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RINGS WHOSE NON-ZERO DERIVATIONS HAVE FINITE KERNELS

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We prove that every infinite ring R is either differentially trivial or has a non-zero derivation d with an infinite kernel $\operatorname{Ker} d$.

Introduction

As usually a derivation d of an associative ring R is an additive mapping $d : R \to R$ which satisfies the Leibnitz rule, i.e.

$$d(ab) = d(a)b + ad(b)$$

for any $a, b \in R$. T. Laffey [3] has proved that an associative ring with finite commutative subrings is finite. From this it follows that a ring in which every non-zero inner derivation has a finite kernel is commutative or finite. We investigate here an associative rings designated in the title and prove the following

Theorem. *If a ring R is not differentially trivial and its every non-zero derivation has a finite kernel, then R is finite.*

Recall that a ring R having no non-zero derivations is called differentially trivial [1]. All rings are assumed to be associative. Henceforth, for any ring R (with an identity element) and its ideal I we denote by $N(R)$ the set of all nilpotent elements, $J(R)$ the Jacobson radical, $U(R)$ the unit group, $\operatorname{Ker} d = \{a \in R \mid d(a) = 0\}$ the kernel of d, $\operatorname{ann} I = \{a \in R \mid aI = \{0\}\}$ the left annihilator of I in R, $\operatorname{char} R$ the characteristic of R, 0_R the zero map.

Any unexplaned terminology is standard as in [4] and [5].

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1 Preliminaries

For the next we need some preliminary results. As defined in [2], a v-ring V is a commutative unramified complete (in the $J(V)$-adic topology) regular local rank one domain of characteristic zero with the residue field $V/J(V)$ of prime characteristic p.

Lemma 1.1. *Let R be a commutative local ring of prime power characteristic p^k for some $k \geq 1$. If $J(R)$ is a nil ideal of finite index in R, then*

$$R = J(R) + V,$$

where V is a finite ring which is a homomorphic image of some v-ring and

$$J(R) \cap V = pV.$$

Proof. Inasmuch as $R/J(R)$ is a finite field, there exists an element $\theta \in R$ such that the unit group

$$U(R/J(R)) = \langle \overline{\theta} \rangle$$

is cyclic generated by element $\overline{\theta} = \theta + J(R)$ of order $p^n - 1$. Then $\theta^{p^n} - \theta \in J(R)$, and so θ is a root of some non-zero polynomial $f \in \mathbb{Z}_{p^k}[X]$. Consequently

$$R = J(R) + \mathbb{Z}_{p^k}[\theta].$$

Since $\mathbb{Z}_{p^k}[\theta]$ is a finite local ring, by results of Cohen [2] (see Theorems 9 and 11)

$$\mathbb{Z}_{p^k}[\theta] = J(\mathbb{Z}_{p^k}[\theta]) + V,$$

where V is a finite ring which is a homomorphic image of some v-ring, $J(\mathbb{Z}_{p^k}[\theta]) \cap V = pV$ and this completes the proof. $\square$

Corollary 1.1. *Let R be a commutative local ring of prime characteristic p. If $J(R)$ is a nil ideal of finite index in R, then*

$$R = J(R) \oplus C$$

is a group direct sum, where C is a finite field.

Lemma 1.2. *Let R be a commutative ring, in which every non-zero derivation has a finite kernel. If R has a non-zero derivation, then $\operatorname{char} R$ is finite.*

Proof. In fact, if d is a non-zero derivation of R, then $1 \in \operatorname{Ker} d$. Since $\operatorname{Ker} d$ is a finite ring, we obtain that $n \cdot 1 = 0$ for some positive integer n. $\square$

Remark 1.1. *If d is a non-zero derivation of R with a finite kernel $\operatorname{Ker} d$, then in view of a group isomorphism*

$$\operatorname{Im} d \cong R/\operatorname{Ker} d$$

we deduce that the image $d(R) = \operatorname{Im} d = \{d(a) \mid a \in R\}$ is infinite.

2 PROOF OF THEOREM

Assume that a ring R has some non-zero derivation d. If R is not commutative, then by Theorem of Laffey [3] it is finite. Therefore in the next without loss of generality we can assume that R is infinite and commutative. As a consequence, $N(R)$ is an ideal of R and $N(R) \subseteq J(R)$. By Lemma 1.2 $\operatorname{char} R$ is finite. Without loss of generality let us $\operatorname{char} R = p^n$ for some prime p and integer $n \geq 1$. A set

$$A = \{a^{p^n} \mid a \in R\} \subseteq \operatorname{Ker} d$$

is finite, and so for any prime ideal P of R the quotient ring R/P is a finite field. This means that

$$pR \subseteq J(R) = N(R)$$

is a nil ideal. Inasmuch as for almost all elements $a, b \in R$ we have

$$\overline{0} = \overline{a}^p - \overline{b}^p = (\overline{a} - \overline{b})^p$$

in the quotient ring $\overline{R} = R/J(R)$, we deduce that $\overline{R}$ is finite. Therefore in the next we also assume that R is a local ring.

Let us $0 \neq v \in J(R)$ and $v^2 = 0$. Suppose that $vd \neq 0_R$. Since $|\operatorname{Ker} vd| < \infty$, we deduce that

$$vd(vd(R)) \neq \{0\} \tag{1}$$

and $vd(v)d(R)$ is an infinite set. By the same argument as above we obtain that

$$vd(v)^m d(R) \neq \{0\} \tag{2}$$

for any positive integer m.

1) If $p \neq 2$, then from $0 = d(v^2) = 2vd(v)$ it follows that $vd(v) = 0$, and this gives a contradiction with (1).

2) Let $p = 2$. If $d(v) \in U(R)$, then $ud(v) = 1$ for some invertible element $u \in U(R)$. If $\delta = ud$, then δ is a derivation of R, $\delta(v) = 1$,

$$0 = \delta(v^2) = 2v \text{ and } 0 = \delta(0) = \delta(2v) = 2\delta(v) = 2.$$

This yields that $n = 1$. By Corollary 1.1 for every element $r \in R$ there are unique elements $j \in J(R)$ and $c \in C$, where C is a finite field, such that

$$r = j + c. \tag{3}$$

Obviously that $\operatorname{ann} J(R) \neq \operatorname{ann}(J(R)^2)$. Then there exist

$$w_0 \in \operatorname{ann}(J(R)^2) \setminus \operatorname{ann} J(R) \text{ and } a \in J(R) \setminus J(R)^2$$

such that $aw_0 \neq 0$. Since $J(R) = \langle a \rangle \oplus K$ is a group direct sum, each element $j \in J(R)$ one can write

$$j = a_1 + k, \tag{4}$$

for unique elements $a_1 \in \langle a\rangle$ and $k \in K$. Then the rule

$$\mu(r + J(R)^2) = a_1 + J(R)^2$$

with r, a_1 as in (3) and (4), is a non-zero derivation of $R/J(R)^2$. The mapping $\chi : R \to R$ given by

$$\chi(r) = w_0 a_1 \ (r \in R)$$

determines a non-zero derivation of R, where $K \subseteq \operatorname{Ker}\chi$ is infinite. This contradiction yields that $d(v) \in J(R)$. But in view of (2) again we have a contradiction. Therefore $vd = 0_R$ for every $v \in J(R)$ such that $v^2 = 0$. By the same argument we can obtain that $J(R)d(R) = \{0\}$ and consequently

$$d(R)^2 = \{0\} \ \text{ and } \ d(J(R)^2) = \{0\}.$$

Hence the ideal $J(R)^2$ is finite. Furthermore,

$$d(pR) = pd(R) = \{0\},$$

and so an ideal pR is finite. By Lemma 1.1 $R = J(R) + V$, where V is a finite ring which is a homomorphic image of some v-ring and $J(R) \cap V = pV$.

a) Suppose that there is some element

$$t_0 \in J(R) \setminus \operatorname{ann}(J(R)^2 + pR) \text{ with } t_0 a \neq 0$$

for some $a \in J(R)$. Clearly the quotient ring $\widehat{R} = R/(J(R)^2 + pR)$ has a non-zero derivation. Since the quotient ring

$$\widehat{R} = \langle\widehat{a}\rangle \oplus \widehat{S} \oplus \widehat{V}$$

is a group direct sum for some subgroup $\widehat{S}$ of $J(\widehat{R})$, every element $\widehat{r} = r + J(R)^2 + pR \in \widehat{R}$ can be uniquely written in the form

$$\widehat{r} = \widehat{b} + \widehat{s} + \widehat{w}$$

for some elements $\widehat{b} = b + J(R)^2 + pR \in \langle\widehat{a}\rangle$, $\widehat{s} \in \widehat{S}$, $\widehat{w} \in \widehat{V}$. The rule

$$\delta(r + J(R)^2 + pR) = b + J(R)^2 + pR$$

determines a non-zero derivation δ. Then the mapping $\gamma : R \to R$ given by $\gamma(r) = t_0 b \ (r \in R)$ is a non-zero derivation of R with infinite $\operatorname{Ker}\gamma$, a contradiction.

b) Let us $J(R) = \operatorname{ann}(J(R)^2 + pR)$. Then $J(R)^3 = \{0\}$, $pJ(R) = \{0\}$ and

$$\overline{R} = R/J(R)^2 = J(\overline{R}) \oplus \overline{V}$$

is a group direct sum, where $\overline{V}$ is a finite field. Obviously $\overline{R} \neq \overline{V}$ and every element $r \in R$ can be uniquely written in the form $\overline{r} = \overline{\jmath} + \overline{z}$ for some $\overline{\jmath} \in J(\overline{R})$ and $\overline{z} \in \overline{V}$. There is $l_0 \in \operatorname{ann}(J(R)^2) \setminus \operatorname{ann} J(R)$ with $l_0 b \neq 0$ for some $b \in J(R)$. Inasmuch as $J(\overline{R}) = \langle\overline{b}\rangle \oplus \overline{N}$ is a group direct sum, element $\overline{\jmath} = \overline{b}_1 + \overline{m}$ for some $\overline{b}_1 = b_1 + J(R)^2 \in \langle\overline{b}\rangle$, $\overline{m} \in \overline{N}$ and the rule

$$\rho(r + J(R)^2) = b_1 + J(R)^2 \ (r \in R) \tag{5}$$

is a non-zero derivation ρ of $\overline{R}$. Then the mapping $\pi : R \to R$ given by $\pi(r) = l_0 b_1 \ (r \in R)$, where r and b_1 are as in (5), is a non-zero derivation of R with an infinite kernel $\operatorname{Ker}\pi$, a contradiction. $\square$

Corollary 2.1. *Let R be an infinite ring. Then R is differentially trivial or has a non-zero derivation d with an infinite kernel* $\operatorname{Ker} d$.

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Доведено, що кожне нескінченне кільце R диференційно тривіальне або має ненульове диференціювання d з нескінченним ядром $\operatorname{Ker} d$.

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Доказано, что каждое бесконечное кольцо R либо является дифференциально тривиальным, либо имеет ненулевое дифференцирование d с бесконечным ядром $\operatorname{Ker} d$.