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MEROMORPHIC MAPPINGS OF TORUS ONTO THE RIEMANN SPHERE

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The meromorphic mappings of the two dimensional torus onto the Riemann sphere are studied. Their connections with loxodromic meromorphic functions in the punctured plane are considered.

Let $\mathcal{T}$ be a two-dimensional torus in $\mathbb{R}^3$ obtained by the rotation of the unit circle in the $\xi O\zeta$ plane centered at $(l, 0)$, $l > 1$, around the ζ axis. Its parametric representation is

$$\begin{aligned} \xi &= (l + \cos\psi)\cos\varphi, \\ \eta &= (l + \cos\psi)\sin\varphi, \\ \zeta &= \sin\psi, \end{aligned} \tag{1}$$

$0 \le \varphi \le 2\pi$, $-\pi \le \psi \le \pi$.

The torus $\mathcal{T}$ can be obtained also by a continuous map τ of $\overline{A}_\rho$, $A_\rho = \{z : \rho < |z| < \frac{1}{\rho}\}$, in $\mathbb{R}^3$ so that τ is homeomorphic to the interior of A_ρ, $\tau(\overline{A}_\rho) = \mathcal{T}$, and $\tau(\rho e^{i\varphi}) = \tau(\frac{1}{\rho}e^{i\varphi})$, for each φ from $[0, 2\pi]$.

A mapping F of $\mathcal{T}$ into the Riemann sphere S is said to be *meromorphic* if there is a meromorphic function f on $\overline{A}_\rho$ such that $f = p\circ F\circ\tau$, where p is the stereographic projection of S on $\overline{\mathbb{C}}$.

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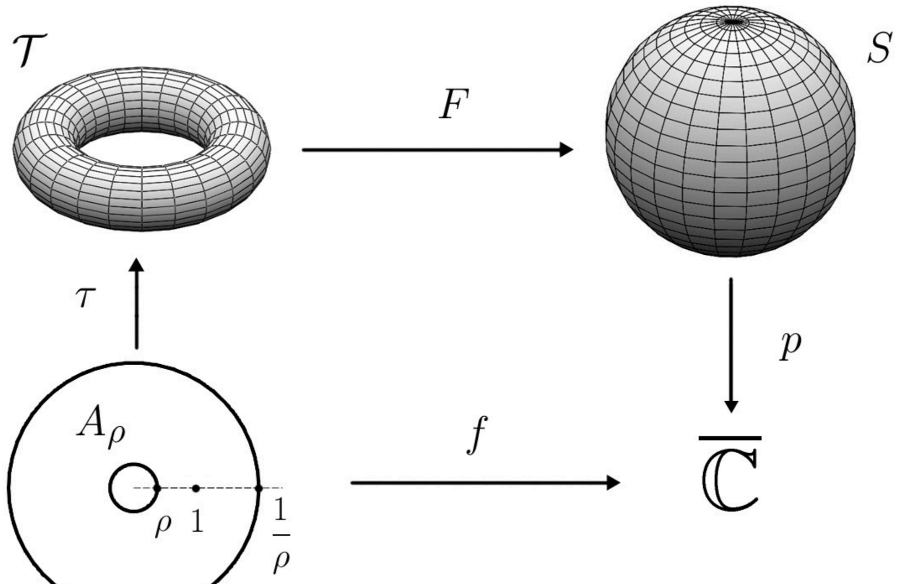

Put $q=\rho^2$. Since $\tau(qz)=\tau(z)$ for $|z|=\frac{1}{\rho}$ then

$$f(qz)=f(z). \tag{2}$$

Let us show that f has the meromorphic extension in $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ by relation (2). Indeed, put $g(z)=f(\frac{z}{\rho})$ and $\widetilde{g}(z)=g(qz)$. Since $g(z)$ is meromorphic in the closure of the annulus $\{z: q<|z|<1\}$ there is $\varepsilon>0$ such that $g(z)$ is meromorphic in the annulus $\{z: q-\varepsilon<|z|<1+\varepsilon\}$ and $\widetilde{g}(z)$ is meromorphic for z satisfying the inequalities $q-\varepsilon< q|z|<1+\varepsilon$, i.e. in the annulus

$$\Big\{z: 1-\frac{\varepsilon}{q}<|z|<\frac{1}{q}+\frac{\varepsilon}{q}\Big\}.$$

However, $g(e^{i\varphi})=g(qe^{i\varphi})=\widetilde{g}(e^{i\varphi})$. By the uniqueness theorem $\widetilde{g}(z)$ coincides with $g(z)$ in $\{z: 1-\frac{\varepsilon}{q}<|z|<1+\varepsilon\}$. Hence, $\widetilde{g}(z)$ is a meromorphic extension of $g(z)$ in the closure of the annulus $\{z: 1<|z|<\frac{1}{q}\}$. By induction f admits the meromorphic extension in $\mathbb{C}^*$ which satisfies (2).

Now we can conclude that to any meromorphic function on a torus corresponds a multiplicatively periodic (loxodromic) meromorphic function of a multiplicator q, $0<q<1$, and vise versa. A slight modification of our construction above shows that this is true not only for positive q but for arbitrary complex q satisfying the inequality $|q|<1$ [4], [5].

An example of loxodromic function of such a multiplicator is

$$f(z)=\sum_{n=-\infty}^{+\infty}\frac{q^n z}{(1-q^n z)^2}.$$

The *mapping* $\tau(z)$ is called of *conformal type* if there exists a positive continuous function $\varkappa(z)$ on $\overline{A}_\rho$ such that

$$ds_{\mathcal{T}}=\varkappa(z)|dz|,\quad z=re^{i\varphi}, \tag{3}$$

where $ds_{\mathcal{T}}$ is the length element on $\mathcal{T}$.

Lemma 1. *There are a unique l and a unique mapping τ of conformal type which maps $\overline{A}_\rho$ onto $\mathcal{T}$.*

Proof. We have $ds^2_{\mathcal{T}} = d\psi^2 + (l + \cos\psi)^2 d\varphi^2$, and $|dz|^2 = dr^2 + r^2 d\varphi^2$. Relation (3) implies

$$\begin{gathered} d\psi^2 + (l+\cos\psi)^2 d\varphi^2 = \varkappa^2(z)(dr^2 + r^2 d\varphi^2), \\ (l+\cos\psi) = \varkappa(z) r, \\ d\psi = \varkappa(z) dr, \end{gathered} \tag{4}$$

and, consequently,

$$\frac{dr}{r} = \frac{d\psi}{l+\cos\psi},$$

what together with the conditions $\psi(\rho) = -\pi$, $\psi(1/\rho) = \pi$, and the first of the relation (4) yields

$$\begin{gathered} \operatorname{tg}\frac{\psi(r)}{2} = \sqrt{\frac{l+1}{l-1}}\operatorname{tg}\left(\frac{\sqrt{l^2-1}}{2}\log r\right), \\ l = \sqrt{1+\frac{\pi^2}{\log^2\rho}}, \end{gathered}$$

and

$$\varkappa(z) = \frac{l+\cos\psi(r)}{r}, \quad r = |z|.$$

Thus, $\tau(z)$ is the mapping of conformal type by which to any $z = re^{i\varphi}$ from $\overline{A}_\rho$ corresponds a point from $\mathcal{T}$ given by (1) with

$$\psi = \psi(r) = 2\operatorname{arctg}\left(\sqrt{\frac{l+1}{l-1}}\operatorname{tg}\left(\frac{l^2-1}{2}\log r\right)\right).$$

□

If a loxodromic function $f(z)$ is not identically constant then it has two essential singularities at $z = 0$ and $z = \infty$ [5], [2]. In order to investigate its behaviour as $|z| \to 0$ and $|z| \to \infty$ consider the following geometric characteristics.

Let F be a nonconstant meromorphic mapping of $\mathcal{T}$ onto S. Let $|dF|_S$ be the length element by this mapping.

The *toroidal derivative* of F is said to be

$$\overset{\circ}{F}_{\mathcal{T}} := \frac{|dF|_S}{ds_{\mathcal{T}}}.$$

Let $d\sigma$, $d\sigma_S$, $d\sigma_{\mathcal{T}}$ be area elements on $\mathbb{C}^*$, S, and $\mathcal{T}$ respectively, $\tau(z)$ be of conformal type,

$$f = p\circ F\circ\tau, \quad G_r = \tau(A_{1/r}), \quad A_{1/r} = \{z : \frac{1}{r} \le |z| \le r\}.$$

The spherical area of the image of G_r is

$$A_{\mathcal{T}}(r,F) = \frac{1}{4\pi}\iint\limits_{G_r}\left(\frac{|dF|_S}{ds_{\mathcal{T}}}\right)^2 d\sigma_{\mathcal{T}} = \frac{1}{4\pi}\iint\limits_{G_r}(\overset{\circ}{F}_{\mathcal{T}})^2\, d\sigma_{\mathcal{T}}. \tag{5}$$

But the connection of F with $w = f(z)$ yields

$$\frac{|dF|_S}{ds_{\mathcal{T}}} = \frac{|dp^{-1}(w)|_S}{ds_{\mathcal{T}}} = \frac{|dp^{-1}(w)|_S}{|dw|}\frac{|dw|}{ds_{\mathcal{T}}} =$$
$$= \frac{|dp^{-1}(w)|_S}{|dw|}\left|\frac{dw}{dz}\right|\frac{1}{\varkappa(z)} = \overset{\circ}{f}_S(z)\frac{1}{\varkappa(z)} ,$$

where $\overset{\circ}{f}_S(z)$ is the spherical derivative of f.

The direct verification shows that $\overset{\circ}{F}_{\mathcal{T}}$ is multiplicatively periodic of multiplicator q.

It follows from (3) that

$$\frac{d\sigma_{\mathcal{T}}}{d\sigma} = \varkappa^2(z).$$

Taking into account (5) we have

$$A_{\mathcal{T}}(r, F) = \frac{1}{4\pi}\iint\limits_{\frac{1}{r}\le|z|\le r}(\overset{\circ}{f}(z))^2 d\sigma(z) = A_S(r, f) \ .$$

Here $A_S(r, f)$ is the spherical area of $A_{1/r}$ by the mapping f.

For any $a \in \overline{\mathbb{C}}$ the non-constant loxodromic function f of multiplicator q has the same number m of a-points, $m \ge 2$ in each annulus $\{z : qr < |z| \le r\}$ [5], [2].

This number m is called the *order* of f.

Thus, denoting by $n(r, a)$ the number of a-points of f in the annulus $\{z : \frac{1}{r} < |z| \le r\}$ we obtain [1]

$$A_S(r, f) = \frac{1}{4\pi}\int\limits_{\overline{\mathbb{C}}} n(r, a)d\sigma(a) \le 2nm \le 2m\frac{\log r}{\log\frac{1}{q}} + 2m \tag{6}$$

for r in the interval $(\frac{1}{q^{n-1}}, \frac{1}{q^n}]$.

Similarly,

$$2m\frac{\log r}{\log\frac{1}{q}} - 2m \le A_S(r, f). \tag{7}$$

The Ahlfors-Shimizu characteristic of f is [1], [3]

$$\overset{\circ}{T}(r, f) = \int\limits_1^r \frac{A_S(t, f)}{t}dt.$$

Relations (6) and (7) yield the following result.

Theorem 1. *Let f be a loxodromic function of multiplicator q and order m. Then its Ahlfors-Shimizu characteristic $\overset{\circ}{T}(r, f)$ satisfies the inequalities*

$$\frac{m}{\log\frac{1}{q}}\log^2 r - 2m\log r \le \overset{\circ}{T}(r, f) \le \frac{m}{\log\frac{1}{q}}\log^2 r + 2m\log r.$$

Note that

$$A_{\mathcal{T}}\left(\frac{1}{\rho}, F\right) = A_S\left(\frac{1}{\rho}, f\right) = m.$$

Therefore, the spherical area of the image of $\mathcal{T}$ is equal to m.

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Вивчаються мероморфні відображення двовимірного тора на сферу Рімана. Розглядаються їх зв'язки з локсодромними мероморфними функціями в проколеній площині.

Кондратюк А.А., Християнин А.Я. *Мероморфные отображения тора на сферу Римана* // Карпатские математические публикации. — 2012. — Т.4, №1. — С. 155–159.

Изучаются мероморфные отображения двухмерного тора на сферу Римана. Рассмотрены их связи с локсодромными мероморфными функциями в проколотой плоскости.