PONOMARCHUK B.S.

METRIC DIMENSION OF METRIC TRANSFORM AND WREATH PRODUCT

Let (X, d) be a metric space. A non-empty subset A of the set X is called *resolving* set of the metric space (X, d) if for two arbitrary not equal points u, v from X there exists an element a from A, such that $d(u, a) \neq d(v, a)$. The smallest of cardinalities of resolving subsets of the set X is called *the metric dimension $md(X)$* of the metric space (X, d).

In general, finding the metric dimension is an NP-hard problem. In this paper, metric dimension for metric transform and wreath product of metric spaces are provided. It is shown that the metric dimension of an arbitrary metric space is equal to the metric dimension of its metric transform.

Key words and phrases: metric dimension, metric transform, wreath product.

National University of Kyiv-Mohyla Academy, 2 Skovorody str., 04070, Kyiv, Ukraine

E-mail: ponomarchuk.bogdan@gmail.com

INTRODUCTION

Let (X, d) be a metric space. It is said that a set A is the *resolving* set of the metric space (X, d) if A is a non-empty subset of X and for an arbitrary different points u, v from X there exists an element a in A, such that distances $d(u, a)$ and $d(v, a)$ are not equal. The smallest of cardinalities of resolving subsets of the set X is called *the metric dimension $md(X)$* of the metric space (X, d).

Definition of the metric dimension for metric spaces was firstly introduced by Blumenthal in 1953 [4]. 20 years later Harari and Melter in [7] applied it to the graphs. After that the metric dimension concept found range of applications, like in combinatorial analysis, robotics, for finding its location, biology, chemistry etc. [14], [9], [13].

In 2013 S. Bau and F. Beardon [2] got the Blumenthal's ideas and proceeded research of the metric spaces metric dimension. They has managed to calculate the metric dimension of the sphere in a k-dimensional Euclidean space. Later, M. Heydarpour and S. Maghsoudi in [8] calculated the metric dimensions of geometric spaces.

As well as metric dimension, Blumenthal has also described metric transforms [3], which was studied further by other researchers, like by Schoenberg and von Neumann in scope of Euclidian subspace metric transforms into Hilbert space subsets [12], [15].

In general, finding of the metric dimension of a finite graph is an NP-hard problem [6]. Following that, metric dimension characterization for a finite metric space is also NP-hard. This is why there are several ways of conducting metric dimension research. One of those is researching metric dimension of constructions of two graphs, if we know metric dimensions of both of them. For example, metric dimensions of wreath products and cartesian products of two finite graphs characterized in [5], [1].

In this paper we characterize metric dimensions of the wreath products of metric spaces which were considered in [11], [10]. This construction of metric spaces was called a wreath product because the isometry group of the wreath product of metric spaces is isomorphic to the wreath product of theirs isometry groups. In particular, we will also show that metric dimension of the metric transform of an arbitrary metric space is equal to the metric dimension of this space.

1   METRIC TRANSFORM

Denote by $\mathbb{R}^+$ the set of all non-negative real numbers. Let s be a continuous monotone increasing function and $s(0) = 0$. Such functions are called *scales*. *Transformation* of metric space (X, d_X) is the space $(X, s(d_X))$, where function $s(d_X)$ might not follow triangle inequality [3]. Transformation is called *metric*, if $s(d_X)$ is metric.

Definition 1 ([3]). *If for metric spaces (X, d_X) and (Y, d_Y) there is a bijection $g : X \to Y$, and scale s that for arbitrary $u, v \in X$ holds:*

$$d_X(u, v) = s(d_Y(g(u), g(v))),$$

then such metric spaces are called isomorphic.

Proposition 1. *Let (X, d) be a metric space and let $s : \mathbb{R}^+ \to \mathbb{R}^+$ be a metric transform. Then metric basis of X is also the metric basis of the metric transform $(X, s(d))$.*

Proof. Let $V = \{v_i, i \in I\}$ be a metric basis of the space (X, d). As follows from the definition of a metric basis, for every $u, w \in X$ there is $v_j \in V$, such that

$$d(u, v_j) \neq d(w, v_j),$$

i. e. v_j resolves points u and w. The function s is monotone increasing, so, we have

$$s(d(u, v_j)) \neq s(d(w, v_j)).$$

Hence, v_j resolves u and w in $(X, s(d))$. Therefore, V is resolving set of $(X, s(d))$.

We need to show, that V is minimal cardinality. Assume, that there is v_l, such that $V \setminus \{v_l\}$ also is a resolving set of $(X, s(d))$. But V is a minimal resolving set of (X, d). Hence, there are points $u, w \in X$ such that for any $v_j \in V \setminus \{v_l\}$ the following condition holds:

$$d(u, v_j) = d(w, v_j).$$

But it means, that $s(d(u, v_j)) = s(d(w, v_j))$. Hence, $V \setminus \{v_l\}$ is not a metric basis of $(X, s(d))$. $\qquad\square$

Corollary 1. *Metric dimension of a metric space (X, d) is equal to the metric dimension of its metric transform $(X, s(d))$ for any scale s.*

2   WREATH PRODUCT

First, we recall the construction of a wreath product of metric spaces.

Definition 2. *A metric space (X, d) is called uniformly discrete if for an arbitrary $u, v \in X$ either $u = v$ or there exists a radius $r > 0$ such that $d(u, v) > r$.*

Let (X, d_X) be a uniformly discrete metric space, and (Y, d_Y) be a bounded metric space. Since space (X, d_X) is uniformly discrete, then there exists r such that for two different arbitrary points x_1, x_2 from set X inequality $d_X(x_1, x_2) \geq r$ holds. Let $s(x)$ be the scale such that

$$diam(s(Y)) < r. \tag{1}$$

Let us define a function ρ_s on the Cartesian product $X \times Y$ by:

$$\rho_s((x_1, y_1), (x_2, y_2)) = \begin{cases} d_X(x_1, x_2), & \text{if } x_1 \neq x_2 \\ s(d_Y(y_1, y_2)), & \text{if } x_1 = x_2. \end{cases}$$

Such a metric space is called *wreath product* of metric spaces (X, d_X) and (Y, d_Y) and denoted as $Xwr_s Y$ [11]. For different scales s_1 and s_2 metric spaces $Xwr_{s_1} Y$ and $Xwr_{s_2} Y$ are isomorphic.

Theorem 1. *Let X be a finite metric space and Y be a bounded metric space, $md(Y) < \infty$. Then, the dimension of wreath product of metric spaces (X, d_X) and (Y, d_Y) is equal to*

$$md(Xwr_s Y) = |X| * md(Y).$$

If $md(Y) = \infty$, then $md(Xwr_s Y) = \infty$.

Proof. Let $v_1, \ldots, v_l$ be a metric basis of (Y, d_Y). We assume that $X = \{x_1, \ldots, x_m\}$. Define a set

$$B = \{(x_j, v_i) | 1 \leq j \leq m, 1 \leq i \leq n\}.$$

We need to show that the set B is a basis of $Xwr_s Y$.

Let (x_1, y_1) and (x_2, y_2) be two different points of $Xwr_s Y$. From the definition of the wreath product of metric spaces follows, that if $x_1 \neq x_2$, then points (x_1, y_1) and (x_2, y_2) are resolved by point (x_2, v_2). Indeed, we will have:

$$\rho((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2), \quad \rho((x_2, y_2), (x_2, v_2)) = s(d_Y(y_2, v_2)).$$

From inequality (1) follows, that

$$\rho((x_1, y_1), (x_2, y_2)) < \rho((x_2, y_2), (x_2, v_2))$$

and therefore points (x_1, y_1) and (x_2, y_2) are resolved by (x_2, v_2).

Let $x_1 = x_2$. In this case, since $v_1, \ldots, v_n$ is the metric basis of Y, exists v_j that resolves y_1 and y_2. Then

$$\rho((x_1, y_1), (x_1, v_j)) = s(d_Y(y_1, v_j)), \quad \rho((x_2, y_2), (x_1, v_j)) = s(d_Y(y_2, v_j)).$$

Since v_j resolves y_1 and y_2, $s(d_Y(y_1, v_j)) \neq s(d_Y(y_2, v_j))$. In this case all elements from the set X are supposed to be included into a basis of the cartesian product.

And now let us show that B is a basis. Assume that $B' = B/\{x_1, v_1\}$ is a basis. Since $v_1, \ldots, v_n$ is the basis of the metric space Y, then there exists $y_1, y_2 \in Y$ which are not resolved by $v_2, v_3, \ldots, v_n$ but are resolved by v_1 only. Then points (x, y_1) and (x, y_2) are not resolved by points from B'. This means that B is the minimal set, therefore B is a metric basis of the space $Xwr_s Y$.

As a result we have that $md(Xwr_s Y) = |X| md(Y)$.   □

Theorem 1 implies the next statement.

Corollary 2. *If the space X is infinite, then $md(Xwr_s Y) = \infty$.*

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Для довільного метричного простору (X, d) множина $A \subset X$ називається розділяючою, якщо для довільних різних елементів u, v, що належать множині X існує такий елемент $a \in A$, що відстані $d(a, u)$ та $d(a, v)$ є різними. Метричною розмірністю $md(X)$ простору (X, d) називається розділяюча множина найменшої потужності.

В загальному випадку пошук метричної розмірності є NP–важкою задачею. В роботі охарактеризовано метричну розмірність метричної трансформації та вінцевого добутку метричних просторів. Також показано, що метрична розмірність довільного метричного простору співпадає з метричною розмірністю його метричної трансформації.

Ключові слова і фрази: метрична розмірність, метрична трансформація, вінцевий добуток.