ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 345–349
doi:10.15330/cmp.11.2.345-349

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2019, Т.11, №2, С.345–349

KARAKAŞ A.

A NEW FACTOR THEOREM FOR GENERALIZED ABSOLUTE RIESZ SUMMABILITY

The aim of this paper is to consider an absolute summability method and generalize a theorem concerning $|\bar{N}, p_n|_k$ summability of infinite series to $\varphi-\mid \bar{N}, p_n; \delta \mid_k$ summability of infinite series by using almost increasing sequence. Furthermore, it is explained that a well known result dealing with $|\bar{N}, p_n|_k$ summability is obtained when this generalization is restricted under special conditions.

Key words and phrases: summability factors, almost increasing sequence, infinite series, Hölder inequality, Minkowski inequality.

Department of Mathematics, Erciyes University, 38039, Kayseri, Turkey
E-mail: ahmetkarakas1985@hotmail.com

INTRODUCTION

A positive sequence (z_n) is said to be almost increasing if there exists a positive increasing sequence (d_n) and two positive constants L and M such that $Ld_n \leq z_n \leq Md_n$ (see [1]).

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (w_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [8]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid w_n - w_{n-1} \mid^k < \infty.$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - \mid \bar{N}, p_n; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [16])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} \mid w_n - w_{n-1} \mid^k < \infty.$$

If we take $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - \mid \bar{N}, p_n; \delta \mid_k$ summability is the same as $\mid \bar{N}, p_n; \delta \mid_k$ summability (see [4]). Also, if we take $\varphi_n = \frac{P_n}{p_n}$ and $\delta = 0$, then we get $|\bar{N}, p_n|_k$ summability.

УДК 517.521.7
2010 *Mathematics Subject Classification:* 26D15, 40D15, 40F05, 40G99.

1 THE KNOWN RESULT

A well known theorem dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series is given below.

Theorem 1 ([3]). *Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (λ_n) and (β_n) such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{3}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{4}$$

If

$$\sum_{n=1}^{m} \frac{1}{n} \mid s_n \mid^k = O(X_m) \quad as \quad m \to \infty \tag{5}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{6}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

2 THE MAIN RESULT

Some works dealing with generalized absolute summability methods have been done (see [5–7, 9, 10, 13–19]). The aim of this paper is to generalize Theorem 1 to $\varphi - |\bar{N}, p_n; \delta|_k$ summability using almost increasing sequence in place of positive non-decreasing sequence.

Theorem 2. *Let (φ_n) be a sequence of positive real numbers such that*

$$\varphi_n p_n = O(P_n), \tag{8}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O(\varphi_v^{\delta k} \frac{1}{P_v}) \quad as \quad m \to \infty. \tag{9}$$

Let (X_n) be an almost increasing sequence. If conditions (1)–(4), (6)–(7) of the Theorem 1 and

$$\sum_{n=1}^{m} \varphi_n^{\delta k} \frac{\mid s_n \mid^k}{n} = O(X_m) \quad as \quad m \to \infty \tag{10}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ is summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $0 \leq \delta k < 1$.

We need the following lemmas for the proof of Theorem 2.

Lemma 1 ([11]). *Under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, we have that*

$$nX_n\beta_n = O(1) \quad as \quad n \to \infty, \tag{11}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{12}$$

Lemma 2 ([12]). *If the conditions (6) and (7) of Theorem 1 are satisfied, then* $\Delta\left(\frac{P_n}{np_n}\right) = O(\frac{1}{n})$.

Remark 1 ([3]). *It should be noted that, from the hypotheses of Theorem 1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$.*

3 PROOF OF THEOREM 2

Proof. Let (J_n) indicate $(\bar{N}, p_n)$ means of the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$. Then, for $n \geq 1$, we obtain

$$\bar{\Delta} J_n = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \frac{a_v P_v \lambda_v}{v p_v}.$$

Applying Abel's formula, we get

$$\begin{aligned}\bar{\Delta} J_n &= \frac{s_n\lambda_n}{n} + \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} \frac{P_{v+1}P_v\Delta\lambda_v}{(v+1)p_{v+1}} s_v + \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} P_v\lambda_v s_v \Delta(\frac{P_v}{vp_v}) - \frac{p_n}{P_nP_{n-1}} \sum_{v=1}^{n-1} P_v\lambda_v s_v \frac{1}{v} \\ &= J_{n,1} + J_{n,2} + J_{n,3} + J_{n,4}.\end{aligned}$$

For the proof of Theorem 2, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \mid J_{n,r} \mid^k < \infty, \quad for \quad r = 1,2,3,4.$$

By using Abel's formula, we have

$$\begin{aligned}\sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \mid J_{n,1} \mid^k &= O(1) \sum_{n=1}^{m} \varphi_n^{\delta k+k-1} \frac{1}{n^k} |\lambda_n|^{k-1} |\lambda_n| |s_n|^k = O(1) \sum_{n=1}^{m} \varphi_n^{\delta k} |\lambda_n| \frac{|s_n|^k}{n} \\ &= O(1) \sum_{n=1}^{m-1} \Delta|\lambda_n| \sum_{v=1}^{n} \varphi_v^{\delta k} \frac{|s_v|^k}{v} + O(1)|\lambda_m| \sum_{n=1}^{m} \varphi_n^{\delta k} \frac{|s_n|^k}{n} \\ &= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1)|\lambda_m| X_m = O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by virtue of (1), (4), (6), (8), (10) and (12).

Now, using Hölder's inequality and (1), (6), (8), we obtain

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid J_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} (\frac{p_n}{P_nP_{n-1}})^k \left(\sum_{v=1}^{n-1} P_v \, |\Delta\lambda_v| \, |s_v| \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} P_v \, |\Delta\lambda_v| \, |s_v| \right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} P_v\beta_v \, |s_v|^k \right) \times \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v\beta_v \right)^{k-1}.\end{aligned}$$

Again, using Abel's formula and (3), (9)–(12), we have

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid J_{n,2} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \beta_v \, |s_v|^k = O(1) \sum_{v=1}^{m} P_v \beta_v |s_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{|s_v|^k}{v} v\beta v = O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \varphi_r^{\delta k} \frac{|s_r|^k}{r} \\ &\quad + O(1) m\beta_m \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{|s_v|^k}{v} = O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) X_v + O(1) m\beta_m X_m \\ &= O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} \beta_v X_v + O(1) m\beta_m X_m = O(1) \text{ as } m \to \infty.\end{aligned}$$

Since $\Delta\left(\frac{P_v}{vp_v}\right) = O(\frac{1}{v})$, as in $J_{n,1}$, we obtain

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid J_{n,3} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} P_v \, |s_v| \, |\lambda_v| \, \frac{1}{v}\right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}^k} \left(\sum_{v=1}^{n-1} \frac{P_v}{p_v} p_v \, |s_v| \, |\lambda_v| \, \frac{1}{v}\right)^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left(\sum_{v=1}^{n-1} \left(\frac{P_v}{vp_v}\right)^k p_v \, |s_v|^k \, |\lambda_v|^k\right) \left(\frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v\right)^{k-1} \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} \left(\frac{P_v}{vp_v}\right)^{k-1} \frac{P_v}{vp_v} p_v \, |s_v|^k \, |\lambda_v|^k \\ &= O(1) \sum_{v=1}^{m} \frac{P_v}{vp_v} p_v \, |s_v|^k \, |\lambda_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \\ &= O(1) \sum_{v=1}^{m} \frac{P_v}{vp_v} p_v \, |s_v|^k \, |\lambda_v|^{k-1} \, |\lambda_v| \, \varphi_v^{\delta k} \frac{1}{P_v} \\ &= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{|s_v|^k}{v} \, |\lambda_v| = O(1) \quad \text{as } m \to \infty,\end{aligned}$$

by means of (1), (4), (6), (8)–(10) and (12).

Finally, as in $J_{n,3}$, we have

$$\begin{aligned}\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \mid J_{n,4} \mid^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left(\sum_{v=1}^{n-1} P_v \, |s_v| \, |\lambda_v| \, \frac{1}{v}\right)^k \\ &= O(1) \quad \text{as } m \to \infty,\end{aligned}$$

in view of (1), (4), (6), (8)–(10) and (12).

Thus, the proof of Theorem 2 is completed. □

4 CONCLUSION

If we take (X_n) as a positive non-decreasing sequence, $\varphi_n = \frac{P_n}{p_n}$ and $\delta = 0$ in Theorem 2, then we get Theorem 1. In this case, condition (10) reduces to condition (5). Also, the conditions (8) and (9) are automatically satisfied.

REFERENCES

[1] Bari N. K., Stečkin S. B. *Best approximations and differential properties of two conjugate functions*. Trudy Moskov. Mat. Obšč. 1956, **5**, 483–522. (in Russian)

[2] Bor H. *On two summability methods*. Math. Proc. Camb. Philos. Soc. 1985, **97**, 147–149.

[3] Bor H. *A note on* $|\bar{N}, p_n|_k$ *summability factors of infinite series*. Indian J. Pure Appl. Math. 1987, **18**, 330–336.

[4] Bor H. *On local property of* $|\bar{N}, p_n; \delta|_k$ *summability of factored Fourier series*. J. Math. Anal. Appl. 1993, **179** (2), 646–649.

[5] Bor H., Seyhan H. *On almost increasing sequences and its applications*. Indian J. Pure Appl. Math. 1999, **30**, 1041–1046.

[6] Bor H., Özarslan H. S. *On absolute Riesz summability factors*. J. Math. Anal. Appl. 2000, **246**, 657–663.

[7] Bor H., Özarslan H. S. *A note on absolute summability factors*. Adv. Stud. Contemp. Math. (Kyungshang) 2003, **6** (1), 1–11.

[8] Hardy G. H. Divergent Series. Oxford University Press, Oxford, 1949.

[9] Karakaş A. *A note on absolute summability method involving almost increasing and* δ*-quasi-monotone sequences*. Int. J. Math. Comput. Sci. 2018, **13** (1), 73–81.

[10] Kartal B. *On generalized absolute Riesz summability method*. Commun. Math. Appl. 2017, **8** (3), 359–364.

[11] Mazhar S. M. *A note on absolute summability factors*. Bull. Inst. Math. Acad. Sinica 1997, **25**, 233–242.

[12] Mishra K. N., Srivastava R. S. L. *On* $|\bar{N}, p_n|$ *summability factors of infinite series*. Indian J. Pure Appl. Math. 1984, **15**, 651–656.

[13] Özarslan H. S. *On almost increasing sequences and its applications*. Int. J. Math. Math. Sci. 2001, **25**, 293–298.

[14] Özarslan H. S. *A note on* $|\bar{N}, p_n; \delta|_k$ *summability factors*. Indian J. Pure Appl. Math. 2002, **33**, 361–366.

[15] Özarslan H. S. *On* $|\bar{N}, p_n; \delta|_k$ *summability factors*. Kyungpook Math. J. 2003, **43**, 107–112.

[16] Seyhan H. *On the local property of* $\varphi - |\bar{N}, p_n; \delta|_k$ *summability of factored Fourier series*. Bull. Inst. Math. Acad. Sinica 1997, **25**, 311–316.

[17] Seyhan H. *A note on absolute summability factors*. Far East J. Math. Sci. 1998, **6** (1), 157-162.

[18] Seyhan H. *On the absolute summability factors of type (A,B)*. Tamkang J. Math. 1999, **30** (1), 59–62.

[19] Seyhan H., Sönmez A. *On* $\varphi - |\bar{N}, p_n; \delta|_k$ *summability factors*. Portugaliae Math. 1997, **54**, 393–398.

Received 21.02.2019

Каракас А. *Нова факторизаційна теорема для узагальненого абсолютного підсумовування за Рісом* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 345–349.

Метою цієї статті є розгляд методу абсолютного підсумовування і узагальнення теореми про $|\bar{N}, p_n|_k$ сумовність нескінченного ряду до $\varphi-|\bar{N}, p_n; \delta|_k$ сумовності, використовуючи майже зростаючі послідовності. Більше того, показано, що добре відомі результати для $|\bar{N}, p_n|_k$ сумовності випливають з цих узагальнень за деяких обмежень.

Ключові слова і фрази: сумовні дільники, майже зростаюча послідовність, ряди, нерівність Гьольдера, нарівність Мінковського.