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POINT-EVALUATION FUNCTIONALS ON ALGEBRAS OF SYMMETRIC FUNCTIONS ON $(L_\infty)^2$

It is known that every continuous symmetric (invariant under the composition of its argument with each Lebesgue measurable bijection of $[0, 1]$ that preserve the Lebesgue measure of measurable sets) polynomial on the Cartesian power of the complex Banach space L_∞ of all Lebesgue measurable essentially bounded complex-valued functions on $[0, 1]$ can be uniquely represented as an algebraic combination, i.e., a linear combination of products, of the so-called elementary symmetric polynomials. Consequently, every continuous complex-valued linear multiplicative functional (character) of an arbitrary topological algebra of the functions on the Cartesian power of L_∞, which contains the algebra of continuous symmetric polynomials on the Cartesian power of L_∞ as a dense subalgebra, is uniquely determined by its values on elementary symmetric polynomials. Therefore, the problem of the description of the spectrum (the set of all characters) of such an algebra is equivalent to the problem of the description of sets of the above-mentioned values of characters on elementary symmetric polynomials.

In this work the problem of the description of sets of values of characters, which are point-evaluation functionals, on elementary symmetric polynomials on the Cartesian square of L_∞ is completely solved. We show that sets of values of point-evaluation functionals on elementary symmetric polynomials satisfy some natural condition. Also we show that for any set c of complex numbers, which satisfies the above-mentioned condition, there exists the element x of the Cartesian square of L_∞ such that values of the point-evaluation functional at x on elementary symmetric polynomials coincide with the respective elements of the set c.

Key words and phrases: symmetric polynomial, point-evaluation functional.

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INTRODUCTION

In general, the problem of the description of the spectrum (the set of continuous complex-valued linear multiplicative functionals, or characters) of a topological algebra of analytic functions on a Banach space is unsolved. But if a topological algebra or its dense subalgebra has a countable algebraic basis (the subset B of the algebra A is called an algebraic basis of A, if every element of A can be uniquely represented as an algebraic combination (a linear combination of products) of elements of B), then the problem of the description of the spectrum simplifies, because in this case every character is uniquely determined by the sequence of its values on elements of the algebraic basis and, consequently, the problem of the description of the spectrum is equivalent to the problem of the description of the set of such sequences. For

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example, in [2] it was constructed an algebraic basis of the algebra of all continuous symmetric (see definition below) polynomials on the complex Banach space L_∞ of all complex-valued Lebesgue measurable essentially bounded functions on $[0,1]$. Also, using this result, in [2] it was described the spectrum of the Fréchet algebra $H_{bs}(L_\infty)$ of all entire symmetric functions of bounded type on L_∞ and it was shown that every character of $H_{bs}(L_\infty)$ is a point-evaluation functional.

Firstly algebraic bases of algebras of symmetric continuous polynomials on real Banach spaces of Lebesgue measurable integrable in a power p functions on $[0,1]$ and $[0,+\infty)$, where $1 \le p < +\infty$, were studied by Nemirovskii and Semenov in [7]. Some of their results were generalized to real separable rearrangement invariant Banach spaces of Lebesgue measurable functions on $[0,1]$ and $[0,+\infty)$ by González, Gonzalo and Jaramillo in [4]. Symmetric polynomials and symmetric analytic functions on the complex Banach spaces of all complex-valued Lebesgue measurable essentially bounded functions on $[0,1]$ and $[0,+\infty)$ were studied in [2] and [3] respectively. Symmetric polynomials on Cartesian products of some Banach spaces were studied in [6, 8–12]. In particular, in [10] it was constructed a countable algebraic basis of the algebra of continuous symmetric polynomials on the Cartesian power of L_∞.

In this work the problem of the description of sequences of values of point-evaluation functionals on the elements of the algebraic basis of the algebra of continuous symmetric polynomials on the Cartesian square of L_∞ is completely solved. We show that the above-mentioned sequences satisfy some natural condition. Also we show that for any sequence c of complex numbers, which satisfies this condition, there exists an element x of the Cartesian square of L_∞ such that the sequence of values of the point-evaluation functional at x coincides with c. We generalize the results of [11].

1 PRELIMINARIES

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers.

A mapping $P : X \to \mathbb{C}$, where X is a Banach space with norm $\|\cdot\|_X$, is called an N-*homogeneous polynomial*, where $N \in \mathbb{N}$, if there exists an N-linear mapping $A_P : X^N \to \mathbb{C}$ such that

$$P(x) = A_P(\underbrace{x, \ldots, x}_{N})$$

for every $x \in X$. It is known that an N-homogeneous polynomial $P : X \to \mathbb{C}$ is continuous if and only if

$$\|P\| = \sup_{\|x\|_X \le 1} |P(x)| < +\infty.$$

Consequently, if P is a continuous N-homogeneous polynomial, then

$$|P(x)| \le \|P\| \|x\|_X^N \tag{1}$$

for every $x \in X$.

A mapping $P = P_0 + P_1 + \ldots + P_N$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for every $j \in \{1, \ldots, N\}$, is called a *polynomial* of degree at most N.

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |x(t)|.$$

Let $(L_\infty)^2$ be the Cartesian square of L_∞ with norm

$$\|x\|_{\infty,2} = \max\{\|x_1\|_\infty, \|x_2\|_\infty\}$$

where $x = (x_1, x_2) \in (L_\infty)^2$.

Let Ξ be the set of all bijections $\sigma : [0,1] \to [0,1]$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure. A function $f : (L_\infty)^2 \to \mathbb{C}$ is called *symmetric* if

$$f(x \circ \sigma) = f(x)$$

for every $x = (x_1, x_2) \in (L_\infty)^2$ and for every $\sigma \in \Xi$, where $x \circ \sigma = (x_1 \circ \sigma, x_2 \circ \sigma)$.

For every multi-index $k = (k_1, k_2) \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ let us define a mapping $R_k : (L_\infty)^2 \to \mathbb{C}$ by

$$R_k(x) = \int_{[0,1]} \prod_{\substack{s=1 \\ k_s>0}}^{2} (x_s(t))^{k_s}\, dt, \tag{2}$$

where $x = (x_1, x_2) \in (L_\infty)^2$. Note that R_k is a continuous symmetric $|k|$-homogeneous polynomial, where $|k| = k_1 + k_2$, and $\|R_k\| = 1$. By [10, Theorem 2], the set of polynomials $\{R_k : \ k \in \mathbb{Z}_+^2 \setminus \{(0,0)\}\}$ is an algebraic basis of the algebra $\mathcal{P}_s((L_\infty)^2)$ of all continuous symmetric polynomials on $(L_\infty)^2$.

Let A be an algebra of functions $f : D \to \mathbb{C}$, where the set D is such that $D \supset (L_\infty)^2$. For $x \in (L_\infty)^2$, let the mapping $\delta_x : A \to \mathbb{C}$ be defined by

$$\delta_x(f) = f(x),$$

where $f \in A$. The mapping δ_x is called a point-evaluation functional at the point x. Note that a point-evaluation functional is linear and multiplicative.

We shall use the following result.

Theorem 1. *(see [2, Section 3]) For every sequence $\xi = \{\xi_n\}_{n=1}^\infty \subset \mathbb{C}$ such that*

$$\sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|} < +\infty,$$

there exists $v_\xi \in L_\infty$ such that

$$\int_{[0,1]} (v_\xi(t))^n\, dt = \xi_n$$

for every $n \in \mathbb{N}$ and $\|x_\xi\|_\infty \le \frac{2}{M} \sup_{n\in\mathbb{N}} \sqrt[n]{|\xi_n|}$, where

$$M = \prod_{n=1}^\infty \cos\left(\frac{\pi}{2}\frac{1}{n+1}\right). \tag{3}$$

2 THE MAIN RESULT

Theorem 2. *For every mapping $c : \mathbb{Z}_+^2 \setminus \{(0,0)\} \to \mathbb{C}$ such that*

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |c(n)|^{1/|n|} < +\infty$$

there exists a function $x_c \in (L_\infty)^2$ such that $R_n(x_c) = c(n)$ for every $n \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ and

$$\|x_c\|_{\infty,2} \le \frac{24}{M^3} \sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |c(n)|^{1/|n|},$$

where M is defined by (3).

Proof. Let ε_k be the kth Rademacher function, that is, $\varepsilon_k(t) = \operatorname{sign}(\sin 2^k \pi t)$. It is well known (see, e.g., [1, p. 162] or [5, Chapter 3]) that the series $\sum_{k=1}^{\infty} \varepsilon_k(t) u_k$ is convergent almost everywhere on $[0,1]$ if and only if the series $\sum_{k=1}^{\infty} |u_k|^2$ converges. Consequently, the series $\sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{k+1}$ converges almost everywhere on $[0,1]$.

For every $n = (n_1, n_2) \in \mathbb{N}^2$ let us define a function $p_n : [0,1] \to \mathbb{C}^2$ by

$$p_n(t) = \left(\exp\Big(\frac{\mathrm{i}\pi}{2n_1} \sum_{k=1}^{\infty} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big), \exp\Big(\frac{\mathrm{i}\pi}{2n_2} \sum_{k=1}^{\infty} \frac{\varepsilon_{2k}(t)}{k+1} \Big) \right).$$

Note that the function p_n belongs to the space $(L_\infty[0,1])^2$ and $||p_n|| = 1$.

The sequence of the functions $\{p_n^{(l)}\}_{l=1}^{\infty}$, where

$$p_n^{(l)}(t) = \left(\exp\Big(\frac{\mathrm{i}\pi}{2n_1} \sum_{k=1}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big), \exp\Big(\frac{\mathrm{i}\pi}{2n_2} \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) \right),$$

converges pointwise to p_n. Therefore, for every $m = (m_1, m_2) \in \mathbb{N}^2$, according to the dominated convergence theorem,

$$R_m(p_n) = \lim_{l\to\infty} R_m(p_n^{(l)}).$$

Note that

$$\begin{aligned}
R_m(p_n^{(l)}) &= \int_{[0,1]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=1}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \frac{1}{2} \Big) \int_{[0,\frac{1}{2}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&+ \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \frac{-1}{2} \Big) \int_{[\frac{1}{2},1]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{2} \Big) \int_{[0,\frac{1}{2}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=1}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= 4 \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{2} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{2} \Big) \\
&\times \int_{[0,\frac{1}{4}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=2}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=2}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt \\
&= 4^2 \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{2} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{2} \Big) \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{3} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{3} \Big) \\
&\times \int_{[0,\frac{1}{4^2}]} \exp\Big(\frac{\mathrm{i}\pi}{2n_1} m_1 \sum_{k=3}^{l} \frac{\varepsilon_{2k-1}(t)}{k+1} \Big) \exp\Big(\frac{\mathrm{i}\pi}{2n_2} m_2 \sum_{k=3}^{l} \frac{\varepsilon_{2k}(t)}{k+1} \Big) dt = \ldots \\
&= 4^l \int_{[0,\frac{1}{4^l}]} dt \prod_{k=1}^{l} \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{k+1} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{k+1} \Big) \\
&= \prod_{k=1}^{l} \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{k+1} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{k+1} \Big).
\end{aligned}$$

Therefore,

$$R_m(p_n) = \prod_{k=1}^{\infty} \cos\Big(\frac{\pi m_1}{2n_1} \frac{1}{k+1} \Big) \cos\Big(\frac{\pi m_2}{2n_2} \frac{1}{k+1} \Big).$$

For $k \in \mathbb{N}$ and $j \in \{1, \dots, k\}$, let

$$a_{j,k} = \exp\Big(\frac{2\pi i j}{k}\Big).$$

For every $k \in \mathbb{N}$ let us define a function $S_k : [0,1] \to \mathbb{C}$ in the following way. For $t \in [\frac{j-1}{k}, \frac{j}{k}]$, where $j \in \{1, \dots, k\}$, let

$$S_k(t) = a_{j,k}.$$

Let $\mathrm{frac}(t)$ be the fractional part of a real number t. For every $n = (n_1, n_2) \in \mathbb{N}^2$ let us define a function $y_n : [0,1] \to \mathbb{C}^2$ by a formula

$$y_n(t) = \Big(S_{n_1}(t) p_{n,1}(\mathrm{frac}(n_1 n_2 t)), S_{n_2}(\mathrm{frac}(n_1 t)) p_{n,2}(\mathrm{frac}(n_1 n_2 t))\Big).$$

Note that $\|y_n\|_{\infty,2} = 1$. For every $m = (m_1, m_2) \in \mathbb{N}^2$, we have

$$R_m(y_n) = \int_{[0,1]} S_{n_1}^{m_1}(t) p_{n,1}^{m_1}(\mathrm{frac}(n_1 n_2 t)) S_{n_2}^{m_2}(\mathrm{frac}(n_1 t)) p_{n,2}^{m_2}(\mathrm{frac}(n_1 n_2 t)) dt =$$

$$= \sum_{j=1}^{n_1} a_{j,n_1}^{m_1} \int_{[\frac{j-1}{n_1}, \frac{j}{n_1}]} S_{n_2}^{m_2}(\mathrm{frac}(n_1 t)) p_{n,1}^{m_1}(\mathrm{frac}(n_1 n_2 t)) p_{n,2}^{m_2}(\mathrm{frac}(n_1 n_2 t)) dt.$$

Let us make the substitution $u = n_1 t - (j-1)$ in the jth integral. Then $n_1 t = u + j - 1$ and, consequently, $\mathrm{frac}(n_1 t) = \mathrm{frac}(u + j - 1) = \mathrm{frac}(u)$ and $\mathrm{frac}(n_1 n_2 t) = \mathrm{frac}(n_2 u + n_2(j-1)) = \mathrm{frac}(n_2 u)$. Therefore,

$$R_m(y_n) = \frac{1}{n_1} \sum_{j=1}^{n_1} a_{j,n_1}^{m_1} \int_{[0,1]} S_{n_2}^{m_2}(\mathrm{frac}(u)) p_{n,1}^{m_1}(\mathrm{frac}(n_2 u)) p_{n,2}^{m_2}(\mathrm{frac}(n_2 u)) du.$$

Note that

$$\int_{[0,1]} S_{n_2}^{m_2}(\mathrm{frac}(u)) p_{n,1}^{m_1}(\mathrm{frac}(n_2 u)) p_{n,2}^{m_2}(\mathrm{frac}(n_2 u)) du$$

$$= \sum_{r=1}^{n_2} a_{r,n_2}^{m_2} \int_{[\frac{r-1}{n_2}, \frac{r}{n_2}]} p_{n,1}^{m_1}(\mathrm{frac}(n_2 u)) p_{n,2}^{m_2}(\mathrm{frac}(n_2 u)) du.$$

Let us make the substitution $v = n_2 u - (r-1)$ in the rth integral. Then $n_2 u = v + r - 1$ and, consequently, $\mathrm{frac}(n_2 u) = \mathrm{frac}(v + r - 1) = \mathrm{frac}(v) = v$. Therefore,

$$R_m(y_n) = \frac{1}{n_1} \sum_{j=1}^{n_1} a_{j,n_1}^{m_1} \frac{1}{n_2} \sum_{r=1} n_2 a_{r,n_2}^{m_2} \int_{[0,1]} p_{n_1}^{m_1}(v) p_{n_2}^{m_2}(v) dv = \Big(\frac{1}{n_1} \sum_{j=1}^{n_1} a_{j,n_1}^{m_1}\Big)\Big(\frac{1}{n_2} \sum_{r=1}^{n_2} a_{r,n_2}^{m_2}\Big)$$

$$\times R_m(p_n) = \Big(\frac{1}{n_1} \sum_{j=1}^{n_1} a_{j,n_1}^{m_1}\Big)\Big(\frac{1}{n_2} \sum_{r=1}^{n_2} a_{r,n_2}^{m_2}\Big) \prod_{k=1}^{\infty} \cos\Big(\frac{\pi m_1}{2 n_1} \frac{1}{k+1}\Big) \cos\Big(\frac{\pi m_2}{2 n_2} \frac{1}{k+1}\Big).$$

If m_1 is not a multiple of n_1, then

$$\sum_{j=1}^{n_1} a_{j,n_1}^{m_1} = 0.$$

Similarly, if m_2 is not a multiple of n_2, then

$$\sum_{r=1}^{n_2} a_{r,n_2}^{m_2} = 0.$$

Let $m_1 = k_1 n_1$ and $m_2 = k_2 n_2$, where $k_1, k_2 \in \mathbb{N}$. Then

$$\frac{1}{n_1}\sum_{j=1}^{n_1} a_{j,n_1}^{m_1} = \frac{1}{n_2}\sum_{r=1}^{n_2} a_{r,n_2}^{m_2} = 1.$$

Therefore,

$$R_m(y_n) = \prod_{k=1}^{\infty} \cos\Big(\frac{\pi k_1}{2}\frac{1}{k+1}\Big)\cos\Big(\frac{\pi k_2}{2}\frac{1}{k+1}\Big).$$

If $k_1 > 1$ or $k_2 > 1$, then there is a multiplier $\cos\frac{\pi}{2} = 0$ in the given product. Thus $R_m(y_n) = 0$, if $m \neq n$. If $m = n$, then $R_m(y_n) = M^2$, where M is defined by (3).

For every $n = (n_1, n_2) \in \mathbb{N}^2$, let us define a function $z_n : [0,1] \to \mathbb{C}^2$ by

$$z_n = \frac{1}{\sqrt[|n|]{M^2}} y_n.$$

Note that

$$\|z_n\|_{\infty,2} = \frac{1}{\sqrt[|n|]{M^2}} \le \frac{1}{M^2}, \tag{4}$$

since $0 < M < 1$. For every $m \in \mathbb{N}^2$,

$$R_m(z_n) = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases} \tag{5}$$

Let us define sequences $\xi = \{\xi_l\}_{l=1}^{\infty}, \eta = \{\eta_l\}_{l=1}^{\infty} \subset \mathbb{C}$ by

$$\xi_l = 4c((l,0)) - 4\sum_{k=1}^{\infty}\frac{1}{k2^{k+1}}\sum_{j=1}^{k}\big(c((j,k-j))k2^{k+1}\big)^{l/k} R_{(l,0)}(z_{(j,k-j)}) \tag{6}$$

and

$$\eta_l = 4c((0,l)) - 4\sum_{k=1}^{\infty}\frac{1}{k2^{k+1}}\sum_{j=1}^{k}\big(c((j,k-j))k2^{k+1}\big)^{l/k} R_{(0,l)}(z_{(j,k-j)})$$

for $l \in \mathbb{N}$. Let us show that $\sup_{l\in\mathbb{N}} |\xi_l|^{1/l} < +\infty$ and $\sup_{l\in\mathbb{N}} |\eta_l|^{1/l} < +\infty$. Let

$$a = \sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |c(n)|^{1/|n|}.$$

Then $|c(n)| \le a^{|n|}$ for every $n \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$. By (1), $|R_{(l,0)}(z_{(j,k-j)})| \le \|R_{(l,0)}\| \|z_{(j,k-j)}\|_{\infty,2}^{l}$. By (4), taking into account the equality $\|R_{(l,0)}\| = 1$,

$$|R_{(l,0)}(z_{(j,k-j)})| \le \frac{1}{M^{2l}}.$$

Therefore,

$$|\xi_l| \le 4a^l + \frac{4a^l}{M^{2l}}\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}\big(k2^{k+1}\big)^{l/k}.$$

Note that $\sup_{k\in\mathbb{N}}(k2^{k+1})^{1/k} = 4$. Therefore, $k2^{k+1} \le 4^k$ for every $k \in \mathbb{N}$. Consequently,

$$\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}\big(k2^{k+1}\big)^{l/k} \le 4^l \sum_{k=1}^{\infty}\frac{1}{2^{k+1}} = \frac{4^l}{2}.$$

Therefore,

$$|\xi_l| \le 4a^l + \frac{2(4a)^l}{M^{2l}}.$$

Taking into account the estimate $0 < M < 1$,

$$4a^l + \frac{2(4a)^l}{M^{2l}} \le \frac{4a^l + 2(4a)^l}{M^{2l}} \le \frac{3(4a)^l}{M^{2l}} \le \frac{(12a)^l}{M^{2l}}.$$

Thus,

$$|\xi_l| \le \frac{(12a)^l}{M^{2l}}.$$

Analogically,

$$|\eta_l| \le \frac{(12a)^l}{M^{2l}}.$$

Since $\sup_{l\in\mathbb{N}} |\xi_l|^{1/l} \le 12a/M^2$ and $\sup_{l\in\mathbb{N}} |\eta_l|^{1/l} \le 12a/M^2$, by Theorem 1, there exist $v_\xi, v_\eta \in L_\infty$ such that

$$\int_{[0,1]} (v_\xi(t))^l\, dt = \xi_l \quad \text{and} \quad \int_{[0,1]} (v_\eta(t))^l\, dt = \eta_l \tag{7}$$

for every $l \in \mathbb{N}$ and

$$\|v_\xi\|_\infty, \|v_\eta\|_\infty \le \frac{24a}{M^3}. \tag{8}$$

For $k \in \mathbb{N}$ and $j \in \{1, \ldots, k\}$, let

$$\Delta_{j,k} = \Big(1 - \frac{1}{2^k} + \frac{j-1}{k2^{k+1}}, 1 - \frac{1}{2^k} + \frac{j}{k2^{k+1}}\Big)$$

and $h_{j,k} : \Delta_{j,k} \to (0,1)$ let be defined by

$$h_{j,k}(t) = \Big(t - \Big(1 - \frac{1}{2^k} + \frac{j-1}{k2^{k+1}}\Big)\Big)k2^{k+1}.$$

Note that $h_{j,k}$ is a bijection. Let us define a function $x_c : [0,1] \to \mathbb{C}^2$ by

$$x_c(t) = \begin{cases} (v_\xi(4t), 0), & \text{if } t \in (0, 1/4), \\ (0, v_\eta(4t-1)), & \text{if } t \in (1/4, 1/2), \\ \big(c((j, k-j))k2^{k+1}\big)^{1/k} z_{(j,k-j)}(h_{j,k}(t)), & \text{if } t \in \Delta_{j,k},\ k \in \mathbb{N},\ j \in \{1, \ldots, k\}, \\ (0,0), & \text{otherwise.} \end{cases}$$

Note that $x_c \in (L_\infty)^2$ and, taking into account estimations (4), (8) and the inequality $\big(c((j,k-j))k2^{k+1}\big)^{1/k} \le 4a$, we obtain

$$\|x_c\|_{\infty,2} \le \max\Big\{\frac{24a}{M^3}, \frac{4a}{M^2}\Big\}.$$

Since $0 < M < 1$, it follows that $4a/M^2 \le 4a/M^3 \le 24a/M^3$. Therefore, $\|x_c\|_{\infty,2} \le 24a/M^3$. Let us show that $R_n(x_c) = c(n)$ for every $n \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$. Consider the case $n = (n_1, n_2) \in \mathbb{N}^2$. In this case, taking into account (5),

$$\begin{aligned} R_n(x_c) &= \int_{(0,1/4)} (v_\xi(4t))^{n_1} 0^{n_2}\, dt + \int_{(1/4,1/2)} 0^{n_1} (v_\eta(4t-1))^{n_2}\, dt + \sum_{k=1}^{\infty} \sum_{j=1}^{k} \big(c((j,k-j))k2^{k+1}\big)^{|n|/k} \\ &\quad \times \int_{\Delta_{j,k}} \big(z_{(j,k-j),1}(h_{j,k}(t))\big)^{n_1} \big(z_{(j,k-j),2}(h_{j,k}(t))\big)^{n_2}\, dt \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{k} \big(c((j,k-j))k2^{k+1}\big)^{|n|/k} \frac{1}{k2^{k+1}} R_n(z_{(j,k-j)}) = \big(c((n_1,n_2))|n|2^{|n|+1}\big)^{|n|/|n|} \frac{1}{|n|2^{|n|+1}} \\ &= c((n_1, n_2)). \end{aligned}$$

Consider the case $n=(l,0)$, where $l\in\mathbb{N}$. In this case, taking into account (6) and (7),

$$\begin{aligned} R_n(x_c) &= \int_{(0,1/4)} (v_\xi(4t))^l\,dt + \int_{(1/4,1/2)} 0^l\,dt \\ &\qquad + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \big(c((j,k-j))k2^{k+1}\big)^{l/k} \int_{\Delta_{j,k}} \big(z_{(j,k-j),1}(h_{j,k}(t))\big)^l\,dt \\ &= \frac{1}{4}\int_{(0,1)} (v_\xi(t))^l\,dt + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \big(c((j,k-j))k2^{k+1}\big)^{l/k}\frac{1}{k2^{k+1}} R_{(l,0)}(z_{(j,k-j)}) \\ &= \frac{1}{4}\xi_l + \sum_{k=1}^{\infty}\sum_{j=1}^{k} \big(c((j,k-j))k2^{k+1}\big)^{l/k}\frac{1}{k2^{k+1}} R_{(l,0)}(z_{(j,k-j)}) = c((l,0)). \end{aligned}$$

Analogically, in the case $n=(0,l)$, where $l\in\mathbb{N}$, we have $R_n(x_c)=c((0,l))$. This completes the proof. □

Corollary 1. *Let A be a topological algebra of complex-valued functions on $(L_\infty)^2$, which contains the algebra $\mathcal{P}_s((L_\infty)^2)$ as a dense subalgebra. Let A be such that for each $x\in L_\infty$ the point-evaluation functional δ_x is continuous on A. Let $\varphi: A\to\mathbb{C}$ be a continuous linear multiplicative functional. Then φ is a point-evaluation functional if and only if*

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |\varphi(R_n)|^{1/|n|} < +\infty.$$

Proof. Let $\varphi: A\to\mathbb{C}$ be a continuous linear multiplicative functional such that

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |\varphi(R_n)|^{1/|n|} < +\infty.$$

By Theorem 2, there exists $x\in(L_\infty)^2$ such that $R_n(x)=\varphi(R_n)$ for every $n\in\mathbb{Z}_+^2\setminus\{(0,0)\}$, that is, $\delta_x(R_n)=\varphi(R_n)$ for every $n\in\mathbb{Z}_+^2\setminus\{(0,0)\}$. Since both δ_x and φ are linear and multiplicative, it follows that $\delta_x(P)=\varphi(P)$ for every $P\in\mathcal{P}_s((L_\infty)^2)$. Since both δ_x and φ are continuous and $\mathcal{P}_s((L_\infty)^2)$ is dense in A, it follows that $\delta_x=\varphi$.

Let $\varphi=\delta_x$ for some $x=(x_1,x_2)\in(L_\infty)^2$. By (1), for every $n=(n_1,n_2)\in\mathbb{Z}_+^2\setminus\{(0,0)\}$, $|\varphi(R_n)|=|R_n(x)|\le\|x\|^{|n|}$. Consequently,

$$\sup_{n\in\mathbb{Z}_+^2\setminus\{(0,0)\}} |\varphi(R_n)|^{1/|n|} \le \|x\|.$$

□

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Василишин Т.В. *Функціонали обчислення значень в точках на алгебрах симетричних функцій на просторі* $(L_\infty)^2$ // Карпатські матем. публ. — 2019. — Т.11, №2. — C. 493–501.

Відомо, що кожен неперервний симетричний (інваріантний відносно дії композиції аргумента з будь-якою вимірною за Лебегом бієкцією відрізка $[0,1]$, яка зберігає міру Лебега вимірних множин) поліном на декартовому степені комплексного банахового простору L_∞ всіх вимірних за Лебегом суттєво обмежених комплекснозначних функцій на відрізку $[0,1]$ може бути єдиним чином подано як алгебраїчну комбінацію, тобто лінійну комбінацію добутків, так званих елементарних симетричних поліномів. Як наслідок, кожен неперервний комплекснозначний лінійний мультиплікативний функціонал (характер) довільної топологічної алгебри функцій на декартовому степені простору L_∞, яка містить алгебру неперервних симетричних поліномів на декартовому степені простору L_∞ як щільну підалгебру, однозначно визначається своїми значенннями на елементарних симетричних поліномах. Тому задача опису спектра (множини всіх характерів) такої алгебри еквівалентна до задачі опису множин вищезгаданих значень характерів на елементарних симетричних поліномах.

В даній роботі розв'язано задачу опису множин значень характерів, які є функціоналами обчислення значення в точках, на елементарних симетричних поліномах на декартовому квадраті простору L_∞. Показано, що множини значень функціоналів обчислення значення в точках на елементарних симетричних поліномах задовольняють деяку природну умову. Також показано, що для кожної множини c комплексних чисел, яка задовольняє вищезгадану умову, існує елемент x декартового квадрата простору L_∞ такий, що значення функціонала обчислення значення в точці x на елементарних симетричних поліномах збігаються з відповідними елементами множини c.

Ключові слова і фрази: симетричний поліном, функціонал обчислення значення в точці.