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THE ANALYTICAL PROPERTIES OF SOLUTIONS OF THE THREE-BODY PROBLEM

The analytic properties of solutions of the three-body problem were investigated.

Introduction

The properties of solutions of differential equations with an analytic right-hand side depend on the singular points of these solutions, lying in the complex plane of time, in a great extent. For example, S.Kovalevskaya's classic solution ([3]) of the Euler – Poisson equations was found just during the investigation of single-valued solutions of this problem. A systematic research of the singular points of the solutions together with compactification of the flow ([10]), defined by the Euler – Poisson equations, allow us not only to find the partial solutions with given properties of their singular points ([9]) but also to investigate some global properties of these solutions ([11]). Therefore it is quite natural to use this approach for studying another classical problem of mechanics, namely, the three-body problem (see, for example, [4], [5], [14]).

The essence of this method lies in getting a compact holomorphic manifold with a structure of a one-dimensional foliation $\mathcal{F}$ with singular points, as the result of factorization of the flow of the phase space. Such a reduction of the problem leads to the loss of a true parametrization of solutions of the initial differential equations. However this loss is not important, because in case if we obtain an effective representation for layers of the foliation $\mathcal{F}$ we will be able to find the starting parametrization by means of single integration.

At the same time, compactification of this problem permits to consider the solutions in the large, that is very important for investigating any nonlinear differential equations. Besides, the singularities of the foliation $\mathcal{F}$ are correspondent to the ones of complex solutions of the initial problem, and can be studied efficiently.

Thus, in this paper we realize the factorization of the flow of the three-body problem, achieve the asymptotics of the singular points of the solution and the classification of these singular points for this problem.

When describing the results of this paper we can't but note that they do not seem to be impressive against the background of the numerous results, already achieved on the way to solving the three-body and the n-body problems (see bibliography in [2], [1]). All our results follow practically from the primary problem setting and do not guarantee drawing the same considerable results as those, that are already known, a priory. Moreover, as it is naturally to expect, some results received within the scope of our approach are known in another form. For example the characteristic system which determines the singular points of the foliation $\mathcal{F}$ is almost equivalent to the system which determines the Euler's and Lagrange's central configurations of n bodies ([12], [13], [16]). The asymptotics of the singular points of the solutions is almost the asymptotics of collisions of the bodies ([15]).

Nevertheless the fundamental difference of the present paper's results from the well known ones is that we begin to investigate systematically the solutions of the three-body problem as analytic functions defined on the whole complex plane that is not only on the stripe, containing the real axis. Thanks to the fact that we consider not only real solutions of the characteristic system we get most general asymptotic behavior of solutions in the singular points.

It is necessary to add that even in the beginning of investigation our method leads to the facts which show its significance. These facts are the classification of the singular points and the absence of the entire solutions. At last we obtain the next surprising result: for any mass of bodies the operators which determine linear approximations of the foliation $\mathcal{F}$ in any types of the singular points, have the identical whole eigenvalues and the identical conditions which determine correspondent eigenspaces.

1   PRELIMINARIES

We consider the problem (see, for example, [4], [5]) on n bodies moving (m_1, r_1), ... (m_n, r_n), $m_i \in \mathbb{R}^+, r_i \in \mathbb{R}^3$ which move by the law of gravity. Kinetic and potential energy are correspondingly equal

$$T = \frac{1}{2} \sum m_i \dot{r}_i^2, \quad U = -\frac{G}{2} \sum_{j,k} \frac{m_j m_k}{|r_j - r_k|}.$$

Lagrangian $L = T - U$ determines the following system of differential equations:

$$m\ddot{r}_i = -Gm_i \sum_j m_j \frac{r_i - r_j}{|r_i - r_j|^3}. \tag{1}$$

In the Hamiltonian form

$$\mathcal{H} = \sum_i \dot{r}_i \frac{\partial L}{\partial \dot{r}_i} - L = \frac{1}{2} \sum m_i \dot{r}_i^2 - \frac{G}{2} \sum_{j,k} \frac{m_j m_k}{|r_j - r_k|},$$

canonical coordinates are

$$q_i = r_i, \quad p_i = \frac{\partial L}{\partial \dot{r}_i} = m_i \dot{r}_i$$

and a Hamiltonian system has the next form:

$$\begin{cases} \dot{q}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} = \dfrac{p_i}{m_i} \\ \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial q_i} = -Gm_i \sum_j m_j \dfrac{q_i - q_j}{|q_i - q_j|^3}. \end{cases} \tag{2}$$

The first integrals of system (2) are

$$\mathcal{H}, \quad \mathcal{I} = \sum m_i \dot{q}_i, \quad \mathcal{M} = \sum q_i \times p_i.$$

In the classic notation (1), (2) of the n-body problem in the right-hand side there is the module which is a local real-analytic function. As we want to consider the n-body problem for complex time, the right-hand side of the differential equations is necessary to be complex-analytical. Therefore we consider the function module for the vector $q \in \mathbb{C}^3$ as a complex-analytical function which is determined by the formula:

$$|q| = \sqrt{q_1^2 + q_2^2 + q_3^2}.$$

We use the next notation for a norm of the vector :

$$\|q\| = \sqrt{q_1 \bar{q}_1 + q_2 \bar{q}_2 + q_3 \bar{q}_3}.$$

2   THE FACTORIZATION OF THE FLOW OF THE n-BODY PROBLEM

Let $p_i(t), q_i(t), \ i = 1, 2, 3$ be a solution of the n-body problem then $\alpha p_i(\alpha^3 t), \ \alpha^{-2} q_i(\alpha^3 t)$ is a solution too. This fact gives a possibility for factorization on the set of trajectories of the system (2).

Remark 2.1. *The flow of the problem (2) allows the factorization by the action of the orthogonal group $So(3, \mathbb{C})$ ([8])*

$$p_i \to A p_i, \ q_i \to A q_i, \ A \in So(3, \mathbb{C}),$$

but using of this does not give any visible technic preferences.

The following proposition is well known.

Proposition 2.1. *Let $\mathbb{C}$ act as a transformation group on $\mathbb{C}^n$ in the following way:*

$$\alpha : (z_1, ..., z_n) \to (\alpha^{k_1} z_1, ..., \alpha^{k_n} z_n),$$

$k = (k_1, ..., k_n) \in \mathbb{N}^n$. Then the factor-space $P_k^{n-1} = \{(z_1^{k_1} : ... : z_n^{k_n})\}$ is a compact holomorphic manifold ([7]) with respect to this action.

Proposition 2.2. *The projection*

$$\pi : \mathbb{C}^{3n} + \mathbb{C}^{3n} \to P_*^{6n-1},$$

where $* = (\overbrace{1,..,1}^{n}, \overbrace{2,..,2}^{n})$, *is determined by the next formula:*

$$\pi : (p_1, ..., p_n, q_1, ..., q_n) \to (p_1 : ... : p_n : \frac{q_1}{|q_1|^2} : ... : \frac{q_n}{|q_n|^2}),$$

and induces the structure of a holomorphic $\mathbb{C}$*-one-dimension foliation ([6])* $\mathcal{F}$ *of the compact holomorphic manifolds* P_*^{6n-1}.

Proof. According to the definition of π the vector

$$(\alpha p_1, ..., \alpha p_n, \alpha^{-2} q_1, ..., \alpha^{-2} q_n)$$

is projected onto

$$(\alpha p_1 : ... : \alpha p_n : \alpha^2 \frac{q_1}{|q_1|^2} : ... : \alpha^2 \frac{q_n}{|q_n|^2}).$$

$\square$

Remark 2.2. *The projection* π *can be defined by more natural means:*

$$\pi : (p_1, ..., p_n, q_1, ..., q_n) \to (p_1 : ... : p_n : q_1 : ... : q_n),$$

but in this case the image of this mapping is a noncompact manifold.

Remark 2.3. *We can use the mapping* π^{-1} *which has the next presentation:*

$$\pi^{-1} : (p_1 : ... : p_n : q_1, ..., q_n) \to (p_1 : ... : p_n : \frac{q_1}{|q_1|^2} : ... : \frac{q_n}{|q_n|^2})$$

if it is necessary.

Remark 2.4. *The foliation* $\mathcal{F}$ *is integrable as there is the invariant mapping* $J : P_*^{6n-1} \backslash X \to P^6$,

$$J : \eta \to (\mathcal{H}(\xi) : \mathcal{I}_1^2(\xi) : \mathcal{I}_2^2(\xi) : \mathcal{I}_3^2(\xi) : \mathcal{M}_1^2(\xi) : \mathcal{M}_2^2(\xi) : \mathcal{M}_3^2(\xi)),$$

where $\xi = \pi^{-1}(\eta)$,

$$X = \{\eta \in P_*^{6n-1} : \mathcal{H}(\pi^{-1}(\eta)) = 0, \mathcal{I}(\pi^{-1}(\eta)) = 0, \mathcal{M}(\pi^{-1}(\eta)) = 0\}.$$

Moreover the surface X *is fiber invariant for the foliation* $\mathcal{F}$ *too.*

Proposition 2.3. *All the singular points of the foliation* $\mathcal{F}$ *are the following:*
   π*-projections of the solutions* $(\tilde{p}^0, \tilde{q}^0)$ *of the characteristic system (3),*
   π*-projections of the singular points of the system (2), i.e. the points*

$$\{\pi(p_1, ..., p_n, q_1, ..., q_n) : \exists i, j \ |q_i - q_j| = 0\}$$

and

$$\{(p_1 : ... : p_n : q_1 : ... : q_n) \in P_*^{6n-1} : \exists i \ |q_i| = 0\}.$$

Proof. Evidently singular points of the equation (2)

$$\{\pi(p_1, ..., p_n, q_1, ..., q_n) : \exists i, j \ \ |q_i - q_j| = 0\}$$

are projected onto singular points of the foliation $\mathcal{F}$. Moreover if the vector $(\dot{p}, \dot{q})$ touches the π-pre-image of $\pi(p, q)$ the point $\pi(p, q)$ will be a singular point of the foliation too. Such points satisfy the system

$$
\begin{cases}
-2\,\widetilde{q}_i + 3\dfrac{\widetilde{p}_i}{m_i} = 0, \\[2mm]
\widetilde{p}_i - 3\,Gm_i \sum_j m_j \dfrac{\widetilde{q}_i - \widetilde{q}_j}{|\,\widetilde{q}_i - \widetilde{q}_j\,|^3} = 0.
\end{cases}
\tag{3}
$$

We call this system characteristic. At last the points of the manifold P_*^{6n-1}, which haven't got a pre-image, i.e. the points $\{(p_1 : ... : p_n : q_1 : ... : q_n) : \exists i \ |q_i| = 0\}$, may be singular points. $\qquad \square$

Remark 2.5. *The solution of the characteristic system (3) designates the central configuration specifying some partial solutions of the three-body problem that were known to Euler and Lagrange ([12], [13], [16]).*

Further we make some calculations for the 3-body problem.

3   THE CHANGE OF VARIABLES FOR THE 3-BODY PROBLEM

In this case the problem has dimension 18. Using the first integrals, this dimension can be lowered to 8 ([17]). However the differential equations derived in such a way are rather inconvenient for any further investigations, and besides, the dimensionality of the reduced problem remains rather high. Therefore we change the variables without lowering the problem's dimensionality trying to make the equations, defining this problem, as simple as possible.

At first we make the change of variables, switching to relative coordinates:

$$
\begin{cases}
x_1 = (q_2 - q_3) \cdot G^{-1/3}, \sigma \\[2mm]
y_1 = \left(\dfrac{p_2}{m_2} - \dfrac{p_3}{m_3}\right) \cdot G^{-1/3}, \sigma
\end{cases}
\tag{4}
$$

here and lower σ denotes the circle permutation of indexes (1, 2, 3). The system (2) takes the next form

$$
\begin{cases}
\dot{x}_1 = y_1, \sigma \\[2mm]
\dot{y}_1 = (m_1(\dfrac{x_2}{|x_2|^3} + \dfrac{x_3}{|x_3|^3}) - (m_2 + m_3)\dfrac{x_1}{|x_1|^3}), \sigma.
\end{cases}
\tag{5}
$$

If the system (5) is solved it will be simple to find p_i, q_i. In this case we express the variables p_i through y_i and the value of the first integral $\mathcal{I} = p_1 + p_2 + p_3$. Then we find $q_1 = \int p_1 dt\sigma$, and we may consider the initial values q_i to be known.

Now let us suppose that

$$\dot{u} = -\sum_{\sigma} \frac{x_1}{|x_1|^3}, \quad m = \sum_{\sigma} m_1, \quad mz_1 = y_1 - m_1 u, \, \sigma,$$

then

$$\begin{cases} \dot{z}_1 = -\dfrac{x_1}{|x_1|^3}, \sigma \\ \dot{x}_1 = mz_1 + m_1 u, \sigma \\ \dot{u} = \sum_{\sigma} \dfrac{x_1}{|x_1|^3}. \end{cases} \tag{6}$$

Since

$$u + \sum_{\sigma} z_1 = \frac{1}{m} \sum_{\sigma} y_1 = \frac{1}{m} \sum_{\sigma} \left(\frac{p_2}{m_2} - \frac{p_3}{m_3} \right) G^{-1/3} \equiv 0,$$

then $u = -\sum_{\sigma} z_1$ and finally we get the following system:

$$\begin{cases} \dot{x}_1 = mz_1 - m_1 \sum_{\sigma} z_1, \sigma \\ \dot{z}_1 = -\dfrac{x_1}{|x_1|^3}, \sigma. \end{cases} \tag{7}$$

Theorem 1. *The system of differential equations (7) is equivalent to the three-body problem (2) and is a canonical Hamiltonian system with coordinates $(\frac{x_i}{m_i}, z_i)$. Its Hamiltonian has the form:*

$$\mathcal{H} = \frac{1}{2} \left(\sum_{\sigma} \frac{mz_1^2}{m_1} - \left(\sum_{\sigma} z_1 \right)^2 \right) - \sum_{\sigma} \frac{1}{m_1 |x_1|}.$$

The system of differential equations (7) has the following first integrals:

$$\mathcal{I}_1 = \sum_{\sigma} x_1, \quad \mathcal{I}_2 = \sum_{\sigma} \frac{1}{m_1} z_1 \times x_1.$$

Proof.

$$\mathcal{I}_1 = \sum_{\sigma} (q_1 - q_2) G^{-1/3} \equiv 0.$$

$$\dot{\mathcal{I}}_2 = \sum_{\sigma} \frac{1}{m_1} (\dot{z}_1 \times x_1 + z_1 \times \dot{x}_1) = \sum_{\sigma} \left[\frac{1}{m_1} \left(-\frac{x_1}{|x_1|^3} \times x_1 + z_1 \times \left(mz_1 - m_1 \sum_{\sigma} z_1 \right) \right) \right]$$

$$= \sum_{\sigma} \left(z_1 \times \sum_{\sigma} z_1 \right) = \sum_{\sigma} z_1 \times \sum_{\sigma} z_1 = 0.$$

The fact that the system (7) is canonical can be tested by straight calculations.   □

Now we make one more change of variables ([10]) which allows to investigate the asymptotics of the singular points of solutions of the three-body problem.

Let $t_* \in \mathbb{C}$ be a singular point of the solution $(x_i(t), z_i(t))$ (i. e. t_* is a singular point of one of the coordinate functions of $(x_i(t), z_i(t))$). Get rid of branching in t_*, if any, by representing $x_i(t) = \hat{x}_i(Ln(t - t_*)^\alpha)$, $z_i(t) = \hat{z}_i(Ln(t - t_*)^\alpha)$, where $\hat{x}_i(\tau)$, $\hat{z}_i(\tau)$ are single-valued functions for $Re\, \tau \to -\infty$.

The system (7) is transformed into

$$
\begin{cases}
\dot{\hat{x}}_1 = \frac{1}{\alpha}\, e^{\tau/\alpha}\,(mz_1 - m_1 \sum_\sigma z_1),\, \sigma \\[2mm]
\dot{\hat{z}}_1 = -\frac{1}{\alpha}\, e^{\tau/\alpha}\, \dfrac{\hat{x}_1}{|\hat{x}_1|^3},\, \sigma;
\end{cases}
$$

where the derivative is taken with respect to τ.

In order to make the right-hand side of the equation independent of τ, we make a replacement of the variable in the following form: $\tilde{x}_i(\tau) = e^{\beta\tau}\, \hat{x}_i(\tau)$, $\tilde{z}_i(\tau) = e^{\gamma\tau}\, \hat{z}_i(\tau)$. Then we have

$$
\begin{cases}
\dot{\tilde{x}}_1 = \beta\, \tilde{x}_1 + \frac{1}{\alpha} e^{\tau\beta + \tau/\alpha}\,(m\, \tilde{z}_1 - m_1 \sum_\sigma \tilde{z}_1),\, \sigma \\[2mm]
\dot{\tilde{z}}_i = \gamma\, \tilde{z}_1 - \frac{1}{\alpha} e^{\tau\gamma + \tau/\alpha}\, \dfrac{\hat{x}_1}{|x_1|^3},\, \sigma.
\end{cases}
$$

We see that the right-hand side is independent of τ, if $\beta + \frac{1}{\alpha} = \gamma$, $\gamma + \frac{1}{\alpha} = -2\beta$, i. e. if $\gamma = \frac{1}{3\alpha}, \beta = -\frac{2}{3\alpha}$. Taking $\alpha = \frac{1}{3}$, we obtain the following system:

$$
\begin{cases}
\dot{\tilde{x}}_1 = -2\, \tilde{x}_1 + 3\,(m\, \tilde{z}_1 - m_1 \sum_\sigma \tilde{z}_1),\, \sigma \\[2mm]
\dot{\tilde{z}}_1 = \tilde{z}_1 - 3\, \dfrac{\tilde{x}_1}{|\, \tilde{x}_1\, |^3},\, \sigma.
\end{cases}
\tag{8}
$$

The dependence between the differential systems (7) and (8) is expressed by the next relations:

$$
z_i(t) = (t - t_*)^{-1/3} \tilde{z}_i\big(\tfrac{1}{3} Ln(t - t_*)\big), \quad x_i(t) = (t - t_*)^{2/3} \tilde{x}_i\big(\tfrac{1}{3} Ln(t - t_*)\big).
\tag{9}
$$

Proposition 3.1. *The solution of the system (7) has not got a singularity at the point t_* if and only if the corresponding solutions of (8) by (9) have the asymptotic behavior $\tilde{x}_i \sim \tilde{x}_{i0} e^{-2\tau}, \tilde{z}_i \sim \tilde{z}_{i0} e^\tau$ for $Re\, \tau \to -\infty$.*

Proof. It is enough to substitute the asymptotics $\tilde{x}_i \sim \tilde{x}_{i0} e^{-2\tau}, \tilde{z}_i \sim \tilde{z}_{i0} e^\tau$ to (9). $\qquad\square$

Remark 3.1. *If we know the asymptotic behavior of the solutions $\tilde{x}_i, \tilde{z}_i$ in the neighbourhood of the singular points, we will be able to obtain the asymptotics of the singular points of the solutions (7) having the representation (9).*

4   The singular points of the foliation $\mathcal{F}$ for the 3-body problem

The projection π from the Proposition 2.2 for the system (7) has the following form:

$$
\pi : (z_1, z_2, z_3, x_1, x_2, x_3)
$$

$$
\to (z_1 : z_2 : z_3 : \frac{x_1}{|x_1|^2} : \frac{x_2}{|x_2|^2} : \frac{x_3}{|x_3|^2}) = (z_1 : z_2 : z_3 : w_1 : w_2 : w_3).
$$

The correspondent system of differential equations for the fiber of $\mathcal{F}$ on the manifold P_*^{17} obtains the following form:

$$\begin{cases} \dot{w}_1 = -2w_1(w_1, mz_1 - m_1 \sum_\sigma z_1) + |w_1|^2(mz_1 - m_1 \sum_\sigma z_1), \sigma \\ \dot{z}_1 = -w_1|w_1|, \sigma. \end{cases} \quad (10)$$

The system (10) is suitable for the investigation of the three-body problem because this system is defined on the compact manifold and its right-hand side is always determined. At the same time the system (7) is more convenient for calculations, as essentially more simple.

Proposition 4.1. *The projections* $\pi(\tilde{x}_i^0, \tilde{z}_i^0)$ *of the solutions of the characteristic system*

$$\begin{cases} -2\,\overset{\sim 0}{x}_1 + 3\,(m\,\overset{\sim 0}{z}_1 - m_1 \sum_\sigma \overset{\sim 0}{z}_1) = 0, \sigma \\ \overset{\sim 0}{z}_1 - 3\,\dfrac{\overset{\sim 0}{x}_1}{|\,\overset{\sim 0}{x}_1\,|^3} = 0, \sigma \end{cases} \quad (11)$$

are the singular points of the foliation $\mathcal{F}$. *In addition to this, the points* $(z_1 : z_2 : z_3 : w_1 : w_2 : w_3)$, *satisfying the condition* $|w_i| = 0$ *for some* i *may also be the singular points of this foliation.*

Proof. This Proposition repeats the Proposition 2.3 for the three-body problem (10).

As for the singular points of the form $\{(z_1 : z_2 : z_3 : w_1 : w_2 : w_3) \in P_*^{17} : \exists i |w_i| = 0\}$, which pretend to be singular points thanks to singularity of the functions $|w_i|$, these points form an invariant surface in P_*^{17}. We can make sure that it is true by finding the derivative of the function $|w_i|^2$ along the vector field (10). This derivative is identically equal to zero on the surface $|w_i|^2 = 0$. $\qquad\square$

The roots of the characteristic system (11) were already known to Euler and Lagrange. We present their finding for completing the paper taking into account that this finding is simple.

Substitute $\tilde{z}_1$ into the first equation of (10). Omitting the sign $\sim$ for simplicity we get

$$\frac{x_1}{m_1} = \frac{9}{2}\left(\frac{m}{m_1}\frac{x_1}{|x_1|^3} - \sum_\sigma \frac{x_1}{|x_1|^3}\right), \sigma.$$

Then we subtract these equations one from another and get:

$$\frac{x_1}{m_1} - \frac{x_2}{m_2} = \frac{9m}{2}\left(\frac{x_1}{|x_1|^3 m_1} - \frac{x_2}{|x_2|^3 m_2}\right), \sigma$$

or

$$\frac{x_1}{m_1}\left(1 - \frac{9m}{2|x_1|^3}\right) = \frac{x_2}{m_2}\left(1 - \frac{9m}{2|x_2|^3}\right) = \frac{x_3}{m_3}\left(1 - \frac{9m}{2|x_3|^3}\right). \quad (12)$$

Let all the vectors x_i be collinear. Let also $x_1, x_2 > 0, x_3 < 0$. Denote $x_2 = \rho x_1, x_3 = -(1 + \rho)x_1$ and substitute x_2, x_3 into (4.3). Then we get following quintic polynomial:

$$(m_2 + m_3)\rho^5 + (3m_2 + 2m_3)\rho^4 + (3m_2 + m_3)\rho^3 -$$

$$(3m_1 + m_3)\rho^2 - (3m_1 + 2m_3)\rho - (m_1 + m_3) = 0. \tag{13}$$

Similarly it is possibly to consider all the rest cases when vectors x_i are collinear.

Now let vectors x_i be non-collinear then

$$|x_1|^3 = |x_2|^3 = |x_3|^3 = \frac{2}{9m}$$

and besides $x_1 + x_2 + x_3 = 0$.

The singular points of the foliation $\mathcal{F}$ are interesting primarily for finding the asymptotic behaviors of the singular points of the three-body problem's solutions.

Definition 4.1. *The singular points $\pi(\tilde{x}^0, \tilde{z}^0)$ of the foliation $\mathcal{F}$ where $(\tilde{x}^0, \tilde{z}^0)$ is a root of the characteristic system with non-collinear vectors x_i (11) will be called α-singular points. The singular points $\pi(\tilde{x}^0, \tilde{z}^0)$ of the foliation $\mathcal{F}$ where $(\tilde{x}^0, \tilde{z}^0)$ is a root of the characteristic system with collinear vectors x_i (11) will be called β-singular points.*

Let $(\tilde{x}^0, \tilde{z}^0)$ be a solution of the characteristic system (11). At the same time it is a singular point of the system of differential equations (8). The linearized system (8) has the following form in the neighbourhood of such a point:

$$\begin{cases} \dot{\tilde{x}}_1 = -2\,\tilde{x}_1 + 3\left(m\,\tilde{z}_1 - m_1 \sum_\sigma \tilde{z}_1\right), \sigma \\ \dot{\tilde{z}}_1 = \tilde{z}_1 - 3\dfrac{\tilde{x}_1}{|\tilde{x}_1^0|^3} + 9\dfrac{\tilde{x}_1^0}{|\tilde{x}_1^0|^5}(\tilde{x}_1^0, \tilde{x}_1), \sigma. \end{cases} \tag{14}$$

5   The asymptotic behavior of α-singular points of the foliation $\mathcal{F}$

Suppose that $|\tilde{x}_1^0|^3 = |\tilde{x}_2^0|^3 = |\tilde{x}_1^0|^3 = \frac{9m}{2} = a^3$ then the linear system for finding the eigenvectors of the right-hand side (14) has the following form:

$$\begin{cases} (m_2 + m_3)z_1 - m_1(z_2 + z_3) = \dfrac{\lambda+2}{3}x_1\sigma \\ -\dfrac{3}{a^3}x_1 + \dfrac{9}{a^5}(\tilde{x}_1^0, x_1)\,\tilde{x}_1^0 = (\lambda - 1)z_1, \sigma. \end{cases} \tag{15}$$

Theorem 2. *The eigenvalues and the eigenvectors, corresponding to these eigenvalues, of the linear system (15) are the following:*

$\lambda = 1$

The eigenvector u_1 has the form:

$$x_1 = 0, \; z_1 = m_1 r, \; \sigma, \; r \in \mathbb{C}^3.$$

The dimension of the eigenspace is equal to 3.

$\lambda(\lambda + 1) = 0$

The eigenvectors u_0, u_{-1} satisfy the conditions:

$$x_1 \perp \tilde{x}_1^0, \; z_1 = \frac{3}{a^3(1 - \lambda)}, \; \sigma.$$

The dimension of the eigenspaces for $\lambda = 0$ and $\lambda = -1$ is equal to 3.

$(\lambda + 3)(\lambda - 2) = 0$.

The eigenvectors u_{-3}, u_2 have the form: $x_1 = \nu\tilde{x}_1^0, z_1 = \mu\tilde{z}_1^0, \ \sigma, \ 2\mu = (\lambda + 2)\nu$. The dimension of the eigenspace for $\lambda = -3$ and $\lambda = 2$ is equal to 1.

The rest of the eigenvalues $\lambda_k, \ k = 1, ..., 4$ are the roots of the equation (compare with §16, [14])

$$\lambda(\lambda + 1)(\lambda + 3)(\lambda - 2) + \frac{27}{m^2} \sum_\sigma m_1 m_2 = 0.$$

The dimension of the eigenspaces for these λ is equal to 1.

It is possible to select a proper basis from all the eigenvectors, mentioned above.

Proof. $\lambda = 1$.

We get the next presentation from the second equation of the system (15):

$$x_1 = \frac{3}{a^2}(\tilde{x}_1^0, x_1)\tilde{x}_1^0 = \mu\tilde{x}_1^0 = \frac{3\mu}{a^2}(\tilde{x}_1^0, \tilde{x}_1^0)\tilde{x}_1^0 = 3\mu\tilde{x}_1^0, \ \sigma.$$

We see that $\mu = 0$ hence $x_1 = 0, \sigma$. We find the vectors z_i from the first equation of the system (15).

$\lambda \neq 1$.

Substitute the presentation of z_i taken from the second equation of the system (15) to the first equation. Then we use the relation $\sum_\sigma \tilde{x}_1^0 = \sum_\sigma x_1 = 0$. As a result we obtain the following system:

$$(16) \quad \begin{cases} m(\tilde{x}_1^{\,0}, x_1)\, \tilde{x}_1^{\,0} + m_1((\tilde{x}_2^{\,0}, x_1 + x_2) + (\tilde{x}_1^{\,0}, x_2))\, \tilde{x}_1^{\,0} \\ + m_1((\tilde{x}_1^{\,0}, x_1 + x_2) + (\tilde{x}_2^{\,0}, x_1))\, \tilde{x}_2^{\,0} = \frac{1}{27}\, a^5\, \lambda(\lambda + 1)x_1 \\ m(\tilde{x}_2^{\,0}, x_2)\, \tilde{x}_2^{\,0} + m_2((\tilde{x}_1^{\,0}, x_1 + x_2) + (\tilde{x}_2^{\,0}, x_1))\, \tilde{x}_2^{\,0} \\ + m_2((\tilde{x}_2^{\,0}, x_1 + x_2) + (\tilde{x}_1^{\,0}, x_2))\, \tilde{x}_1^{\,0} = \frac{1}{27}\, a^5\, \lambda(\lambda + 1)x_2. \end{cases}$$

$\lambda = -1$ or $\lambda = 0$.

Multiply the first and the second equations of the system (16) by m_2 and m_1 correspondingly and then subtract the second equation from the first one. We get:

$$m_2\, \tilde{x}_1^{\,0}\, (\tilde{x}_1^{\,0}, x_1) - m_1\, \tilde{x}_2^{\,0}\, (\tilde{x}_2^{\,0}, x_2) = 0 \Rightarrow (\tilde{x}_1^{\,0} \perp x_1), \ \sigma. \quad (17)$$

Now we can state with assurance that the system (15) is true if $\lambda(\lambda + 1) = 0$ and the orthogonal condition (17) is held true.

The vectors x_i have two components: the first component lies on the plane $\{\tilde{x}_1^0, \ \sigma\}$ and the second component lies on the plane that is orthogonal to the first plane. There is only one free parameter for the components lying in the first plane taking into account that $\sum_\sigma x_1 = 0$ and there are two free parameters for the components, lying on the second plane on the same condition.

$\lambda = -3$ or $\lambda = 2$.

Now let us find the eigenvectors having the following presentation: $x_1 = \nu\tilde{x}_1^0, z_1 = \mu\tilde{z}_1^0, \ \sigma$. Using the characteristic system (11) we obtain the next relations:

$$\begin{cases} 2\mu = (\lambda + 2)\nu \\ 2\nu = (\lambda - 1)\mu \end{cases}$$

from which we get $(\lambda + 3)(\lambda - 2) = 0$.

$\lambda \neq -1, 0, 1$.

In this case (as it follows from (16)) vectors x_i lie on the plane $\{\tilde{x}_1^0, \ \sigma\}$. Denote the expression $a^3\lambda(\lambda + 1)/27$ by μ. Then the linear system (16) in the basis

$$
e_1 = \begin{pmatrix} \tilde{x}_2^{\,0} \\ 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 \\ \tilde{x}_1^{\,0} \end{pmatrix}, \ e_3 = \begin{pmatrix} \tilde{x}_1^{\,0} \\ \tilde{x}_2^{\,0} \end{pmatrix}, \ e_4 = \begin{pmatrix} \tilde{x}_1^{\,0} + 2\,\tilde{x}_2^{\,0} \\ -2\,\tilde{x}_1^{\,0} - \tilde{x}_2^{\,0} \end{pmatrix}
$$

will be presented by the following matrix:

$$
\begin{pmatrix}
\frac{1}{2}(-m_1 + m_2 + m) & \frac{1}{2}(2m_1 + m_2 - m) & 0 & 0 \\
\frac{1}{2}(m_1 + 2m_2 - m) & \frac{1}{2}(m_1 - m_2 + m) & 0 & 0 \\
\frac{1}{4}(2m_1 + m_2 - m) & \frac{1}{4}(m_1 + 2m_2 - m) & m & 0 \\
\frac{1}{2}(2m_1 - m_2 - m) & \frac{1}{2}(m_1 - 2m_2 + m) & 0 & 0
\end{pmatrix} - \mu E,
$$

where E is a unitary matrix and its characteristic polynomial may be easily found.

Being equal to 15 the total dimension of the eigen-spaces is equal to dimension of the subspace $\mathbb{C}^{18}$, which is defined by the condition $x_1 + x_2 + x_3 = 0$.   □

6   The asymptotic behavior of β-singular points of the foliation $\mathcal{F}$

Thus, we try to find the solution of the following system:

$$
\begin{cases}
(m_2 + m_3)z_1 - m_1(z_2 + z_3) = \frac{\lambda+2}{3}x_1, \ \sigma \\
-\dfrac{3}{\mid \tilde{x}_1^{\,0} \mid^3}x_1 + \dfrac{9}{\mid \tilde{x}_1^{\,0} \mid^5}(\tilde{x}_1^{\,0}, x_1)\,\tilde{x}_1^{\,0} = (\lambda - 1)z_1, \ \sigma.
\end{cases} \tag{18}
$$

Without any restrictions of generality we can suppose that the vectors $\tilde{x}_i^0, \tilde{z}_i^0$ have the forms $(a_i, 0, 0), (b_i, 0, 0)$ correspondingly. In this case the operator (6.1) has three following eigenspaces: $x_i, z_i \in V_1 = \{(*, 0, 0)\}$, $x_i, z_i \in V_2 = \{(0, *, 0)\}$ and $x_i, z_i \in V_3 = \{(0, 0, *)\}$.

Considering the problem (18) in the every proper subspace we get the following theorem.

Theorem 3. *The eigenvalues and the eigenvectors, that are corresponding to these eigenvalues of the linear system (18) are the following:*

$\lambda = 1$

The eigenvector u_1 has the form:

$$
x_1 = 0, \ z_1 = m_1 r, \ \sigma, \ r \in \mathbb{C}^3.
$$

The dimension of the eigenspace is equal to 3.

In the space V_1.

$(\lambda + 3)(\lambda - 2) = 0$.

The eigenvectors u_{-3}, u_2 have the following form: $x_1 = \nu \tilde{x}_1^0, z_1 = \mu \tilde{z}_1^0, \ \sigma, \ 2\mu = (\lambda + 2)\nu$. The dimension of the eigenspaces for $\lambda = -3, \lambda = 0$ and $\lambda = 2$ is equal to 1.

$\lambda \neq -3, 1, 2$.

The eigenvalues λ_1, λ_2 are found as the roots of the equation

$$\lambda^2 + \lambda + 2 - 18 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0.$$

The dimension of the eigenspaces for every root is equal to 1.

In the space V_2.

$\lambda(\lambda + 1) = 0.$

The eigenvectors u_0, u_{-1} have the following form: $x_1 = \nu(0, a_1, 0)$, $z_1 = \mu(0, b_1, 0), \sigma$, $2\mu = (\lambda - 1)\nu$. The dimension of the eigenspaces for $\lambda = -1, \lambda = 0$ and $\lambda = 1$ is equal to 1.

$\lambda \neq 0, 1, 2.$

The eigenvalues λ_3, λ_4 are found as the roots of the equation

$$\lambda^2 + \lambda - 4 + 9 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0.$$

The dimension of the eigenspace for every root is equal to 1.

In the space V_3 the eigenvalues are the same as in the space V_2. The eigenvectors are found similarly.

It is possible to select a proper basis from all the eigenvectors, mentioned above.

Proof. First of all, consider the case $\lambda = 1$. Then we'll be able to substitute z_i from the second equation (18) to the first one. The second equation of the system (18) for the space V_1 and the spaces V_2, V_3 has the form

$$\frac{6}{|\tilde{x}_1^0|^3} x_1 = (\lambda - 1)z_1, \ \sigma$$

and

$$\frac{-3}{|\tilde{x}_1^0|^3} x_1 = (\lambda - 1)z_1, \ \sigma$$

correspondingly. In either case $x_i = 0$ and we get the presentation of the eigenvectors for the eigenvalue $\lambda = 1$.

Consider the space V_1.

Repeating the proof of the Theorem 2 let us find the eigenvectors in such a form: $x_1 = \nu\tilde{x}_1^0, z_1 = \mu\tilde{z}_1^0, \ \sigma$. In this case we get the equation $(\lambda + 3)(\lambda - 2) = 0$. In order to find the rest of the roots λ, substitute z_i from second equation (18) to the first one. At the same time, denote the product $(\lambda - 1)(\lambda + 2)$ by μ and replace $-x_1 - x_2$ by x_3.

We shall get a quadratic polynomial by μ. Finally, taking into account that the derived polynomial is exactly divided by $(\lambda + 3)(\lambda - 2) = \mu + 4$, we get the first power polynomial

$$\mu + 4 - 18 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0.$$

Now consider the space V_2.

Now let us pay our attention to the fact that the system (18) for the spaces V_1 and V_2 doesn't differ much from the same one for the space V_1. That is why the eigenvectors for V_2

can be found in the form $x_i = \nu(0, a_i, 0)$, $z_i = \mu(0, b_i, 0)$ and similarly - for V_3. So we get the following system:

$$\begin{cases} 2\mu = (\lambda + 2)\nu \\ -\nu = (\lambda - 1)\mu \end{cases}$$

from where it follows that $\lambda^2 + \lambda = 0$.

Repeating the same calculations as for the space V_1 we get the polynomial

$$\mu - 2 + 9 \sum_\sigma \frac{m_1 + m_2}{a_3^3} = 0,$$

where $\mu = (\lambda - 1)(\lambda + 2)$.

The calculations for the space V_3 exactly coincide with the case of V_2.

Being equal to 15 the total dimension of the eigenspaces is equal to the dimension of the subspace $\mathbb{C}^{18}$, which is defined by the condition $x_1 + x_2 + x_3 = 0$. □

Undoubtedly, we can't but pay our attention to the surprising coincidence of properties of the eigenvectors of α and β-points.

Theorem 4. *The operators (15), (18), which linearize the differential equations (14) in the neighbourhoods of α and β-points, have the following entire eigenvalues for any masses m_i and the following eigenvectors, corresponding to these eigenvalues:*

$\lambda = 1$.

The eigenvector u_1 has the following form:

$$x_1 = 0, \ z_1 = m_1 r, \ \sigma, \ r \in \mathbb{C}^3.$$

$\lambda(\lambda + 1) = 0$.

The eigenvectors u_0, u_{-1} satisfy the following conditions:

$$x_1 \perp \tilde{x}_1^0, \ z_1 = \frac{3}{|\tilde{x}^0|^3 (1 - \lambda)}, \ \sigma.$$

$(\lambda + 3)(\lambda - 2) = 0$.

The eigenvector u_1 has the following form:

$x_1 = \nu \tilde{x}_1^0, z_1 = \mu \tilde{z}_1^0, \ \sigma, \ 2\mu = (\lambda + 2)\nu$.

Proof. The correctness of the theorem follows from Theorems 2 and 3. □

Suppose that the functions $x_1(t), x_2(t)$ from the system (7) are known. Then $x_3(t) = -x_1(t) - x_2(t)$, $z_1(t) = \int \frac{x_1(t)}{|x_1(t)|^3} dt$, σ.

If we know the functions $w_1(t), w_2(t)$ we can find $x_1(t), x_2(t)$ as $x_1(t) = \frac{w_1(t)}{|w_1(t)|^2}$, σ.

At last if we know the functions $z_1(t), z_2(t)$, then $x_1(t) = \frac{\dot{z}_1(t)}{|\dot{z}_1(t)|^{3/2}}$, σ, according to (7).

Taking into account the reasoning adduced above we call any collection $x(t) = ((x_1(t), x_2(t))$, $w(t) = (w_1(t), w_2(t))$, $z(t) = (z_1(t), z_2(t))$, σ the solution of the three-body problem.

The properties of the foliation $\mathcal{F}$ are important for profound studying the three-body problem but we'll try to investigate the singular points of the solutions $x(t)$, $w(t)$, $z(t)$.

Proposition 6.1. *If t_* is the singular point of the solution of the three-body problem (10) then $max_i\{|w_i|\} \to \infty$, $t \to t_*$. At the same time $\|w(t)\| \to \infty$.*

Proof. Let the functions $|w_i(t)|$ be bounded in the neighbourhood of the singular point t_*. It means that the solution $(w(t), z(t))$ of the differential equation (10) is holomorphic in this neighbourhood. We get the contradiction hence $max_i\{|w_i|\} \to \infty$, $t \to t_*$ and $\|w(t)\| \to \infty$. □

Theorem 5. *There exist the singular points of the solutions*

$$w_1(t) = \frac{x_1(t)}{|x_1(t)|^2}, \ \sigma$$

of the three-body problem (10) with the asymptotic behavior (compare with 2.4. [2]):

$$x(t) = \tilde{x}^0 t^{2/3} + \kappa_1 u_1 t + \kappa_2 u_2 t^{4/3} + \sum_k \mu_k v_k t^{(2+\lambda_k)/3} + ...,$$

where $\tilde{x}^0$ is the solution of the characteristic system (11), $\kappa_1, \kappa_2, \mu_k$ are free parameters, u_1, u_2 are the eigenvectors of the operators (15), (18), v_k are the eigenvectors of these operators for the eigenvalues $\lambda_k > 0$, $k = 1, ..., 4$ (see Theorems 2, 3).

Proof. Using the results of Theorems 2, 3, we get the first approximation of the asymptotic behavior of the solutions (7) and then apply the Piccard operator. □

Remark 6.1. *If $\lambda < 0$, $\tau \to -\infty$, because of (8), the trajectories, outgoing from the singular points specify the expansion of $x(t)$ by powers $t^{-1/3}$ due to (8), i.e. the asymptotic behavior when $t \to \infty$.*

According to the proved theorem, the singular points (8) don't have an asymptotic behavior of a general position. Undoubtedly, this fact is quite natural, for the reason that these singular points characterize triple collisions or movements of the bodies, making some stable configurations.

All other singular points of the foliation $\mathcal{F}$ can be obtained when the right hand side (7) is not defined: these are binary collisions when $\|x_i\| = 0$ or $|x_i| = 0$, $\|x_i\| \neq 0$.

Definition 6.1. *Let us call the singular points of the solutions of the system (7), that are specified by the condition $|x_i| = 0$, $\|x_i\| \neq 0$ the points of quasi-collisions.*

Certainly, all the types of collisions and solutions, defining them, are interesting for us, even if the measure of such solutions is equal to zero. Nevertheless, we pass to considering the singular points of quasi-collisions that are the singular points of a general position taking into account the problem of the solar system stability.

7   THE ASYMPTOTIC BEHAVIOR OF QUASI-COLLISIONS

Let $|x_1| = 0$, $\|x_1\| \neq 0$ in the system (7). Then the variables z_2, z_3 are small in comparison with z_1, hence the system (2.4) is approximately described by the system

$$\begin{cases} \dot{x}_1 = mz_1, \\ \dot{z}_1 = -\dfrac{x_1}{|x_1|^3}. \end{cases} \tag{19}$$

We can also get this system, if the mass of one of these three bodies is equal to zero. This is natural because it is clear that we speak about the situation when the influence of one of the bodies on the two others is minimal.

The problem (19) is plane; denote the coordinates of the vector $x_1 = (\chi_1, \chi_2)$, $z_1 = (\xi_1, \xi_2)$, then

$$(\dot{\chi}_1 \chi_2 - \chi_1 \dot{\chi}_2)^{\cdot} = \ddot{\chi}_1 \chi_2 - \chi_1 \ddot{\chi}_2 = m\left(\frac{\chi_1}{|x_1|^3}\chi_2 - \chi_1\frac{\chi_2}{|x_1|^3}\right) = 0.$$

It means that in the polar coordinates

$$\dot{\chi}_1 \chi_2 - \chi_1 \dot{\chi}_2 = (r\cos(\varphi))^{\cdot} r\sin(\varphi) - r\cos(\varphi)(r\sin(\varphi))^{\cdot} = -r^2\dot{\varphi} = C.$$

The integral from Theorem 1 takes the following form:

$$\mathcal{H} = \frac{m}{2} z_1^2 - \frac{1}{|x_1|} \tag{20}$$

and then we have:

$$\dot{\chi}_1 = \dot{r}\cos(\varphi) - r\,\dot{\varphi}\sin(\varphi),$$

$$\dot{\chi}_2 = \dot{r}\sin(\varphi) - r\,\dot{\varphi}\cos(\varphi),$$

$$\begin{cases} \dot{r} = \sqrt{-\dfrac{C^2}{r^2} + 2m\left(\mathcal{H} + \dfrac{1}{r}\right)} \\ \dot{\varphi} = -\dfrac{C}{r^2}. \end{cases}$$

The derived system can be solved by quadratures, but regretfully, $r(t) = |x_1(t)|$ has a "bad" asymptotic behavior when $r \to 0$. Applying the iterations of the Piccard operator to $r(t) = \sqrt{2Cit} + ...$ we get the summands of the form $t^k \ln(t)^m$, $k, m \to \infty$.

Another reason, for which the polar coordinates are inconvenient is that they do not have a natural generalization for the three-body problem. That is why trying to get the required asymptotic behavior in the current coordinates (z, x), being convenient so far, is absolutely reasonable.

It will be recalled that besides the energy integral there is also the moment integral:

$$\mathcal{M} = \chi_1 \xi_2 - \chi_2 \xi_1. \tag{21}$$

We're interested in considering the easiest case, when one of the points moves on a parabola relative to the other one. Thus we consider the motion of the following form:

$$\chi_2 = k\chi_1^2 - a. \tag{22}$$

Further representations are obtained, using the relations (19)–(22), being quite simple. That is the reason for which we do not comment them at length.

$$(19), (22) \Rightarrow \dot{\chi}_2 = 2k\chi_1\dot{\chi}_1 = 2mk\chi_1\xi_1 = m\,\xi_2 \Rightarrow \xi_2 = 2k\chi_1\xi_1, \tag{23}$$

$$(23), (19), (22) \Rightarrow \dot{\xi}_2 = 2km\,\xi_1^2 - \frac{2k\chi_1^2}{|\chi_1|^3} = \frac{a - k\chi_1^2}{|\chi_1|^3} \Rightarrow 2km\,\xi_1^2 = \frac{a + k\chi_1^2}{|\chi_1|^3}. \tag{24}$$

$$(22), (23) \Rightarrow \mathcal{M} = 2k\chi_1^2\xi_1 - (k\chi_1^2 - a)\xi_1 \Rightarrow \xi_1 = \frac{\mathcal{M}}{a + k\chi_1^2}. \tag{25}$$

$$(24), (25) \Rightarrow \frac{2km\mathcal{M}}{(a + k\chi_1^2)^2} = \frac{a + k\chi_1^2}{|\chi_1|^3} = \frac{a + k\chi_1^2}{(\chi_1^2 + (a - k\chi_1^2)^2)^{3/2}}$$

$$\Rightarrow |x_1| = (\chi_1^2 + (a - k\chi_1^2)^2)^{1/2} = a + k\chi_1^2 \tag{26}$$

$$\Rightarrow 4ka = 1. \tag{27}$$

$$(24) - (27) \Rightarrow 2\mathcal{M}^2 km = 1 \Rightarrow a = \frac{\mathcal{M}^2 m}{2}, \quad k = \frac{1}{2\mathcal{M}^2 m}.$$

Thus, we have the solution in the following form:

$$\xi_1 = \frac{2\mathcal{M}^3 m}{\chi_1^2 + \mathcal{M}^4 m^2}, \quad \xi_2 = \frac{4\mathcal{M}^3 km\chi_1}{\chi_1^2 + \mathcal{M}^4 m^2}, \quad \chi_2 = \frac{\chi_1^2}{2\mathcal{M}^2 m} - \frac{\mathcal{M}^2 m}{2}, \tag{28}$$

$$\dot{\chi}_1 = \frac{2\mathcal{M}^3 m^2}{\chi_1^2 + \mathcal{M}^4 m^2}, \quad \frac{\chi_1^3}{3\mathcal{M}^4 m^2} + \chi_1 = \frac{2t}{\mathcal{M}}, \tag{29}$$

where the last relation follows from (19), (25).

On the assumption of the formulae, we have just obtained, it is easy to get the asymptotic behavior of the solution $\chi_1(t)$.

$$\chi_1 = -\mathcal{M}^2 mi + \sqrt{2i\mathcal{M}m}\,t^{1/2} + \frac{t}{3\mathcal{M}} + \frac{5\,t^{3/2}}{18\mathcal{M}^2\sqrt{2i\mathcal{M}m}} + \frac{4\,it^2}{27\mathcal{M}^4 m} + ...,$$

$$\chi_2 = -\mathcal{M}^2 m - i\sqrt{2i\mathcal{M}m}\,t^{1/2} + \frac{2it}{3\mathcal{M}} + \frac{7\,i\,t^{3/2}}{18\mathcal{M}^2 m\sqrt{2i\mathcal{M}m}} + \frac{5t^2}{27\mathcal{M}^4 m} + ...,$$

$$\xi_1 = \frac{\dot{\chi}_1}{m} = \frac{\sqrt{\mathcal{M}i}}{\sqrt{2mt}} + \frac{1}{3\mathcal{M}m} + \frac{5\,t^{1/2}}{12\mathcal{M}^2\sqrt{2i\mathcal{M}m}} + \frac{8\,it}{27\mathcal{M}^4 m^2} + ...,$$

$$\xi_2 = \frac{\dot{\chi}_2}{m} = -i\frac{\sqrt{\mathcal{M}i}}{\sqrt{2mt}} + \frac{2i}{3\mathcal{M}m} + \frac{7\,i\,t^{1/2}}{12m\mathcal{M}^2\sqrt{2i\mathcal{M}m}} + \frac{10\,t^2}{27\mathcal{M}^4 m^2} + ... \tag{30}$$

Now let us consider the hyperbolic motion.

$$\sqrt{(\chi_1 + a)^2 + \chi_2^2} - \sqrt{\chi_1^2 + \chi_2^2} = b,$$

$$k^2\chi_2^2 - \chi_1(\chi_1 + a) = \frac{c^2}{4}, \quad a^2 - b^2 = c^2, \quad k = \frac{b}{c}. \tag{31}$$

After having differentiated (31) by t, we get:

$$2k^2\chi_2\xi_2 - 2\chi_1\xi_1 - a\xi_1 = 0,$$

$$2k^2 m\xi_2^2 - \frac{2k^2\chi_2^2}{|\chi|^3} - 2m\xi_1^2 + \frac{2\chi_1^2}{|\chi|^3} + \frac{a\chi_1}{|\chi|^3} = 0.$$

After having expressed ξ_2 by $\mathcal{H}$, we get:

$$2m(1+k^2)\xi_1^2 = \frac{2(1+k^2)\chi_1^2 + a\chi_1}{|\chi|^3} + 4\mathcal{H}k^2m,$$

$$|\chi|^2 = \chi_1^2 + \chi_2^2 = \chi_1^2 + \frac{1}{k^2}\left(\chi_1^2 + a\chi_1 + \frac{c^2}{4}\right) = \frac{1+k^2}{k^2}\left(\chi_1 + \frac{a}{2(1+k^2)}\right)^2,$$

$$(\dot{\chi}_1)^2 = \frac{4mk^3(1+k^2)^{1/2}\chi_1}{(2(1+k^2)\chi_1 + a)^2},$$

$$\chi_1 = -\frac{c^2}{2a} + \kappa_1\sqrt{t} + ..., \ \kappa_1 \in \mathbb{C}, \tag{32}$$

that is we get the asymptotic behavior $\chi_1 = \chi_{10} + \chi_{11}\sqrt{t} + ...$ in the point of a quasi-collision as well as in the parabolic case.

Then from (31) we get:

$$\chi_2 = \frac{ic^2}{2a} + \kappa_2\sqrt{t} + ..., \ \kappa_2 \in \mathbb{C}. \tag{33}$$

Thus, a one-parameter family of solutions of the two-body problem is obtained. It is quite evident, that if the third body participates in a quasi-collision, the asymptotic behavior (32),(33) will change only a little and moreover, it will be possible to get it by means of the Piccard iterations.

Theorem 6. *For almost all the initial conditions the trajectory of solutions of the three-body problem* $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{C}^9$

$$\begin{cases} \dot{x}_1 = mz_1 - m_1 \sum_\sigma z_1, \sigma \\ \dot{z}_1 = -\dfrac{x_1}{|x_1|^3}, \sigma \end{cases}$$

specifies a smooth manifold, immersed into $\mathbb{C}^9$.

Theorem 7. *(see [9]) There are no entire solutions of three-body problem.*

Proof. Let the solution $(w(t), z(t))$ be an arbitrary solution of the problem (10) and $Y = \pi(w(t), z(t))$ be a fiber of the foliation $\mathcal{F}$. Assume that the solution $(w(t), z(t))$ has no singular points $t_* \in \mathbb{C}$. Then the leaf Y has no singular points for otherwise the leaf Y would have a singular point $\pi(\tilde{w}^0, \tilde{z}^0)$ and the solution $(w(t), z(t))$ would have singular α or β-points or would be bounded in infinity, that is impossible for entire solution.

Let $y \in Y$ be an arbitrary point and $\pi(w(t_0), z(t_0)) = y$, $\|w(t_0), z(t_0)\| = max_i\{\|w_i(t_0)\|, \|z_i(t_0)\|\} = 1$. One can move along a path $\gamma \subset \mathbb{C}$ from the point t_0 to the point t_1 where $\|w(t_0), z(t_0)\| > 2$.

And what is more, there exists an neighbourhood U in P_*^{17} such as for all $\acute{y} \in U$ if $\pi(w(t_0), z(t_0)) = \acute{y}$, $\|w(t_0), z(t_0)\| = 1$ then along γ we have $\|w(t_0), z(t_0)\| > 2$. So we have an open covering of the closure of Y and choose a finite sub-covering U_i. Let $|t_{0i} - t_{1i}| < T$ for all i. Now construct a path Γ which contains the points $t_0, t_1, ..., t_n, ...$ where t_k can be found from t_{k-1} in the same way as t_1 was found by t_0. Then the points $t_0, t_1, ..., t_n, ...$ must be in the circle with the radius $T + \frac{T}{2} + \frac{T}{2^2} + ... = 2T$ and there exist the limit point $t_* \in \mathbb{C}$ which must be a singular point. $\square$

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