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NYKYFORCHYN KH.O.

# CONDITIONAL OPERATIONS ON FUZZY NUMBERS

An estimate of a joint possibility distribution of two fuzzy numbers, based on fuzzy relations with a third fuzzy variable, is suggested. This leads to the operations of conditional addition and conditional multiplication, of interactive fuzzy numbers for which joint possibility distributions with a fuzzy variable are known.

*Key words and phrases:* fuzzy arithmetic, possibility distribution.

Taras Shevchenko National University, 64/13 Volodymyrska str., 01601, Kyiv, Ukraine

## INTRODUCTION

Simulation and prognosis for natural, economic, and social processes should take into account unavoidable imprecision in input data and parameters, as well as non-determinism, which is inherent to such processes. Therefore fuzzy sets, fuzzy numbers, and fuzzy logic are used extensively in this class of problems [1].

Extension of functions and operations from crisp sets onto fuzzy ones follows the widely known *extension principle* by Lofty Zadeh [4]. In particular, when a function of several variables is extended, a fuzzy joint distribution of the variables is constructed with the assumption of non-interactivity. The latter property is analogous to independence of random variables. Not only such an assumption can be unrealistic, but difficulties arise if the domain of a function of several variables is not the cartesian product of domains for the respective variables.

In the presented paper we construct operations on one- and many-dimensional fuzzy variables that take into account their simultaneous fuzzy relations with other variables. The omitted proofs are straightforward and can be easily reproduced by the reader.

## 1 JOINT DISTRIBUTION OF INDIRECTLY FUZZY RELATED VARIABLES

In the sequel $\mathbb{I} = [0;1]$ is the unit segment with the standard metric, $\mathbf{1}_M : M \to M$ is the identity mapping from a set $M$ onto itself. Recall that a metric space is called *proper* if all its closed bounded sets are compact. This implies completeness and local compactness, but the converse is not true. From now on all metric space are considered proper, unless otherwise is stated explicitly.

Let $(X, d)$ be such a metric space. A mapping $A : X \to \mathbb{I}$ is called a *(normal upper semicontinuous) fuzzy $X$-valued variable* if:

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(i) $A$ is upper semicontinuous;

(ii) there is $x \in X$ such that $A(a) = 1$.

A fuzzy $X$-valued variable $A$ is called *weakly bounded* if the following is also valid:

(iii) the $\alpha$-cuts $A_\alpha = \{x \in X \mid A(x) \geqslant \alpha\}$ for all $\alpha \in (0;1]$ are bounded sets.

The set of all fuzzy $X$-valued variables is denoted by $\mathcal{F}(X)$, and $\mathcal{F}_0(X)$ is the subset of all weakly bounded fuzzy $X$-valued variables.

Observe that (i) implies that all $\alpha$-cuts are closed sets, hence they are *compact*. Moreover, the hypograph of the function $A$, i.e., the set $\operatorname{hypo} A = \{(x,\alpha) \in X \times \mathbb{I} \mid \alpha \leqslant A(x)\}$, is closed as well. This allows to define the distance between $X$-valued fuzzy variables as the Hausdorff distance between their hypographs:

$$\begin{aligned} d_H(A,B) = \inf\{\varepsilon \geqslant 0 \mid & \text{ for all } x \in X \text{ there are } x_1, x_2, \\ & \text{such that } d(x,x_1) \leqslant \varepsilon, A(x) \leqslant B(x_1) + \varepsilon, d(x,x_2) \leqslant \varepsilon, B(x) \leqslant A(x_2) + \varepsilon\}. \end{aligned}$$

Thus we avoid negative effects, which appeared in the previous attempts to apply Hausdorff distance to fuzzy numbers [2].

**Proposition 1.1.** *The metric space $(\mathcal{F}(X), d_H)$ is complete, and the subspace $(\mathcal{F}_0(X), d_H)$ is closed in $(\mathcal{F}(X), d_H)$, therefore is complete.*

Observe that the metric $d_H$ is bounded from the above by 1, hence neither $(\mathcal{F}(X), d_H)$ nor $(\mathcal{F}_0(X), d_H)$ is proper for an unbounded space $X$.

The number $A(x)$ is usually interpreted as the plausibility of the event "The fuzzy variable $A$ attains the value $x$", and the function $A$ itself is considered as a possibility distribution for the variable in question. If an $Y$-valued fuzzy variable $B$ is uniquely determined with an $X$-valued fuzzy number $A$ via a mapping $f : X \to Y$, then, by Zadeh' extension principle, the fuzzy distribution of $B$ is estimated as follows: $B(y) = \sup\{A(x) \mid x \in X, f(x) = y\}$ for all $y \in Y$. The latter distribution function may fail to be upper semicontinuous, but the following statement holds.

**Proposition 1.2.** *If the values of an $Y$-valued fuzzy variable $B$ are determined with the values of $A \in \mathcal{F}_0(X)$ via a continuous mapping $f$ by the above formula, then $B \in \mathcal{F}_0(Y)$,*

$$\operatorname{hypo} B = (f \times \mathbf{1}_{\mathbb{I}})(\operatorname{hypo} A) \cup (Y \times \{0\}),$$

*and the correspondence $A \mapsto B$ is a continuous mapping $\mathcal{F}_0(X) \to \mathcal{F}_0(Y)$ (which is denoted by $\mathcal{F}_0(f)$ from now on).*

Let $X_1, \dots, X_k$ be proper metric spaces, $A_1, \dots, A_k$ weakly bounded fuzzy variables with the values in these spaces. Then $(A_1, \dots, A_k)$ attains the values in $X_1 \times \dots \times X_k$. If joint non-interactivity of $A_1, \dots, A_k$ is assumed, then the standard inference on the possibility distribution if $(A_1, \dots, A_k)$, by the extension principle, is of the form

$$A_1 \otimes \ldots \otimes A_k(x_1, \dots, x_k) = A_1(x_1) \wedge \dots \wedge A_k(x_k),$$

where $\wedge$ is pairwise minimum of real numbers. For interactive variables a joint distribution can differ, but necessarily

$$\mathcal{F}_0(\mathrm{pr}_i)((A_1,\dots,A_k)) = A_i, \quad i = 1,\dots,k.$$

Assume that, for weakly bounded fuzzy variables $A_1 \in \mathcal{F}_0(X_1)$, $A_2 \in \mathcal{F}_0(X_2)$, $B \in \mathcal{F}_0(Y)$, joint possibility distributions of $(A_1, B) \in \mathcal{F}_0(X_1 \times Y)$ and $(A_2, B) \in \mathcal{F}_0(X_2 \times Y)$ are known. They describe fuzzy relations between the values of the variables $A_1, A_2$ and the values of $B$. Following the arguments for the extension principle, we analogously deduce that a reasonable estimate for the distribution of $(A_1, A_2, B)$ is given by

$$A_1 \otimes_B A_2(x_1, x_2, y) = (A_1, B)(x_1, y) \wedge (A_2, B)(x_2, y), \quad x_1 \in X_1, x_2 \in X_2, y \in Y.$$

**Proposition 1.3.** *The introduced above mapping*

$$\otimes_{\dots} : \left\{(C_1, C_2) \in \mathcal{F}_0(X_1 \times Y) \times \mathcal{F}_0(X_2 \times Y) \;\middle|\; \mathcal{F}_0(\mathrm{pr}_2)(C_1) = \mathcal{F}_0(\mathrm{pr}_2)(C_2)\right\} \to \mathcal{F}_0(X_1 \times X_2 \times Y)$$

*is continuous.*

**Remark.** *The equalities*

$$\mathcal{F}_0(\mathrm{pr}_{13})(A_1 \otimes_B A_2) = (A_1, B), \quad \mathcal{F}_0(\mathrm{pr}_{23})(A_1 \otimes_B A_2) = (A_2, B),$$

*are obviously valid, i.e., the suggested estimate is compatible with the input information.*

## 2 CONDITIONAL FUZZY ARITHMETIC

From now on we consider the sets $\mathbb{R}^n$, $n \in \mathbb{N}$, with the metrics that are determined by any of the equivalent norms

$$\|(a_1, a_2, \dots, a_n)\|_p = (|a_1|^p + |a_2|^p + \dots + |a_n|^p)^{\frac{1}{p}},$$

for $p \in [1, +\infty)$, or

$$\|(a_1, a_2, \dots, a_n)\|_\infty = \max\{|a_1|, |a_2|, \dots, |a_n|\}.$$

The choice of norms does not affect the validity of the following statements. The obtained metric space $\mathbb{R}^n$ is proper, consequently, $(\mathcal{F}_0(\mathbb{R}^n), d_H)$ is a complete space.

Let $X$ be a proper metric space, and let weakly bounded joint possibility distributions of $\mathbb{R}$-valued variables $A_1, A_2$ with an $X$-valued variable $B$ be given. Having estimated a joint distribution of $(A_1, A_2, B)$ by the function $A_1 \otimes_B A_2$ that was constructed above, we define the sum and the product of $A_1$ and $A_2$ over $B$ as the weakly bounded $\mathbb{R}$-valued fuzzy variables with the following joint distributions with $B$:

$$A_1 \oplus_B A_2 = \mathcal{F}_0((+) \times \mathbf{1}_{\mathbb{I}})(A_1 \otimes_B A_2), \qquad A_1 \odot_B A_2 = \mathcal{F}_0((\cdot) \times \mathbf{1}_{\mathbb{I}})(A_1 \otimes_B A_2),$$

where $(+) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $(\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are respectively the addition and the multiplication of real numbers.

**Theorem 1.** *The operations*

$$\oplus_{\dots}, \odot_{\dots} : \left\{(C_1, C_2) \in \mathcal{F}_0(\mathbb{R} \times X) \times \mathcal{F}_0(\mathbb{R} \times X) \;\middle|\; \mathcal{F}(\mathrm{pr}_2)(C_1) = \mathcal{F}(\mathrm{pr}_2)(C_2)\right\} \to \mathcal{F}_0(\mathbb{R} \times X)$$

*are continuous, associative, and commutative.*

We call the defined operations the *conditional addition* and the *conditional multiplication* of weakly bounded fuzzy numbers considering their joint distributions with a third weakly bounded fuzzy variable.

**Remark.** *Obviously the conditional sum of fuzzy vectors, i.e., of $\mathbb{R}^n$-valued fuzzy variables, can be defined analogously. Subtraction is obtained as addition of $A_1$ and $(-1) \cdot A_2$. To construct division, it is necessary to restrict ourselves to the divisors with zero possibility of the value* $0$.

## 3 CONCLUDING REMARKS

The obtained conditional generalizations of the operations of fuzzy arithmetic can improve efficiency of fuzzy models, e.g., in network planning, fuzzy optimization etc.

We should also note that, along with the simplest fuzzy conjunction $\wedge$, a wide spectrum of so called triangular norms ($t$-norms) is used for the construction of fuzzy arithmetic [3]. They can be more adequate for specific classes of problems. Such generalizations and their analysis will be the topic of a future publication.

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Запропоновано оцінку спільного розподілу можливості для двох нечітких чисел, яка спирається на нечіткі відношення з третьою нечіткою величиною. Як наслідок, отримано операції умовного додавання і умовного множення взаємодіючих нечітких чисел, для яких відомі спільні розподіли можливості з нечіткою величиною.

*Ключові слова і фрази:* нечітка арифметика, розподіл можливості.

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Предложена оценка совместного распределения возможности для двух нечетких чисел, основанная на нечетких отношениях с третьей нечеткой величиной. Как следствие, получены операции условного сложения и условного умножения взаимодействующих нечетких чисел, для которых известны совместные распределения возможности с нечеткой величиной.

*Ключевые слова и фразы:* нечеткая арифметика, распределение возможности.