag{5}$$

and $x_k^{'(j)} = y_k^{'(j)}$ for every $k \in \mathbb{N}$.

Proof. Let j be a minimal number such that $x_k^{(j)} = 0$ and $y_k^{(j)}$ for every $k \in \mathbb{N}$. Using a finite number of mappings $S_k^{\pm(n,m)}$ and Lemma 1 we can find $z' = (y'|x')$, $z' \sim z$ which satisfies (5). So, for every $m \in \mathbb{N}$

$$\sum_{k=1}^{\infty} \left(y_k^{\prime(j)}\right)^m = \sum_{k=1}^{\infty} \left(x_k^{\prime(j)}\right)^m.$$

From [1] it follows that vectors $\left(y_k^{\prime(j)}\right)_k$ and $\left(x_k^{\prime(j)}\right)_k$ coincide up to a permutation σ of coordinates $(x_1, \ldots, x_k, \ldots)$. So, applying $S_\sigma^{(j)}$ to z' we have $x_k^{\prime(j)} = y_k^{\prime(j)}$ for every $k \in \mathbb{N}$. □

Corollary 2. *Let $z = (y|x) \in \Lambda_1^{1/N}$ for some $N \in \mathbb{N}$, $N > 1$, and z has a finite support. Then there is an element $z' = (y'|x') \in \Lambda_1^{1/N}$ such that $z \sim z'$ and z' has the following property: if ${y'}_i^{(j)} \neq 0$, then ${x'}_k^{(n)} \neq {y'}_i^{(j)}$ for all $k \in \mathbb{N}$, $n \in \mathbb{Z}_+$.*

Proof. To get a proof it is enough to apply Proposition 4 to $z \bullet z'^{-} = (y \bullet x'|x \bullet y')$. □

Due to Theorem 1, we can introduce an alternative multiplication by *real* constants in $\mathcal{M}^\omega$, at least for the case $\omega = 1/N$, $N \in \mathbb{N}$, $N > 1$.

Theorem 2. *Let $N \in \mathbb{N}$, $N > 1$. Then $\mathcal{M}^{1/N}$ is a real linear commutative unital algebra with respect to the operations of addition and multiplication defined in Proposition 3 and the following multiplication by constants:*

$$a[z] := [z_{\{a\}}][z] = [z_{\{a\}} \diamond z], \quad a \in \mathbb{R},$$

where $z_{\{a\}}$ is as in Theorem 1.

Proof. Note first that from Theorem 1 and Proposition 2 it follows that for every $m \in \mathbb{N}$, $T_m^\omega(z_{\{a\}} \diamond z) = aT_m^\omega(z)$. So $\mathbb{I} = z_{\{1\}}$ is the unity in $\mathcal{M}^{1/N}$ and $[z_{\{a_1+a_2\}}] = [z_{\{a_1\}}] + [z_{\{a_2\}}]$, $a_1, a_2 \in \mathbb{R}$. Thus,

$$a([z] + [r]) = a[z] + a[r] \qquad \text{and} \qquad (a_1 + a_2)[z] = a_1[z] + a_2[z],$$

where $a, a_1, a_2 \in \mathbb{R}$ and $[z], [r] \in \mathcal{M}^{1/N}$. □

Let us denote by Ω the class of functions $\gamma\colon \mathbb{C} \to \mathbb{C}$ such that the mappings $\Phi_\gamma\colon \Lambda_1^\omega \to \Lambda_1^\omega$ defined by

$$\Phi_\gamma(z) = \Phi_\gamma(y|x) = \begin{pmatrix} \ldots\gamma(y_k^{(0)})\ldots\gamma(y_1^{(0)}) & | & \gamma(x_1^{(0)})\ldots\gamma(x_k^{(0)})\ldots \\ \ldots & | & \ldots \\ \ldots\gamma(y_k^{(n)})\ldots\gamma(y_1^{(n)}) & | & \gamma(x_1^{(n)})\ldots\gamma(x_k^{(n)})\ldots \\ \ldots & | & \ldots \end{pmatrix}$$

are well defined and $z \sim z'$ implies $\Phi_\gamma(z) = \Phi_\gamma(z')$. Such class is nonempty, for example, $\gamma(t) = t^m \in \Omega$, $m \in \mathbb{N}$.

Theorem 3. *Let $\gamma \in \Omega$. Then Φ_γ generates a linear operator $\widehat{\Phi}_\gamma\colon \mathcal{M}^{1/N} \to \mathcal{M}^{1/N}$ defined by $\widehat{\Phi}_\gamma([z]) = \Phi_\gamma(z)$.*

Proof. From the definition of Ω it follows that $\widehat{\Phi}_\gamma$ is well defined. Also, it is clear

$$\widehat{\Phi}_\gamma([z]+[r]) = \Phi_\gamma(z \bullet r) = \Phi_\gamma(z) \bullet \Phi_\gamma(r) = \widehat{\Phi}_\gamma([z]) + \widehat{\Phi}_\gamma([r]),$$

$z, r \in \Lambda_1^{1/N}$. Let now $z_{\{a\}} = (y_{\{a\}}|x_{\{a\}})$ be as in Theorem 1, that is,

$$x_{\{a\}} = \sum_{n=0}^{\infty}\sum_{i=1}^{a_n} e_i^{(n)}, \quad y_{\{a\}} = 0 \text{ if } a \geq 0 \qquad \text{and} \qquad y_{\{a\}} = \sum_{n=0}^{\infty}\sum_{i=1}^{a_n} e_i^{-(n)}, \quad x_{\{a\}} = 0 \text{ if } a < 0,$$

where

$$|a| = \sum_{j=0}^{\infty} \frac{a_j}{N^j}, \quad a_j \in \mathbb{N}.$$

If $a \geq 0$, then $[z_{\{a\}}][z] = a[z]$, $a \in \mathbb{R}$, $z = (y|x) \in \Lambda_1^{1/N}$ and

$$\Phi_\gamma(z_{\{a\}} \diamond z) = \Phi_\gamma\big((\underbrace{z \bullet \ldots \bullet z}_{a_0}) \diamond e_1^{(0)} \bullet \ldots \bullet (\underbrace{z \bullet \ldots \bullet z}_{a_n}) \diamond e_1^{(n)} \bullet \ldots\big)$$

$$= \big(\underbrace{\Phi_\gamma(z) \bullet \ldots \bullet \Phi_\gamma(z)}_{a_0}\big) \diamond e_1^{(0)} \bullet \ldots \bullet \big(\underbrace{\Phi_\gamma(z) \bullet \ldots \bullet \Phi_\gamma(z)}_{a_n}\big) \diamond e_1^{(n)} \bullet \ldots = z_{\{a\}} \diamond \Phi_\gamma(z).$$

If $a < 0$, we have to replace $e_1^{(n)}$ by $e_1^{-(n)}$, $n \in \mathbb{Z}_+$. So $\widehat{\Phi}_\gamma\big(a[z]\big) = a\widehat{\Phi}_\gamma\big([z]\big)$. Therefore, $\widehat{\Phi}_\gamma$ is a linear operator. $\square$

Let us denote $\tau_m\big([z]\big) = T_m^{1/N}(z)$, $[z] \in \mathcal{M}^{1/N}$, $m \in \mathbb{N}$. Clearly, τ_m are complex valued real-linear and multiplicative functions, that is, τ_m are homomorphisms from $\mathcal{M}^{1/N}$ to $\mathbb{C}$. By the definition of $\mathcal{M}^{1/N}$ we have that functionals $\tau_m\colon m \in \mathbb{N}$ separate points of $\mathcal{M}^{1/N}$. Let us denote by $\bar{z} = \Phi_\gamma(z)$, where $\gamma(t) = \bar{t}$ is the complex conjugate of t. It is easy to check that $\tau_m\big([\bar{z}]\big) = \overline{\tau_m\big([z]\big)}$ and so $\gamma(t) = \bar{t}$ belongs to Ω. So $[z] \mapsto \tau_m\big([\bar{z}]\big)$ is a complex valued functional for every $m \in \mathbb{N}$. Thus $\tau_m + \overline{\tau}_m$ and $-i\big(\tau_m - \overline{\tau}_m\big)$ are real valued linear functionals on $\mathcal{M}^{1/N}$.

Corollary 3. *If $\gamma \in \Omega$ is multiplicative, then $\widehat{\Phi}_\gamma$ is an algebra homomorphism.*

Proof. Let $[z], [r] \in \mathcal{M}^{1/N}$,

$$z = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} z_{+k}^{(n)} e_k^{(n)} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} z_{-k}^{(n)} e_k^{-(n)}$$

and

$$r = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} r_{+k}^{(n)} e_k^{(n)} + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} r_{-k}^{(n)} e_k^{-(n)}.$$

Since $\Phi_\gamma\big(z_{+k}^{(n)} e_k^{(n)}\big) = \gamma\big(z_{+k}^{(n)}\big) e_k^{(n)}$, we have

$$\Phi_\gamma\big(z_{\pm k}^{(n)} e_k^{\pm(n)} \diamond r_{\pm i}^{(j)} e_i^{\pm(j)}\big) = \gamma\big(z_{\pm k}^{(n)} r_{\pm i}^{(j)}\big) e_k^{\pm(n)} \diamond e_i^{\pm(j)},$$

$k, i \in \mathbb{N}$, $n, j \in \mathbb{Z}_+$. From the linearity and multiplicativity of τ_m it follows

$$\tau_m\big(\widehat{\Phi}_\gamma([z])\big)\tau_m\big(\widehat{\Phi}_\gamma([r])\big) = \tau_m\big(\widehat{\Phi}_\gamma([z])\widehat{\Phi}_\gamma([r])\big) = \tau_m\big(\widehat{\Phi}_\gamma([z][r])\big).$$

Since it is true for every m, we have

$$\widehat{\Phi}_\gamma([z])\widehat{\Phi}_\gamma([r]) = \widehat{\Phi}_\gamma([z][r]).$$

$\square$

Proposition 5. *Let $\gamma \in \Omega$ and $\gamma(0) = 0$. Then the following formula defines a seminorm on $\mathcal{M}^{1/N}$:*

$$p_\gamma\big([z]\big) = \inf_{(y|x)\in[z]} \sum_{n=0}^{\infty} \frac{1}{N^n} \sum_{k=1}^{\infty} \Big(\big|\gamma\big(x_k^{(n)}\big)\big| + \big|\gamma\big(y_k^{(n)}\big)\big|\Big).$$

Proof. Since the infimum is taken over all representations $(y|x) \in [z]$, the norm is well defined. It is easy to check that p_γ is nonnegative and satisfies the triangle inequality and is homogeneous. □

Definition 3. *Let us define the following seminorms on $\mathcal{M}^{1/N}$:*

$$p_m\big([z]\big) = p_{\gamma_m}\big([z]\big) \text{ for } \gamma_n(t) = t^m.$$

It is clear that $\big|\tau_m\big([z]\big)\big| \le p_m\big([z]\big)$, $[z] \in \mathcal{M}^{1/N}$ and so, if $[z] \neq 0$, then there is $m \in \mathbb{N}$ such that $p_m\big([z]\big) > 0$.

Let us denote $\big(\mathcal{M}^{1/N}, (p_m)\big)$ the linear space $\mathcal{M}^{1/N}$ endowed with the projective topology, generated by seminorms (p_m). So we have the following proposition.

Proposition 6. *The space $\big(\mathcal{M}^{1/N}, (p_m)\big)$ is a locally convex metrisable topological vector space and each functional τ_m is continuous on $\big(\mathcal{M}^{1/N}, (p_m)\big)$.*

Let us denote by $\mathcal{D}$ the following subset of $\mathcal{M}^{1/N}$:

$$\mathcal{D} = \Big\{u \in \mathcal{M}^{1/N}\colon \text{ there is } z \in u \text{ such that } \big|z_k^{(n)}\big| \le 1,\ n \in \mathbb{Z}_+,\ k \in \mathbb{N}\Big\}.$$

Theorem 4. *$\mathcal{D}$ is a subalgebra in $\mathcal{M}^{1/N}$ and the restriction of the topology of $\big(\mathcal{M}^{1/N}, (p_n)\big)$ to $\mathcal{D}$ is generated by a norm on $\mathcal{D}$.*

Proof. From the definition of addition and multiplication in $\mathcal{M}^{1/N}$ it follows that $u + v \in \mathcal{D}$ and $uv \in \mathcal{D}$ for all $u, v \in \mathcal{D}$. Also, for every $a \in \mathbb{R}$, $[z_{\{a\}}] \in \mathcal{D}$ and so $au = [z_{\{a\}}]u \in \mathcal{D}$. Hence, $\mathcal{D}$ is a subalgebra in $\mathcal{M}^{1/N}$. Note that for every $u \in \mathcal{D}$ and $m \in \mathbb{N}$, $p_m(u) \le p_1(u)$. Also, p_1 is a norm on $\mathcal{D}$. Indeed, if $u \neq 0$, then there is $m \in \mathbb{N}$ such that $\tau_m(u) \neq 0$. So

$$0 \neq |\tau_m(u)| \le p_m(u) \le p_1(u).$$

So $(\mathcal{D}, p_1)$ is a normed space and all p_m are continuous with respect to p_1. So the restriction of topology of $\big(\mathcal{M}^{1/N}, (p_n)\big)$ to $\mathcal{D}$ coincides with the norm topology of $(\mathcal{D}, p_1)$. □

REFERENCES

[1] Alencar R., Aron R., Galindo P., Zagorodnyuk A. *Algebra of symmetric holomorphic functions on ℓ_p*. Bull. Lond. Math. Soc. 2003, **35**, 55–64. doi:10.1112/S0024609302001431

[2] Aron R.M., Cole B.J., Gamelin T.W. *Spectra of algebras of analytic functions on a Banach space*. J. Reine Angew. Math. 1991, **415**, 51–93.

[3] Aron R., Galindo P., Pinasco D., Zalduendo I. *Group-symmetric holomorphic functions on a Banach space*. Bull. Lond. Math. Soc. 2016, **48** (5), 779–796. doi:10.1112/blms/bdw043

[4] Chernega I.V. *A semiring in the spectrum of the algebra of symmetric analytic functions in the space ℓ_1*. J. Math. Sci. 2016, **212**, 38–45. doi:10.1007/s10958-015-2647-3

[5] Chernega I., Galindo P., Zagorodnyuk A. *A multiplicative convolution on the spectra of algebras of symmetric analytic functions*. Rev. Mat. Complut. 2014, **27** (2), 575–585. doi:10.1007/s13163-013-0128-0

[6] Chernega I., Galindo P., Zagorodnyuk A. *Some algebras of symmetric analytic functions and their spectra*. Proc. Edinb. Math. Soc. 2012, **55**, 125–142. doi:10.1017/S0013091509001655

[7] Chernega I., Galindo P., Zagorodnyuk A. *The convolution operation on the spectra of algebras of symmetric analytic functions*. J. Math. Anal. Appl. 2012, **395** (2), 569–577. doi:10.1016/j.jmaa.2012.04.087

[8] Chernega I., Holubchak O., Novosad Z., Zagorodnyuk A. *Continuity and hypercyclicity of composition operators on algebras of symmetric analytic functions on Banach spaces*. Eur. J. Math. 2019, 12 P. doi:10.1007/s40879-019-00390-z

[9] Dineen S. Complex analysis on infinite-dimensional spaces. Springer-Verlag, London, 1999.

[10] Jawad F. *Note on separately symmetric polynomials on the Cartesian product of ℓ_1*. Mat. Stud. 2018, **50** (2), 204–210. doi:10.15330/ms.50.2.204-210

[11] Jawad F., Zagorodnyuk A. *Supersymmetric Polynomials on the Space of Absolutely Convergent Series*. Symmetry 2019, **11** (9), 1111 (19 p.). doi:10.3390/sym11091111

[12] Galindo P., Vasylyshyn T., Zagorodnyuk A. *The algebra of symmetric analytic functions on L_∞*. Proc. Roy. Soc. Edinburgh Sect. A. 2107, **147** (4), 743–761. doi:10.1017/S0308210516000287

[13] Galindo P., Vasylyshyn T., Zagorodnyuk A. *Symmetric and finitely symmetric polynomials on the spaces ℓ_∞ and $L_\infty[0,+\infty)$*. Math. Nachr. 2018, **291** (11-12), 1712–1726. doi:10.1002/mana.201700314

[14] García D., Maestre M., Zalduendo I. *The spectra of algebras of group-symmetric functions*. Proc. Edinb. Math. Soc. 2019, **62** (3), 609–623. doi:10.1017/S0013091518000603

[15] Gonzaléz M., Gonzalo R., Jaramillo J. *Symmetric polynomials on rearrangement invariant function spaces*. J. London Math. Soc. 1999, **59**, 681–697.

[16] Kravtsiv V., Vasylyshyn T., Zagorodnyuk A. *On Algebraic Basis of the Algebra of Symmetric Polynomials on $\ell_p(\mathbf{C}^n)$*. J. Funct. Spaces 2017, **2017** (7), Article ID 4947925, 8 p. doi:10.1155/2017/4947925

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Нехай X — зважена пряма сума нескінченної кількості копій комплексного простору $\ell_1 \bigoplus \ell_1$. Ми розглядаємо алгебру, яка складається з поліномів на X, котрі є суперсиметричними на кожному доданку $\ell_1 \bigoplus \ell_1$. Функціонали значень в точках на цій алгебрі задають відношення еквівалентності '$\sim$' на X. У роботі досліджено фактор-множину $X/\sim$ і показано, що за деяких умов на цій множині є структура дійсної топологічної алгебри.

Ключові слова і фрази: симетричні і суперсиметричні поліноми на банахових просторах, алгебри аналітичних функцій на банахових просторах, спектри алгебр аналітичних функцій.