Карпатські математичні публікації. Т.3, №1

Carpathian Mathematical Publications. V.3, №1

УДК 517.98

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EXPONENTIAL TYPE DISTRIBUTIONS AND A GENERALIZED FUNCTIONAL CALCULUS FOR GENERATORS OF C_0-GROUPS

Lozynska V.Ya. *Exponential type distributions and a generalized functional calculus for generators of C_0-groups*, Carpathian Mathematical Publications, **3**, 1 (2011), 73–84.

The properties of a dual space to a space of entire functions of exponential type of many complex variables, that on the real subspace belongs to $L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ are described. A functional calculus for generators of strongly continuous groups of bounded linear operators on an arbitrary Banach space in a Fourier-image of such dual space is constructed.

Introduction

In this paper we consider a space of entire functions of exponential type for which their restriction onto the real subspace belong to $L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$. This space has a property to be invariant with respect to the action of partial differential operators. This property allows us to introduce in the dual space of linear continuous functionals (so called exponential type distributions) a convolution operation and we can consider this space as a convolution topological algebra. In the Fourier-image of such algebra we construct a functional calculus for generators of strongly continuous multi-parameter groups on a Banach space. This functional calculus is a generalization of the well-known Fourier operator transform for convolution algebras of measures [8],[2] and the calculus for generators of nonquasianalytic groups in algebras of entire functions of exponential type [12]. This approach gives an effective method for investigation of differential operators and functions of them. We construct the functional calculus as generalized functions of generators of C_0–groups. In practice some generalized functions (δ–functions) of concrete operators appear in the Quantum theory [3], [4].

The existence of the structure of the convolution algebra on the space of exponential type distributions follows from the invariant properties in this space with respect to differential operators and plays a crucial role to construct the functional calculus. The invariant properties of subspaces of exponential type of entire functions in a wide context exponential type vectors of unbounded linear operators on the Banach spaces are used in the operator calculus [14], [10], [5], in the theory of Differential equations [14], [6] and in the Approximation theory in Banach spaces [6], [15].

2000 *Mathematics Subject Classification:* 47A60.
Key words and phrases: exponential type distributions, generalized functional calculus, Fourier-image.

1 Algebras of exponential type distributions

1.1 Spaces of entire functions of exponential type

We define a space of test functions and prove its basic properties. Let $L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ be a complex Banach space of functions $\varphi(t)$, $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, with the norm

$$\|\varphi\|_{L_p} := \Big(\int_{\mathbb{R}^n} |\varphi(t)|^p \, dt\Big)^{1/p}.$$

We use next notations $k = (k_1, \ldots, k_n) \in \mathrm{Z}_+^n$, $|k| = k_1 + \ldots + k_n$, $k! = k_1! \cdot \ldots \cdot k_n!$, $kr = (k_1 r, \ldots, k_n r)$ for $r \in \mathrm{C}$, $D^k = D_1^{k_1} \ldots D_n^{k_n}$, where $D_j = -i\partial/\partial t_j$ for all $j = 1, \ldots, n$. The domain of the operator of partial differentiation $D_j^{k_j}$ is: $\mathrm{dom}\,(D_j^{k_j}) \equiv \big\{\varphi \in \mathrm{dom}\,(D_j^{k_j-1}) : D_j\varphi \in \mathrm{dom}\,(D_j^{k_j-1})\big\}$ for $k_j \geq 1$, $\mathrm{dom}\,(D_j^0) = L_p$ for $k_j = 0$ for all $j = 1, \ldots, n$. Hence, $\mathrm{dom}\,(D^k) = \bigcap_{j=1}^{n} \mathrm{dom}\,(D_j^{k_j})$ is the domain of the operator D^k.

If $\varphi \in L_p(\mathbb{R}^n)$ and $\psi \in L_q(\mathbb{R}^n)$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then convolution is defined by $(\varphi * \psi)(t) := \int_{\mathbb{R}^n} \varphi(s)\psi(t-s)\, ds$. For $p = 1$ the space $L_1(\mathbb{R}^n)$ is a Banach algebra with respect to the convolution.

Let us consider on $L_p(\mathbb{R}^n)$ the following isometric shift group

$$T_s = e^{-i(s_1 D_1 + \ldots + s_n D_n)} : \varphi(t) \longrightarrow \varphi(t - s), \qquad s = (s_1, \ldots, s_n) \in \mathbb{R}^n,$$

where $D_1 = -i\partial/\partial t_1, \ldots, D_n = -i\partial/\partial t_n$ are the operators of partial differentiation.

For an arbitrary vector $\nu = (\nu_1, \ldots, \nu_n)$, $\nu_j > 0$ $(j = 1, \ldots, n)$ we define the space

$$\mathcal{E}_p^\nu := \Big\{\varphi \in \bigcap_{k \in \mathrm{Z}_+^n} \mathrm{dom}\,(D^k) : \|\varphi\|_{\mathcal{E}_p^\nu} = \sup_{k \in \mathrm{Z}_+^n} \frac{\|D^k\varphi\|_{L_p}}{\nu^k} < \infty\Big\},$$

where $k = (k_1, \ldots, k_n)$, $\nu^k = \nu_1^{k_1} \cdot \ldots \cdot \nu_n^{k_n}$, $D^k = D_1^{k_1} \ldots D_n^{k_n}$. From the next inequality $\|\varphi\|_{L_p} \leq \|\varphi\|_{\mathcal{E}_p^\nu}$ which is true for an arbitrary $\varphi \in \mathcal{E}_p^\nu$ it follows that the embedding $\mathcal{E}_p^\nu \subset L_p(\mathbb{R}^n)$ is continuous.

In the class of entire analytic functions of n complex variables $\mathbb{C}^n \ni t + i\tau \longrightarrow \Phi(t + i\tau) \in \mathbb{C}$ we consider a subspace $\mathcal{M}_p^\nu = \mathcal{M}_p^\nu(\mathbb{C}^n)$ of functions Φ such that for each fixed vector $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n$ a corresponding function of n real variables $\mathbb{R}^n \ni t \longrightarrow \Phi(t + i\tau)$ belongs to the space $L_p(\mathbb{R}^n)$ such that the norm

$$\|\Phi\|_{\mathcal{M}_p^\nu} = \sup_{\tau \in \mathbb{R}^n} \exp\Big(\sum_{j=1}^{n} -\nu_j|\tau_j|\Big)\Big[\int_{\mathbb{R}^n} |\Phi(t + i\tau)|^p dt\Big]^{1/p}$$

is finite. It is know [13] that the spaces $\mathcal{M}_p^\nu$ consist of the functions of exponential type.

Functions $\Phi(t + i\tau)$ from the class $\mathcal{M}_p^\nu$ such that each $\varphi(t) = \Phi(t + i0) \in L_p(\mathbb{R}^n)$ satisfies the Bernstein's inequality ([13], III, 3.2.2) ([1], IV, 8.3) on $\mathbb{R}^n$:

$$\|D^k\varphi\|_{L_p} \leq \nu^k \, \|\varphi\|_{L_p}, \tag{1}$$

where $\nu^k = \nu_1^{k_1} \cdot \ldots \cdot \nu_n^{k_n}$.

Theorem 1. (i) *The mapping* $\mathcal{M}_p^\nu \ni \Phi(t+i\tau) \longrightarrow \varphi(t) := \Phi(t+i0) \in \mathcal{E}_p^\nu$ *is an isometry of the normed spaces.*

(ii) *The embeddings* $\mathcal{E}_p^\nu \subset L_p(\mathbb{R}^n)$ *are isometric.*

(iii) *The spaces* $\mathcal{E}_p^\nu$ *are invariant with respect to the action of the group* T_{-s}*, and the restriction* $T_{-s} : \mathcal{E}_p^\nu \longrightarrow \mathcal{E}_p^\nu$ *is an isometry of the normed spaces.*

Proof. (i) Let $\Phi \in \mathcal{M}_p^\nu$. A restriction $\varphi(t) = \Phi_{|\mathbb{R}^n}$ of a functions $\Phi(t+i\tau) \in \mathcal{M}_p^\nu$ on the real subspace $\mathbb{R}^n$ satisfies the Bernstein's inequality (1)

$$\|D^k\varphi\|_{L_p} \le \nu^k \|\varphi\|_{L_p} \qquad (\forall \Phi \in \mathcal{M}_p^\nu), \tag{2}$$

where $\nu^k \equiv \nu_1^{k_1} \cdot \ldots \cdot \nu_n^{k_n}$. From (2) we obtain $\|\varphi\|_{\mathcal{E}_p^\nu} \le \|\varphi\|_{L_p}$. From the definition of the norm of the space $\mathcal{M}_p^\nu$ it follows $\|\varphi\|_{L_p} \le \|\Phi\|_{\mathcal{M}_p^\nu}$ $(\forall \Phi \in \mathcal{M}_p^\nu)$, i.e. $\mathcal{M}_p^\nu{}_{|\mathbb{R}^n} \subset \mathcal{E}_p^\nu$.

Conversely, let $\varphi \in \mathcal{E}_p^\nu$. Let us consider the power series $\varphi(t+i\tau) = \sum_{|k|=0}^{\infty} \frac{(i\tau)^k\, D^k\varphi}{k!}$. The following inequalities

$$\Bigg(\int_{\mathbb{R}^n} |\varphi(t+i\tau)|^p dt\Bigg)^{1/p} \le \sum_{|k|=0}^{\infty} \frac{|\tau^k|\, \|D^k\varphi\|_{L_p}}{k!} \le \|\varphi\|_{L_p} \exp\Bigg(\sum_{j=1}^{n} \nu_j|\tau_j|\Bigg)$$

are valid so $\|\Phi\|_{\mathcal{M}_p^\nu} \le \|\varphi\|_{L_p}$. The series is convergent and the function $\varphi(t+i\tau)$ is an entire function of class $\mathcal{M}_p^\nu$. Hence $\mathcal{E}_p^\nu \subset \mathcal{M}_p^\nu{}_{|\mathbb{R}^n}$ and we obtain $\mathcal{E}_p^\nu = \mathcal{M}_p^\nu{}_{|\mathbb{R}^n}$.

(ii) Since $\|\varphi\|_{L_p} \le \|\varphi\|_{\mathcal{E}_p^\nu}$, then $\|\varphi\|_{L_p} \le \|\varphi\|_{\mathcal{E}_p^\nu} \le \|\varphi\|_{L_p} \le \|\Phi\|_{\mathcal{M}_p^\nu} \le \|\varphi\|_{L_p}$ $(\forall \varphi \in \mathcal{E}_p^\nu)$ and a necessary isometric isomorphism is $\mathcal{E}_p^\nu = \mathcal{M}_p^\nu{}_{|\mathbb{R}^n}$. In particular $\mathcal{E}_p^\nu \subset L_p(\mathbb{R}^n)$.

(iii) For all $k \in \mathbb{Z}_+^n$ and $s \in \mathbb{R}^n$ next equality $\|T_{-s}D^k\varphi\|_{L_p} = \|D^k\varphi\|_{L_p}$ is valid. From the identity $D^k\psi(s) = T_{-t}D^k\varphi(s)$, where $\psi : \mathbb{R}^n \ni s \to T_{-s}\varphi(t)$, we obtain $\|D^k\psi(s)\|_{L_p} = \|D^k\varphi(s)\|_{L_p}$. Then the inequality (1) has the view $\|D^k\psi\|_{L_p} \le \nu^k\, \|\varphi\|_{L_p}$. From this we obtain the inequality

$$\Bigg(\int_{\mathrm{R}^n} |\psi(t+i\tau)|^p dt\Bigg)^{1/p} \le \|\varphi\|_{L_p} \exp\Bigg(\sum_{j=1}^{n} \nu_j|\tau_j|\Bigg),$$

then $\psi \in \mathcal{M}_p^\nu$. □

Theorem 2. $\mathcal{E}_p^\nu$ *are Banach spaces.*

Proof. Each of operators D_j on $L_p(\mathbb{R}^n)$ $(j = 1, \ldots, n)$ is a generator of a one-parameter isometric shift group [9]

$$\varphi(t) \longrightarrow \varphi(t_1, \ldots, t_{j-1},\, t_j - \xi_j,\, t_{j+1}, \ldots, t_n).$$

We use the inequality $\|D^k\varphi\|_{L_p} \le \nu^k\|\varphi\|_{\mathcal{E}_p^\nu}$, $\varphi \in \mathcal{E}_p^\nu$ for all $k \in \mathbb{Z}_+^n$. If $\{\varphi_m\}$ is a Cauchy sequence in $\mathcal{E}_p^\nu$, then $\{D^k\varphi_m\}$ is the same sequence in $L_p(\mathbb{R}^n)$ for every fixed k. To the induction (by k) and that D_j in $L_p(\mathbb{R}^n)$ is closed it follows that there is a function $\varphi \in L_p(\mathbb{R}^n)$, for which

$$\lim_{m\to\infty} \|D^k\varphi_m - D^k\varphi\|_{L_p} = 0 \tag{3}$$

for any k. Then for any $\varepsilon > 0$ there is a number $m(\varepsilon)$ such that

$$\|\varphi_m - \varphi_l\|_{\mathcal{E}_p^\nu} < \max_{0\le|k|\le m(\varepsilon)} \frac{\|D^k\varphi_m - D^k\varphi_l\|_{L_p}}{\nu^k} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{4}$$

for all $m, l \ge m(\varepsilon)$. Thus $\|\varphi_l\|_{\mathcal{E}_p^\nu} \le \|\varphi_{m(\varepsilon)}\|_{\mathcal{E}_p^\nu} + \|\varphi_{m(\varepsilon)} - \varphi_l\|_{\mathcal{E}_p^\nu} < \|\varphi_{m(\varepsilon)}\|_{\mathcal{E}_p^\nu} + \varepsilon$ for all $l \ge m(\varepsilon)$.We take a limit in the last inequality for $l \to \infty$ and use the inequality (4), we obtain $\|\varphi\|_{\mathcal{E}_p^\nu} \le \|\varphi_{m(\varepsilon)}\|_{\mathcal{E}_p^\nu} + \varepsilon$. Thus, $\varphi \in \mathcal{E}_p^\nu$. We take a limit in (4) for $l \to \infty$ and use (3), we obtain $\|\varphi_m - \varphi\|_{\mathcal{E}_p^\nu} \le \varepsilon$ for all $m \ge m(\varepsilon)$. The theorem is proved. □

Let

$$\mathcal{E}_p := \bigcup\nolimits_\nu \mathcal{E}_p^\nu = \lim_{\nu}\operatorname{ind} \mathcal{E}_p^\nu$$

be the union of spaces endowed with a topology of the inductive limit, where the embeddings $\mathcal{E}_p^\nu \subset \mathcal{E}_p^\mu$ are continuous. The vector $\mu = (\mu_1, \dots, \mu_n)$ is such that $\nu_1 \le \mu_1, \dots, \nu_n \le \mu_n$. The locally convex space $\mathcal{E}_p$ we will call a space of test functions. The space $\mathcal{E}_p$ belongs to the domain of differential operators D_j and is invariant relatively to their action. From a property of regular inductive limits (see [10], [14]) it follows that every bounded subset S of the space $\mathcal{E}_p$ is bounded in some $\mathcal{E}_p^\nu$.

1.2 Distributions of exponential type

We introduce exponential type distributions. We show that the space of exponential type distributions is a convolution algebra.

By $\mathcal{E}_p'$ we denote a dual space of $\mathcal{E}_p$ with a weak topology. The duality $\langle \mathcal{E}_p' \mid \mathcal{E}_p\rangle$ can be determined by a bilinear form $\langle f \mid \varphi\rangle := \langle f_\nu \mid \varphi\rangle$, where ν is an arbitrary vector such that $\varphi \in \mathcal{E}_p^\nu$ and $f_\nu := f_{|\mathcal{E}_p^\nu}$. Functionals $f \in \mathcal{E}_p'$ will be called exponential type distributions.

For any $f \in \mathcal{E}_p'$ and $\varphi \in \mathcal{E}_p$ the following relation

$$\langle D^k f \mid \varphi\rangle = (-1)^{|k|}\langle f \mid D^k\varphi\rangle \qquad (k \in \mathbb{Z}_+^n)$$

correctly defines an operation of a generalized differentiation of distributions.

Theorem 3. *[11] The continuous and dense embeddings $\mathcal{E}_p \subset L_p(\mathbb{R}^n)$, $L_p(\mathbb{R}^n) \subset \mathcal{E}_q'$, where $\frac{1}{p} + \frac{1}{q} = 1$ are valid.*

A convolution of a distribution $f \in \mathcal{E}_p'$ and a function $\varphi \in \mathcal{E}_p$ will be defined as the relation

$$(f \star \varphi)(t) := \langle f(s) \mid \varphi(t+s)\rangle = \langle f(s) \mid T_{-s}\varphi(t)\rangle = \langle f(s) \mid T_{-t}\varphi(s)\rangle,$$

where $f(s)$ denotes an action of a functional f on a function $T_{-s}\,\varphi(t)$ by s.

Let $\mathcal{L}(\mathcal{E}_p)$ be an algebra of linear continuous operators on the space $\mathcal{E}_p$ with a strong operator topology.

Theorem 4. *Let $f, g \in \mathcal{E}_p'$ and $\varphi \in \mathcal{E}_p$. The space $\mathcal{E}_p'$ is a commutative algebra with respect to a convolution defined by the relation*

$$(f * g) \star \varphi := f \star (g \star \varphi).$$

The mapping $\mathcal{E}_p' \ni f \longrightarrow K_f \in \mathcal{L}(\mathcal{E}_p)$, where $K_f\varphi := f \star \varphi$, is an algebraic isomorphism on a commutant of the group T_{-s} in the algebra $\mathcal{L}(\mathcal{E}_p)$. The convolution has properties

$$D^k(f \star \varphi) = f \star (D^k\varphi) = (-1)^{|k|}(D^k f) \star \varphi,$$

$$D^k(f * g) = (D^k f) * g = f * (D^k g)$$

for any $k \in \mathbb{Z}_+^n$.

Proof. For $\varphi \in \mathcal{E}_p^\nu$ we have $\|K_f\varphi\|_{\mathcal{E}_p^\nu} \le \|f_\nu\| \, \|T_{-s}\varphi\|_{\mathcal{E}_p^\nu}$. From Theorem 1 (iii) $\|T_{-s}\varphi\|_{\mathcal{E}_p^\nu} = \|\varphi\|_{\mathcal{E}_p^\nu}$, then $K_f \in \mathcal{L}(\mathcal{E}_p^\nu)$, $\forall\, \nu$. Thus, $K_f \in \mathcal{L}(\mathcal{E}_p)$.

From Theorem 3 there are functions $g_\gamma \in L_q(\mathbb{R}^n)$ such that $\lim_\gamma g_\gamma = g$ in $\mathcal{E}_p'$. The duality $\langle L_q(\mathbb{R}^n) \mid L_p(\mathbb{R}^n)\rangle$ is defined by $\langle g_\gamma \mid \varphi\rangle = \int_{\mathbb{R}^n} g_\gamma(r)\varphi(r)\, dr$, then $(g_\gamma \star \varphi)(t) = \int_{\mathbb{R}^n} g_\gamma(r) T_{-t}\varphi(r)\, dr$. The function $\mathrm{R}^n \ni r \longrightarrow T_{-s-t}\varphi(r) \in \mathcal{E}_p^\nu$ is continuous for fixed t. And then

$$f \star (g_\gamma \star \varphi) = \big\langle f(s) \mid T_{-s}(g_\gamma \star \varphi)(t)\big\rangle = \Big\langle f(s) \mid \int_{\mathrm{R}^n} g_\gamma(r) T_{-s-t}\varphi(r)\, dr\Big\rangle =$$

$$\int_{\mathbb{R}^n} g_\gamma(r)\langle f(s) \mid T_{-s-t}\varphi(r)\rangle\, dr = \big\langle g_\gamma(r) \mid \langle f(s) \mid T_{-s-t}\varphi(r)\rangle\big\rangle = g_\gamma \star (f \star \varphi).$$

From this $f \star (g \star \varphi) = \lim_\gamma f \star (g_\gamma \star \varphi) = \lim_\gamma g_\gamma \star (f \star \varphi) = g \star (f \star \varphi)$.

Let us prove an isomorphism of the space $\mathcal{E}_p'$ to a commutant of the group T_{-s}. Let $f \in E_p'$ and $\varphi \in \mathcal{E}_p^\nu$. From the definition of the convolution and Theorem 1, we obtain $\|f \star \varphi\|_{\mathcal{E}_p^\nu} \le \|f\|_{\mathcal{E}_p^\nu} \|\varphi\|_{L_p}$, where $\|f\|_{\mathcal{E}_p^\nu}-$ the norm of restriction of a functional f on $\mathcal{E}_p^\nu$. Then from $D^k(f \star \varphi)(t) = \big\langle f(s)\, , T_{-s}\, D^k\varphi(t)\big\rangle = (f \star D^k\varphi)(t)$ it follows $\|f \star \varphi\|_{\mathcal{E}_p^\nu} = \sup\limits_k \dfrac{\|f \star D^k\varphi\|_{L_p}}{\nu^k} \le \|f\|_{\mathcal{E}_p^\nu}\|\varphi\|_{\mathcal{E}_p^\nu}$. The embeddings $\mathcal{E}_p^\nu \subset \mathcal{E}_p$ are continuous then $F \in \mathcal{L}(\mathcal{E}_p^\nu)$ and we have $F \in \mathcal{L}(\mathcal{E}_p)$. The relation

$$K_f T_{-s}\varphi = T_{-s} K_f \varphi \qquad (\forall \varphi \in \mathcal{E}_p, \quad s \in \mathbb{R}^n) \tag{5}$$

follows from the equalities $(f \star T_{-s}\varphi)(t) = (f \star \varphi)(t + s) = T_{-s}(f \star \varphi)(t)$.

To prove the converse, let $\varphi \in \mathcal{E}_p$. The mapping $f : \varphi \to (F\varphi)(0)$ is a functional $f \in \mathcal{E}_p'$. From this we obtain $(F\varphi)(0) = \langle f\, , \varphi\rangle = (f \star \varphi)(0)$. Replacing φ by $T_{-t}\varphi$ and using (5), we have $K_f : \mathcal{E}_p \ni \varphi \longrightarrow f \star \varphi$.

Now we prove differential properties of the convolution. Obviously, $D^k(f \star \varphi)(t) = \big\langle f(s) \mid T_{-s}D^k\varphi(t)\big\rangle = (f \star D^k\varphi)(t)$. Next relations

$$(f \star \varphi)(t) = \langle f(s) \mid T_{-t}\varphi(s)\rangle = \langle f(s + t - t) \mid \varphi(s + t)\rangle = \langle T_{-r} f(-t) \mid \varphi(r)\rangle$$

are valid. Then we have $D^k(f \star \varphi)(t) = (-1)^{|k|}\big\langle T_{-r}(D^k f)(-t) \mid \varphi(r)\big\rangle = (-1)^{|k|}(D^k f \star \varphi)(t)$ and $D^k(f * g) \star \varphi = (-1)^{|k|}(f * g) \star D^k\varphi = (-1)^{|k|} f \star (g \star D^k\varphi) = f \star (D^k g \star \varphi) = (f * D^k g) \star \varphi$. Using the commutation we obtain $D^k(f * g) = D^k(g * f) = g * D^k f$. □

Corollary 1.1. *For an arbitrary distribution $f \in \mathcal{E}'_p$ and a vector ν the subspace $\mathcal{E}^\nu_p$ is invariant relatively to K_f and $K_{f_\nu} \in \mathcal{L}(\mathcal{E}^\nu_p)$.*

Corollary 1.2. *Let $f \in \mathcal{E}'_p$ and $\varphi \in \mathcal{E}_p$, $\psi \in \mathcal{E}_q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then $(f \star \varphi) * \psi = f \star (\varphi * \psi)$.*

Proof. Since $f \star \varphi \in \mathcal{E}_p$ then $[(f \star \varphi) * \psi](t) = \int\limits_{\mathbb{R}^n} \langle f(s) \mid T_{-s}\varphi(r)\rangle T_r\psi(t)\,dr = \langle f(s) \mid T_{-s} \int\limits_{\mathbb{R}^n} \varphi(r)T_r\psi(t)\,dr\rangle = [f \star (\varphi * \psi)](t)$. □

1.3 The Fourier transformation

We introduce the Fourier transformation onto the space of exponential type distribution. For $p = 1$ let us denote $\widehat{\mathcal{E}}_1 := \left\{\widehat{\varphi}(\xi) = \int\limits_{\mathbb{R}^n} e^{-it\cdot\xi}\varphi(t)\,dt \,:\, \varphi \in \mathcal{E}_p\right\}$, for $1 < p < \infty$ $\widehat{\mathcal{E}}_p := \left\{\widehat{\varphi}(\xi) = \mathcal{F}(\varphi) : \varphi \in \mathcal{E}_p\right\}$, where $t \cdot \xi := t_1\xi_1 + \ldots + t_n\xi_n$ for any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. The Fourier transformation is a linear isomorphism $\mathcal{F} : \mathcal{E}_p \ni \varphi(t) \longrightarrow \widehat{\varphi}(\xi) \in \widehat{\mathcal{E}}_p$. We endowed $\widehat{\mathcal{E}}_p$ with a topology relatively to the mapping $\mathcal{F}$.

Using the isometry $\mathcal{E}^\nu_p \simeq \mathcal{M}^\nu_p(\mathbb{C}^n)$ from Theorem 1 and the fact that Fourier-images of exponential type functions are finite [13] we can define an inverse transformation by the formula for $p = 1$

$$\mathcal{F}^{-1} : \widehat{\mathcal{E}}_1 \ni \widehat{\varphi}(\xi) \longrightarrow \varphi(t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} e^{it\cdot\xi}\widehat{\varphi}(\xi)\,d\xi \in \mathcal{E}_1,$$

for $1 < p < \infty$ that is an inverse mapping $\mathcal{F}^{-1} : \widehat{\mathcal{E}}_p \ni \widehat{\varphi}(\xi) \longrightarrow \varphi(t) \in \mathcal{E}_p$.

The duality $\langle \mathcal{E}'_p \mid \mathcal{E}_p\rangle$ defines an adjoint mapping to the inverse one

$$\mathcal{F}^{\#} := 2\pi(\mathcal{F}^{-1})' : E'_p \ni f \longrightarrow \widehat{f} \in \widehat{E}'_p, \quad \text{where} \quad \langle \widehat{f} \mid \widehat{\varphi}\rangle := (2\pi)^n\langle f \mid \varphi\rangle.$$

Its image $\widehat{\mathcal{E}}'_p$, that generates a duality $\langle \widehat{\mathcal{E}}'_p \mid \widehat{\mathcal{E}}_p\rangle$, we endow with a weak topology that coincides with an inductive topology relatively to $\mathcal{F}^{\#}$.

We use the symbols $\varphi_-(t) := \varphi(-t)$.

Theorem 5. *For any $f, g \in \mathcal{E}'_p$, $\varphi \in \mathcal{E}_p$, $\psi \in \mathcal{E}_q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, the Fourier transform has properties $\widehat{\varphi * \psi} = \widehat{\varphi} \cdot \widehat{\psi}$, $\widehat{f \star \varphi} = \widehat{f} \cdot \widehat{\varphi}_-$, where $\langle \widehat{f} \cdot \widehat{\varphi}_- \mid \widehat{\psi}\rangle = \langle \widehat{f} \mid \widehat{\varphi}_- \cdot \widehat{\psi}\rangle$ and the space $\widehat{\mathcal{E}}'_p$ is a commutative algebra with respect to the multiplication, that is defined by the relation $\langle \widehat{g} \cdot \widehat{f} \mid \widehat{\varphi}\rangle = \langle \widehat{g} \mid \widehat{f} \cdot \widehat{\varphi}\rangle$. Moreover, the following equalities $\widehat{g * f} = \widehat{g} \cdot \widehat{f}$, $\widehat{D^k f} = (-\xi)^k\widehat{f}$, $(\forall\, k \in \mathbb{Z}_+)$ are valid.*

Proof. Using the Corollary 1.1, we have

$$\langle \widehat{f \star \varphi} \mid \widehat{\psi}\rangle = (2\pi)^n\langle f \star \varphi \mid \psi\rangle = (2\pi)^n[(f \star \varphi) \star \psi](0) = (2\pi)^n[f \star (\varphi \star \psi)](0) =$$

$$(2\pi)^n\langle f \mid \varphi \star \psi\rangle = \langle \widehat{f} \mid \widehat{\varphi \star \psi}\rangle = \langle \widehat{f} \mid \widehat{\varphi}_- \cdot \widehat{\psi}\rangle = \langle \widehat{f} \cdot \widehat{\varphi}_- \mid \widehat{\psi}\rangle.$$

The correctness of definition of the multiplication follows from next equalities

$$\langle \widehat{g \star f} \mid \widehat{\varphi} \rangle = (2\pi)^n \langle g \star f \mid \varphi \rangle = (2\pi)^n [(g \star f) \star \varphi](0) = (2\pi)^n [g \star (f \star \varphi)](0) =$$
$$(2\pi)^n \langle g \mid f \star \varphi \rangle = \langle \widehat{g} \mid \widehat{f \star \varphi} \rangle = \langle \widehat{g} \mid \widehat{f} \cdot \widehat{\varphi}_- \rangle = \langle \widehat{g} \cdot \widehat{f} \mid \widehat{\varphi}_- \rangle.$$

Since $D^k \varphi \in \mathcal{E}_p(\mathbb{R}^n)$, then $\widehat{(D^k \varphi)}(\xi) = \xi^k \widehat{\varphi}(\xi) \in \widehat{\mathcal{E}}_p(\mathbb{R}^n)$. Hence

$$\langle \widehat{D^k f} \mid \widehat{\varphi} \rangle = (2\pi)^n \langle D^k f \mid \varphi \rangle = (2\pi)^n (-1)^k \langle f \mid D^k \varphi \rangle = (-1)^k \langle \widehat{f} \mid \widehat{D^k \varphi} \rangle =$$
$$(-1)^k \langle \widehat{f} \mid \xi^k \widehat{\varphi} \rangle = \langle (-\xi)^k \widehat{f} \mid \widehat{\varphi} \rangle.$$

□

2 Functional calculus

2.1 Finite functions of the generators of C_0-groups

We construct finite functions of generators of strongly continuous groups of bounded linear operators on an arbitrary Banach space.

Let $\{X, \|\cdot\|\}$ be a complex Banach space, $\mathcal{L}(X)$ be an algebra of linear bounded operators on X with a uniform norm $\|\cdot\|_{\mathcal{L}(X)}$ and $U : \mathbb{R}^n \ni t \longrightarrow U_t \in \mathcal{L}(X)$ be an n-parameter C_0–group on X. For every index $j = 1, \dots, n$ generators are determined by $D_j U_t x_{|t=0} = -A_j x$, where $x \in \mathcal{D}(A_j)$. Let the operators $A_j : \mathcal{D}(A_j) \subset X \longrightarrow X$ be closed and densy determined. We denote $A := (A_1, \dots, A_n)$. An example: if $U_t = T_t$ and $X = L_p(\mathbb{R}^n)$ then $D_j T_t \varphi = -\partial/\partial t_j T_t \varphi$ for $\varphi \in L_p(\mathbb{R}^n)$.

Let us assume

$$\|U_t x\|_q := \left(\int_{\mathbb{R}^n} \|U_t x\|^q \, dt \right)^{1/q}, \quad \|U\|_q := \inf \Big\{ c : \|U_t x\|_q \le c \, \|x\|, \ x \in X \Big\},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Let the group satisfies the condition

$$\|U_t x\|_q \le \|U\|_q \|x\| \qquad (\forall x \in X). \tag{6}$$

In the case $p = 1$ the condition (6) is equivalent to a uniform bound of the group. For example, the condition (6) is true for the shift group $U_t = T_t$ on $X = L_p(\mathbb{R}^n)$.

Theorem 6. *Let $\varphi \in \mathcal{E}_p$ and the group U satisfy the condition* (6)*. Then the operators that are defined by the formula*

$$\widehat{\varphi}(A) := \int_{\mathbb{R}^n} U_t \varphi(t) \, dt$$

belong to the Banach algebra $\mathcal{L}(X)$ and satisfies the relation $\widehat{(D_j \varphi)}(A) = A_j \widehat{\varphi}(A)$ $(j = 1, \dots, n)$. If $p = 1$ and the group U_t is uniformly bounded then $\widehat{(\varphi * \psi)}(A) = \widehat{\varphi}(A) \cdot \widehat{\psi}(A)$ $\big(\forall \, \widehat{\varphi}, \widehat{\psi} \in \widehat{E}_1\big)$.

Proof. Let $\varphi \in \mathcal{E}_p^\nu$. By Theorem 1 and condition (6), we have

$$\|\widehat{\varphi}(A)x\| \leq \int\limits_{\mathbb{R}^n} \|U_t x\|\,|\varphi(t)|\,dt \leq \|U_t x\|_q\,\|\varphi\|_{\mathcal{E}_p^\nu} \leq \|U\|_q \|\varphi\|_{\mathcal{E}_p^\nu}\|x\| \quad (\forall x \in X). \tag{7}$$

It is know ([13],III,3.2.5) that functions $\varphi \in \mathcal{E}_p$ have the property $\lim\limits_{t\to\infty} \varphi(t) = 0$. Then integrating by parts and using the property, that a generator of group is closed, we obtain

$$\widehat{(D_j\varphi)}(A) = \int\limits_{\mathbb{R}^n} U_t(D_j\varphi)(t)\,dt = -\int\limits_{\mathbb{R}^n} D_j U_t \varphi(t)\,dt = A_j\widehat{\varphi}(A).$$

For $p = 1$ we determine convolution of functions and next equalities

$$\widehat{\varphi}(A)\cdot\widehat{\psi}(A) = \int\limits_{\mathbb{R}^n} U_t\varphi(t)\,dt\Big[\int\limits_{\mathbb{R}^n} U_s\psi(s)\,ds\Big] = \int\limits_{\mathbb{R}^n}\varphi(t)\Big[\int\limits_{\mathbb{R}^n} U_{s+t}\psi(s)\,ds\Big]\,dt =$$

$$\int\limits_{\mathbb{R}^n}\Big[\int\limits_{\mathbb{R}^n} U_r\varphi(r-s)\psi(s)\,ds\Big]\,dr = \int\limits_{\mathbb{R}^n} U_r\Big[\int\limits_{\mathbb{R}^n}\varphi(r-s)\psi(s)\,ds\Big]\,dr = \widehat{(\varphi*\psi)}(A)$$

are valid. □

2.2 Functional calculus in algebras of exponential type distributions

We construct the functional calculus as generalized functions for generators of C_0-groups.

Let $X\widetilde{\otimes}L_p(\mathbb{R}^n)$ be a completion of a projective tensor product of X and $L_p(\mathbb{R}^n)$. In the case $p = 1$, as it is known ([16], III, 3.2.5), we have the isometric representation $L_1(\mathbb{R}^n;X) \simeq X\widetilde{\otimes}L_1(\mathbb{R}^n)$, where $L_1(\mathbb{R}^n;X)$ is a Banach space of all X–valued functions $\mathbb{R}^n \ni t \longrightarrow x(t) \in X$ with the norm $\|x\|_{L_1(X)} := \int\limits_{\mathbb{R}^n} \|x(t)\|\,dt$.

Using the isometric embedding $\mathcal{E}_p^\nu(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$ for every vector ν we can determine the subspace $\mathcal{E}_p^\nu(\mathbb{R}^n;X) := X\widetilde{\otimes}\mathcal{E}_p^\nu(\mathbb{R}^n) \subset X\widetilde{\otimes}L_p(\mathbb{R}^n)$. The embeddings $\mathcal{E}_p^\nu(\mathbb{R}^n;X) \subset \mathcal{E}_p^\mu(\mathbb{R}^n;X)$ $(\nu_1 \leq \mu_1,\ldots,\nu_n \leq \mu_n)$ are continuous, then on a union of all these spaces we can define a structure of an inductive limit space

$$\mathcal{E}_p(\mathbb{R}^n;X) := \bigcup_\nu \mathcal{E}_p^\nu(\mathbb{R}^n;X) = \lim_\nu \operatorname{ind}\, \mathcal{E}_p^\nu(\mathbb{R}^n;X) \subset X\widetilde{\otimes}L_p(\mathbb{R}^n).$$

A convolution of an arbitrary distribution $f \in \mathcal{E}_p'(\mathbb{R}^n)$ and a vector-valued function $x(t) \in \mathcal{E}_p(\mathbb{R}^n;X)$ is used to define by $(f \star x)(t) := (I \otimes K_f)x(t)$, where I is the identity operator in X.

The topological isomorphism $\lim\limits_\nu \operatorname{ind}\, \mathcal{E}_p^\nu(\mathbb{R}^n;X) \simeq X\widetilde{\otimes}\lim\limits_\nu \operatorname{ind}\, \mathcal{E}_p^\nu(\mathbb{R}^n) = X\widetilde{\otimes}\mathcal{E}_p(\mathbb{R}^n)$ is valid. This assertion is a corollary of a known property of the inductive limit and a projective tensor products [7]. From this assertion and known Grothendieck's representation ([16], III, §6) it follows that for every function $x(t) \in \mathcal{E}_p(\mathbb{R}^n;X)$ there is a vector ν such that $x(t)$ belongs to the space $\mathcal{E}_p^\nu(\mathbb{R}^n;X)$ and is represented in the form of the absolute convergent series

$$x(t) = \sum_{j=1}^\infty x_j \otimes \varphi_j(t), \quad \text{where} \quad x_j \in X, \quad \varphi_j(t) \in \mathcal{E}_p^\nu(\mathbb{R}^n). \tag{8}$$

Using this representation for any distribution $f \in \mathcal{E}'_p(\mathbb{R}^n)$ and a vector-valued function $x(t) \in \mathcal{E}_p(\mathbb{R}^n; X)$, we obtain $(f \star x)(t) = \sum_{j=1}^{\infty} x_j \otimes (f \star \varphi_j)(t)$.

Lemma 2.1. *The convolution has properties*

$$(f * g) \star x = f \star (g \star x),$$

$$D^k(f \star x) = f \star (D^k x) = (-1)^{|k|}(D^k f) \star x$$

for any $f, g \in \mathcal{E}'_p(\mathbb{R}^n)$, $x = x(t) \in \mathcal{E}^\nu_p(\mathbb{R}^n; X)$ *and* $k \in \mathbb{Z}^n_+$.

Proof. From the definition of the convolution and by Theorem 4 next equalities follow

$$(f * g) \star x = \sum_{j=1}^{\infty} x_j \otimes (f * g) \star \varphi_j = f \star \sum_{j=1}^{\infty} x_j \otimes (g \star \varphi_j) = f \star (g \star x),$$

$$D^k(f \star x) = \sum_{j=1}^{\infty} x_j \otimes D^k(f \star \varphi_j) = \sum_{j=1}^{\infty} x_j \otimes (-1)^{|k|}(D^k f) \star \varphi_j = (-1)^{|k|}(D^k f) \star x.$$

□

Lemma 2.2. *Let the group U satisfies the condition* (6). *Then each of the subspaces*

$$\widehat{\mathcal{E}}^\nu_p(X) := \left\{ \widehat{x} = \int_{\mathbb{R}^n} (U_t \otimes I) x(t)\, dt : \ x(t) \in \mathcal{E}^\nu_p(\mathbb{R}^n; X) \right\}$$

is a Banach space respectively to the norm induced by the mapping

$$\mathcal{E}^\nu_p(\mathbb{R}^n; X) \ni x(t) \longrightarrow \widehat{x} \in \widehat{\mathcal{E}}^\nu_p(X).$$

Proof. Let us show that the mapping $\mathcal{E}^\nu_p(\mathbb{R}^n; X) \ni x(t) \longrightarrow \widehat{x} \in X$ is continuous. From (8) we obtain

$$\widehat{x} = \int_{\mathbb{R}^n} \Big[\sum_{j=1}^{\infty} U_t x_j \otimes \varphi_j(t) \Big] dt = \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} U_t x_j \otimes \varphi_j(t)\, dt = \sum_{j=1}^{\infty} \widehat{\varphi}_j(A) x_j.$$

From this and using the estimate (7), we have

$$\|\widehat{x}\| \le \sum_{j=1}^{\infty} \|x_j\| \, \|\widehat{\varphi_j}(A)\|_{\mathcal{L}(\mathcal{X})} \le \|U\|_q \sum_{j=1}^{\infty} \|x_j\| \, \|\varphi_j\|_{\mathcal{E}^\nu_p}.$$

Using the arbitrary presentation $x(t)$ by absolute convergent series we obtain

$$\|\widehat{x}\| \le \|U\|_q \, \|x(t)\|_{\mathcal{E}^\nu_p(\mathbb{R}^n; X)}$$

and the continuity is proved. A kernel of the continuous mapping $\mathcal{E}^\nu_p(\mathbb{R}^n; X) \longrightarrow X$ is closed then a corresponding factor-space in this kernel is a Banach space. By the definition of the norm in the space $\widehat{\mathcal{E}}^\nu_p(X)$, it is isometric to the constructed factor-space. □

Lemma 2.3. *Let the group U satisfies the condition* (6). *Then each of the subspaces $\widehat{\mathcal{E}}_p^\nu(X)$ is invariant respectively to the operator*

$$\widehat{K}_f : \widehat{\mathcal{E}}_p^\nu(X) \ni \widehat{x} \longrightarrow \widehat{K}_f\widehat{x} := \int\limits_{\mathbb{R}^n} (U_t \otimes K_f)x(t)\,dt.$$

Proof. Taking into account (8) for elements $x(t) \in \mathcal{E}_p^\nu(\mathbb{R}^n; X)$ for any $f \in \mathcal{E}_p'(\mathbb{R}^n)$ we obtain

$$\|(I \otimes K_f)x(t)\| \le \sum_{j=1}^{\infty} \|x_j\|\,\|K_f\varphi_j\|_{\mathcal{E}_p^\nu} \le \|K_f\|_{\mathcal{L}(\mathcal{E}_p^\nu)} \sum_{j=1}^{\infty} \|x_j\|\,\|\varphi_j\|_{\mathcal{E}_p^\nu}$$

or $\|(I \otimes K_f)x(t)\| \le \|K_f\|_{\mathcal{L}(\mathcal{E}_p^\nu(\mathbb{R}^n))}\,\|x(t)\|_{\mathcal{E}_p^\nu(\mathbb{R}^n;X)}$. Thus the space $\mathcal{E}_p^\nu(\mathbb{R}^n; X)$ is invariant with respect to the acting operator $I \otimes K_f$.

Since $U_t \otimes K_f = (U_t \otimes I)(I \otimes K_f)$ and $I \otimes K_f : \mathcal{E}_p^\nu(\mathbb{R}^n; X) \longrightarrow \mathcal{E}_p^\nu(\mathbb{R}^n; X)$ then by Lemma 2.2 for any vector-valued function $x(t) \in \mathcal{E}_p^\nu(\mathbb{R}^n; X)$ we have $\widehat{K}_f\widehat{x} \in \widehat{\mathcal{E}}_p^\nu(X)$. Then $\widehat{K}_f : \widehat{\mathcal{E}}_p^\nu(X) \longrightarrow \widehat{\mathcal{E}}_p^\nu(X)$. The lemma is proved. □

The embeddings $\mathcal{E}_p^\nu(\mathbb{R}^n; X) \subset \mathcal{E}_p^\mu(\mathbb{R}^n; X)$ are continuous. From this it follows that the next embeddings $\widehat{\mathcal{E}}_p^\nu(X) \subset \widehat{\mathcal{E}}_p^\mu(X)$ are also continuous. Then a union of these spaces can be represented as an inductive limit

$$\widehat{\mathcal{E}}_p(X) := \bigcup_\nu \widehat{\mathcal{E}}_p^\nu(X) = \lim_\nu \operatorname{ind}\, \widehat{\mathcal{E}}_p^\nu(X).$$

Let us define by $\mathcal{L}(\widehat{\mathcal{E}}_p(X))$ an algebra of all linear continuous operators on $\widehat{\mathcal{E}}_p(X)$ with a strong operator topology.

Theorem 7. *Let the group U satisfies the condition* (6). *Then the mapping $\widehat{\mathcal{E}}_p'(\mathbb{R}^n) \ni \widehat{f} \longrightarrow \widehat{f}(A) \in \mathcal{L}(\widehat{\mathcal{E}}_p(X))$, where the linear operator $\widehat{f}(A)$ is defined by the relation*

$$\widehat{f}(A) : \widehat{\mathcal{E}}_p(X) \ni \widehat{x} \longrightarrow \widehat{f}(A)\widehat{x} := \int\limits_{\mathbb{R}^n} (U_t \otimes K_f)x(t)\,dt \in \widehat{\mathcal{E}}_p(X),$$

is a continuous homomorphism of the algebra of symbols $\widehat{\mathcal{E}}_p'(\mathbb{R}^n)$ onto a subalgebra of algebra $\mathcal{L}(\widehat{\mathcal{E}}_p(X))$ of operators

$$\widehat{K} : \widehat{\mathcal{E}}_p(X) \ni \widehat{x} \longrightarrow \widehat{K}\widehat{x} := \int\limits_{\mathbb{R}^n} (U_t \otimes K)x(t)\,dt\ ,$$

where the operator $K \in \mathcal{L}(\mathcal{E}_p(\mathbb{R}^n))$ belongs to the commutant of the group T_s. And we have $\widehat{(D_j f)}(A) = A_j\widehat{f}(A)\,(j = 1, \dots, n)$.

Proof. From Theorem 4 and Corollary 1.1 any operator, that belongs to a commutant of group T_s, has the form K_f, where $f \in \mathcal{E}_p'(\mathbb{R}^n)$. From Lemma 2.3 and the definition of the space $\widehat{\mathcal{E}}_p^\nu(X)$ it follows that $\widehat{K}_f : \widehat{\mathcal{E}}_p^\nu(X) \longrightarrow \widehat{\mathcal{E}}_p^\nu(X)$ for every ν. From the definition of the norm in $\widehat{\mathcal{E}}_p^\nu(X)$ we have $\widehat{x}_m \to \widehat{x}$ if and only if $x_m(t) \to x(t)$ in the space $\mathcal{E}_p^\nu(\mathbb{R}^n; X)$. If

$x_m(t) \to x(t)$ then from continuity of K_f it follows, that $(I \otimes K_f)x_m(t) \to (I \otimes K_f)x(t)$ in the space $\mathcal{E}_p^\nu(\mathbb{R}^n; X)$. From this by the definition of the norm in the space $\widehat{\mathcal{E}}_p^\nu(X)$ we obtain $(I \otimes K_f)x_m(t) \to (I \otimes K_f)x(t)$. Thus, $\widehat{K}_f \in \mathcal{L}(\widehat{\mathcal{E}}_p^\nu(X))$ for any ν and therefore $\widehat{K}_f \in \mathcal{L}(\widehat{\mathcal{E}}_p(X))$.

From the equality $K_{f*g} = K_f \cdot K_g$ for arbitrary $f, g \in \mathcal{E}'_p(\mathbb{R}^n)$ it follows $\widehat{K}_{f*g} = \widehat{K}_f \cdot \widehat{K}_g$. Thus, the functional calculus is an algebraic homomorphism from the convolution algebra of distribution onto an algebra of continuous operators on the space $\widehat{\mathcal{E}}_p(X)$.

We now prove a continuity of the functional calculus. As in the space $\widehat{\mathcal{E}}'_p(\mathbb{R}^n)$ a topology is inducted from $\mathcal{E}'_p(\mathbb{R}^n)$, and the space $\mathcal{E}'_p(\mathbb{R}^n)$ is given a weak topology, then it is enough to show a continuity of the mapping $\mathcal{E}'_p(\mathbb{R}^n) \ni f \longrightarrow \widehat{f}(A)\widehat{x} \in \widehat{\mathcal{E}}_p(X)$ for every $\widehat{x} \in \widehat{\mathcal{E}}_p^\nu(X)$. The continuity of $\mathcal{E}'_p(\mathbb{R}^n) \ni f \longrightarrow K_f \in \mathcal{L}(\mathcal{E}_p^\nu(\mathbb{R}^n))$ we obtain from $\|f \star \varphi\|_{\mathcal{E}_p^\nu} \le \|f_\nu\| \, \|\varphi\|_{\mathcal{E}_p^\nu}$, where f_ν is a restricted functional on the subspace $\mathcal{E}_p^\nu(\mathbb{R}^n)$. Then it will be continuous the mapping $\mathcal{E}'_p(\mathbb{R}^n) \ni f \longrightarrow (I \otimes K_f)x(t) \in \mathcal{E}_p^\nu(\mathbb{R}^n; X)$ for every $x(t) \in \mathcal{E}_p^\nu(\mathbb{R}^n; X)$. Let $f_m \to f$ in the space $\mathcal{E}'_p(\mathbb{R}^n)$. Then $(I \otimes K_{f_m})x(t) \to (I \otimes K_f)x(t)$ in the space $\mathcal{E}_p^\nu(\mathbb{R}^n; X)$. By the definition of the norm in $\widehat{\mathcal{E}}_p^\nu(X)$, we have $\widehat{K}_{f_m}\widehat{x} \to \widehat{K}_f\widehat{x}$ in the space $\widehat{\mathcal{E}}_p^\nu(X)$. Thus, the mapping $\mathcal{E}'_p(\mathbb{R}^n) \ni f \longrightarrow \widehat{K}_f\widehat{x} \in \widehat{\mathcal{E}}_p(X)$ is continuous.

The rest assertions of the theorem follow from Lemmas 2.1, 2.3 and Theorem 6. The theorem is proved. □

Thus, we found the integral image of the Fourier operator transform in a case of the algebra of linear continuous functionals on a space of entire function of exponential type, that on the real subspace belongs to $L_p(\mathbb{R}^n)$ $(1 \le p < \infty)$. We also established its multiplicate properties relatively to the convolution and described its image by way of commutant of the scalar many-parameter shift group.

References

1. Akhiezer N. Lectures on the Theory of Approximation, Moscow, 1965.
2. Balakrishnan A.V. *Fractional powers of closed operators and the semigroups generated by them*, Pacific J. Math., **10**, 2 (1960), 419–439.
3. Bogolyubov N.N., Shyrkov D.V. Introduction in the Theory of Quantum Fields, Moscow, 1984.
4. Bogolyubov N.N., Logunov A.A., Todorov I.T. Basis of Axiomatic Approach in Quantum Field Theory, Moscow, 1969.
5. Dmytryshyn M.I., Lopushansky O.V. *Operator calculus on the exponential type vectors of the operator with point spectrum*, General topology in Banach spaces. – Huntington, New York: NOVA Sci. Publ., (2001), 137–145.
6. Gorbachuk M.L., Gorbachuk V.I. *On the approximation of smooth vectors of a closed operator by entire vectors of exponential type*, Ukr. Math. Journal., **47**, 5 (1995), 616–628.
7. Grothendieck A. *Produits tensoriels topologigues et espases nucleaire*, Mem. Amer. Math. Soc., 16 (1955), 1–140.
8. Hille E., Phillips R. Functional Analysis and Semi-Groups. AMS, 1957.
9. Iosida K. Functional Analysis. Springer–Verlag, 1965.
10. Lopushansky O.V. *Operator calculus on ultrasmooth vectors*, Ukr. Math. Journal, **44**, 4 (1992), 502–513.

11. Lopushansky O.V., Lozynska V.Ya. *Analytic distributions of exponential type*, Mathematical Methods and Physicomechanical fields, **42**, 4 (1999), 46–55.

12. Lyubich Yu.I., Matsaev V.I. *Operators with a Separable Spectrum*, Amer. Math. Soc. Transl. (2), **47** (1965), 89–129.

13. Nikolsky S.M. Approximation of Functions of Several Variables and Embeddings Theorems, Moscow, 1977.

14. Radyno Ya.V. *Vectors of exponential type in operator calculus and differential equations*, Differential equations, **21**, 9 (1985), 1559–1569.

15. Radzievsky G.V. *Direct and inverse theorems in problems of approximation by vectors of finite degree*, Math.Sb., **189**, 4 (1998), 83–124.

16. Schaefer H. Topological Vector Spaces, Moscow, 1971.

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Received 20.11.2009

Revised 15.02.2011

Лозинська В.Я. *Розподіли експоненціального типу і узагальнене функціональне числення для генераторів C_0-груп* // Карпатські математичні публікації. — 2011. — Т.3, №1. — С. 73–84.

Описано властивості простору спряженого до простору цілих функцій експоненціального типу багатьох комплексних змінних, що на дійсному підпросторі належать до $L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$. У Фур'є–образі цього простору побудовано функціональне числення для генераторів сильно неперервних груп обмежених лінійних операторів, що діють на довільному банаховому просторі.

Лозинская В.Я. *Распределения экспоненциального типа и обобщенное функциональное исчисление для генераторов C_0-групп* // Карпатские математические публикации. — 2011. — Т.3, №1. — С. 73–84.

Описаны свойства сопряженного пространства к пространству целых функций экспоненциального типа многих комплексных переменных, которые на дейсвительном подпространстве принадлежат к $L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$. В Фурье–образе этого пространства построено функциональное исчисление для генераторов сильно непрерывных груп ограниченных линейных операторов, которые действуют на произвольном банаховом пространстве.